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FINITE-DIMENSIONAL CONTROL OF NONLINEAR PARABOLIC PDE SYSTEMS WITH TIME-DEPENDENT SPATIAL DOMAINS USING EMPIRICAL EIGENFUNCTIONS

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This article presents a methodology for the synthesis of finite-dimensional nonlinear output feedback controllers for nonlinear parabolic partial differential equation (PDE) systems with time-dependent spatial domains. Initially, the nonlinear parabolic PDE system is expressed with respect to an appropriate time-invariant spatial coordinate, and a representative (with respect to different initial conditions and input perturbations) ensemble of solutions of the resulting time-varying PDE system is constructed by computing and solving a high-order discretization of the PDE. Then, the Karhunen-Loève expansion is directly applied to the ensemble of solutions to derive a small set of empirical eigenfunctions (dominant spatial patterns) that capture almost all the energy of the ensemble of solutions. The empirical eigenfunctions are subsequently used as basis functions within a Galerkin model reduction framework to derive low-order ordinary differential equation (ODE) systems that accurately describe the dominant dynamics of the PDE system. The ODE systems are subsequently used for the synthesis of nonlinear output feedback controllers using geometric control methods. The proposed control method is used to stabilize an unstable steady-state of a diffusion-reaction process with nonlinearities, spatially-varying coefficients and time-dependent spatial domain, and is shown to lead to the construction of accurate low-order models and the synthesis of low-order controllers. The performance of the low-order models and controllers is successfully tested through computer simulations.

Keywords: Karhunen-Loève expansion, Galerkin's method, nonlinear control, diffusion-reaction processes with moving boundaries

1. Introduction

There is a large number of industrial control problems which involve highly nonlinear transport-reaction processes with moving boundaries, such as crystal growth, metal casting, gas-solid reaction systems and coatings. In these processes, nonlinear behavior typically arises from complex reaction mechanisms and their Arrhenius dependence

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on the temperature, while the motion of boundaries is usually a result of a phase change, such as a chemical reaction, mass and heat transfer, and melting or solidification. The mathematical models of transport-reaction processes with moving boundaries are usually obtained from the dynamic conservation equations and consist of nonlinear parabolic partial differential equations (PDEs) with time-dependent spatial domains. Research on control of linear/quasi-linear parabolic PDEs has been extensive in the past and has mainly focused on systems with fixed spatial domains (see, for example, the review papers (Balas, 1982; Lasiecka, 1995) and the book (Christofides, 2001) for discussion and references). The main feature of parabolic PDE systems is that the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement (Friedman, 1976). This implies that the dominant dynamic behavior of such systems can be approximately described by finite-dimensional systems. Therefore, the standard approach to the control of linear/quasi-linear parabolic PDE systems (e.g., Balas, 1979; Chen and Chang, 1992) involves the application of standard Galerkin's method to the parabolic PDE system to derive ordinary differential equation (ODE) systems that accurately describe the dominant dynamics of the PDE system, which are subsequently used as the basis for controller synthesis.

Unfortunately, the developed control methods for quasi-linear parabolic PDE systems cannot be directly employed for the design of low-dimensional controllers for systems that include nonlinear spatial differential operators. The reason is that, in general, the eigenvalue problem of nonlinear spatial differential operators cannot be solved analytically, and thus, it is difficult to choose *a priori* (without having any information about the solution of the system) an optimal (in the sense that it will lead to a low-dimensional ODE system) basis to expand the solution of the PDE system. An approximate way to address this problem (Ray, 1981) is to linearize the nonlinear spatial differential operator around a steady state and address the controller design problem on the basis of the resulting quasi-linear system. However, this approach is only valid in a small neighborhood of the steady state where the linearization takes place. An alternative approach which is not based on linearization is to utilize detailed finite difference/element simulations of the PDE system to compute a set of empirical eigenfunctions (dominant spatial patterns) of the system through Karhunen-Loève expansion (also known as proper orthogonal decomposition and principal component analysis), using the method of snapshots (Sirovich, 1987). The use of empirical eigenfunctions as basis functions in Galerkin's method has been shown to lead to the derivation of accurate nonlinear low-dimensional approximations of several dissipative PDE systems arising in the modeling of diffusion-reaction processes and fluid flows (Bangia *et al.*, 1997; Park and Cho, 1996). Recently, empirical eigenfunctions were computed using K-L expansion for PDE systems whose solutions contain traveling structures by exploiting symmetry (Rowley and Marsden, 2000). In the area of control, linear feedback controllers were synthesized in (Shvartsman and Kevrekidis, 1998; Theodoropoulou *et al.*, 1999) for specific diffusion-reaction systems on the basis of low-dimensional models obtained using empirical eigenfunctions as basis functions in Galerkin's method, and a method for the design of nonlinear output feedback controllers for nonlinear parabolic PDE systems was developed in (Baker and Christofides, 2000) by combining Galerkin's method with empirical eigenfunctions.

Despite the recent progress on nonlinear control of parabolic PDE systems with fixed spatial domains, few results are available on control and estimation of parabolic PDE systems with time-dependent spatial domains. In this area, important contributions include work on the synthesis of linear optimal controllers (e.g., Wang, 1967; 1990), as well as the synthesis of nonlinear distributed state estimators using stochastic methods in (Ray and Seinfeld, 1975), and the design of nonlinear and robust controllers on the basis of finite-dimensional models obtained using a combination of Galerkin's method with approximate inertial manifolds (Armaou and Christofides, 1999; 2001).

The objective of this paper is to present a methodology for the synthesis of finite-dimensional nonlinear output feedback controllers for nonlinear parabolic PDE systems with time-dependent spatial domains. Initially, the nonlinear parabolic PDE system is expressed with respect to an appropriate time-invariant spatial coordinate and a representative (with respect to different initial conditions and input perturbations) ensemble of solutions of the resulting time-varying PDE system is constructed by computing and solving a high-order discretization of the PDE. Then, Karhunen-Loève expansion is directly applied to the ensemble of solutions to derive a small set of empirical eigenfunctions (dominant spatial patterns) that capture almost all the energy of the ensemble of solutions. The empirical eigenfunctions are subsequently used as basis functions within a Galerkin model reduction framework to derive low-order ODE systems that accurately describe the dominant dynamics of the PDE system. The ODE systems are subsequently used for the synthesis of nonlinear output feedback controllers using geometric control methods. The proposed control method is used to stabilize an unstable steady-state of a diffusion-reaction process with nonlinearities, spatially-varying coefficients and time-dependent spatial domain, and is shown to lead to the construction of accurate low-order models and the synthesis of low-order controllers. The performance of the low-order models and controllers is successfully tested through computer simulations.

2. Nonlinear Parabolic PDE Systems with Moving Domains

2.1. Description of PDE Systems

We consider nonlinear parabolic PDE systems in one spatial dimension with the following state space description:

$$
\begin{aligned}
\frac{\partial \bar{x}}{\partial t} &= L(\bar{x}) + wb(z,t)u + f(t,\bar{x}), \\
y_{c_i} &= \int_0^{l(t)} c_i(z,t)k\bar{x}\,\mathrm{d}z, \qquad i = 1,\dots,l, \\
y_{m_\kappa} &= \int_0^{l(t)} s_\kappa(z,t)\omega\bar{x}\,\mathrm{d}z, \quad \kappa = 1,\dots,q,
\end{aligned}
\tag{1}
$$

subject to the boundary conditions

$$\begin{aligned} C_1\bar{x}(0,t) + D_1\frac{\partial \bar{x}}{\partial z}(0,t) &= R_1, \\ C_2\bar{x}\big(l(t),t\big) + D_2\frac{\partial \bar{x}}{\partial z}\big(l(t),t\big) &= R_2, \end{aligned} \tag{2}$$

and the initial condition

$$\bar{x}(z,0) = \bar{x}_0(z), \tag{3}$$

where the rate of change of the length of the domain, $l(t)$, is governed by the following ordinary differential equation:

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \mathcal{G}\left(t, l, \int_0^{l(t)} \bar{a}\left(z, t, l, \bar{x}, \frac{\partial \bar{x}}{\partial z}\right)\mathrm{d}z\right), \tag{4}$$

where $\bar{x}(z,t) = [\bar{x}_1(z,t)\ \cdots\ \bar{x}_n(z,t)]^T$ denotes the vector of state variables, $[0,l(t)] \subset \mathbb{R}$ is the domain of definition of the process, $z \in [0,l(t)]$ is the spatial coordinate, $t \in [0,\infty)$ is the time, $u = [u_1\ u_2\ \cdots\ u_l]^T \in \mathbb{R}^l$ denotes the vector of manipulated inputs, $y_{c_i} \in \mathbb{R}$ denotes the i-th controlled output, and $y_{m_\kappa} \in \mathbb{R}$ denotes the κ-th measured output. $L(\bar{x})$ is a nonlinear differential operator which involves first- and second-order spatial derivatives, $f(t,\bar{x})$, $\mathcal{G}(t, l, \int_0^{l(t)} \bar{a}(z,t,l,\bar{x},\frac{\partial \bar{x}}{\partial z})\,\mathrm{d}z)$ are nonlinear vector functions, $\bar{a}(z,t,l,\bar{x},\frac{\partial \bar{x}}{\partial z})$ is a nonlinear scalar function, k, w, ω are constant vectors, A, B, C_1, D_1, C_2, D_2 are constant matrices, R_1, R_2 are column vectors, and $\bar{x}_0(z)$ is the initial condition.

$b(z,t)$ is a known smooth vector function of (z,t) of the form $b(z,t) = [b_1(z,t)\ b_2(z,t)\ \cdots\ b_l(z,t)]$, where $b_i(z,t)$ describes how the control action $u_i(t)$ is distributed in the interval $[0,l(t)]$ (e.g., point/distributed actuation), $c_i(z,t)$ is a known smooth function of (z,t) which is determined by the desired performance specifications in the interval $[0,l(t)]$ (e.g., regulation of the entire temperature profile of a crystal or regulation of the temperature at a specific point), and $s_\kappa(z,t)$ is a known smooth function of (z,t) which is determined by the location and type of the κ-th measurement sensor (e.g., point/distributed sensing). In the case of point actuation (i.e., where the control action enters the system at a single point z_0, with $z_0 \in [0,l(t)]$), the function $b_i(z,t)$ is taken to be nonzero in a finite spatial interval of the form $[z_0-\epsilon, z_0+\epsilon]$, where ϵ is a small positive real number, and zero elsewhere in $[0,l(t)]$. We note that in contrast to the case of parabolic PDE systems defined on a fixed spatial domain, we allow the actuator, performance specification and measurement sensor functions to depend explicitly on time (i.e., moving control actuators and objectives, and measurement sensors). The value of using moving control actuators and sensors in applications with time-dependent domains was illustrated in (Armaou and Christofides, 2001). Throughout the paper, we will use the order of magnitude notation $O(\epsilon)$ (i.e., $\delta(\epsilon) = O(\epsilon)$ if there exist positive real numbers k_1 and k_2 such that $|\delta(\epsilon)| \le k_1|\epsilon|$, $\forall\ |\epsilon| < k_2$). In order to simplify the notation of this manuscript, we assume that $l(t)$ is a known and smooth function of time. Specifically, our assumptions on the properties of $l(t)$ are precisely stated below:

Assumption 1. $l(t)$ is a known smooth (i.e., $\dot{l}$ exists and is bounded, $\forall\, t \in [0, \infty)$) function of time which satisfies $l(t) \in (0, l_{\max}]$, $\forall\, t \in [0, \infty)$, where $l_{\max}$ denotes the maximum length of the spatial domain.

Finally, throughout the paper, we assume that the nonlinear parabolic PDE systems under consideration, with and without feedback control, possess a unique solution which is also sufficiently smooth (i.e., all the spatial and temporal derivatives in the PDEs are smooth functions of space and time); the reader may refer to the books (Curtain and Zwart, 1995; Pazy, 1983) for techniques and results for studying the mathematically delicate questions of existence, uniqueness and regularity of solutions for various classes of PDE systems. In addition, we focus our attention on nonlinear parabolic PDE systems for which the manipulated inputs, measured and controlled outputs are bounded. From a practical point of view, this means that we do not deal with control problems that involve boundary actuation, measurements and control objectives, even though several problems of this kind can be addressed by the proposed method. Linear infinite dimensional systems with unbounded manipulated inputs, measurements and control objectives have been studied extensively (see, for example, (Curtain, 1982; van Keulen, 1993)).

2.2. Formulation of the Parabolic PDE System in Hilbert Space

We formulate the system of eqns. (1)–(3) in a Hilbert space $\mathcal{H}(t)$ consisting of n-dimensional vector functions defined on $[0, l(t)]$ that satisfy the boundary conditions of eqn. (2), with inner product and norm

$$\begin{aligned} (\omega_1, \omega_2) &= \int_0^{l(t)} \big(\omega_1(z), \omega_2(z)\big)_{\mathbb{R}^n}\, \mathrm{d}z, \\ \|\omega_1\|_2 &= (\omega_1, \omega_1)^{\frac{1}{2}}, \end{aligned} \tag{5}$$

where ω_1, ω_2 are two elements of $\mathcal{H}(t)$ and the notation $(\cdot, \cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$. To this end, we define the state function x on $\mathcal{H}(t)$ as

$$x(t) = \bar{x}(z, t), \quad t > 0, \quad z \in \big[0, l(t)\big], \tag{6}$$

the time-varying operator

$$\begin{aligned} &\mathcal{L}(t, x) = L(\bar{x}) + \frac{\dot{l}}{l(t)} z \frac{\partial \bar{x}}{\partial z}, \\ &x \in D(\mathcal{L}) = \bigg\{ x \in \mathcal{H}(t) : \;\; C_1 \bar{x}(0, t) + D_1 \frac{\partial \bar{x}}{\partial z}(0, t) = R_1, \\ &\qquad\qquad C_2 \bar{x}(l(t), t) + D_2 \frac{\partial \bar{x}}{\partial z}\big(l(t), t\big) = R_2 \bigg\}, \end{aligned} \tag{7}$$

and the input, controlled output and measurement operators as

$$\mathcal{B}(t)u = wb(t)u, \quad \mathcal{C}(t)x = \big(c(t), kx\big), \quad \mathcal{S}(t)x = \big(s(t), \omega x\big), \tag{8}$$

where $c(t) = [c_1(t)\ c_2(t)\ \cdots\ c_l(t)]^T$ and $s(t) = [s_1(t)\ s_2(t)\ \cdots\ s_q(t)]^T$, and $c_i(t) \in \mathcal{H}(t)$, $s_\kappa(t) \in \mathcal{H}(t)$. The system of eqns. (1)–(3) can then be written as

$$\begin{aligned} \dot{x} &= \mathcal{L}(t,x) + \mathcal{B}(t)u + f(t,x), \quad x(0) = x_0, \\ y_c &= \mathcal{C}(t)x, \ \ y_m = \mathcal{S}(t)x, \end{aligned} \tag{9}$$

where $f(t,x(t)) = f(t,\bar{x}(z,t))$ and $x_0 = \bar{x}_0(z)$. We assume that the nonlinear term $f(t,x)$ satisfies $f(t,0) = 0$ and is also locally Lipschitz continuous uniformly in t, i.e., there exist positive real numbers a_0, K_0 such that for any $x_1, x_2 \in \mathcal{H}(t)$ that satisfy $\max\{\|x_1\|_2, \|x_2\|_2\} \le a_0$, we have

$$\|f(t,x_1) - f(t,x_2)\|_2 \le K_0 \|x_1 - x_2\|_2, \quad \forall\, t \in [0,\infty). \tag{10}$$

Remark 1. In the formulation of the PDE system of eqns. (1)–(3) in $\mathcal{H}(t)$, the time-varying term $\frac{\dot{l}(t)}{l(t)} z \frac{\partial \bar{x}}{\partial z}$ in the expression of $\mathcal{L}(t,x)$ (eqn. (7)) accounts for convective transport owing to the motion of the domain. This term was not present in the expression of the differential operator in the case of parabolic PDE systems with fixed spatial domains (where $\dot{l}(t) \equiv 0$), and makes $\mathcal{L}(t,x)$ an explicit function of time.

2.3. Methodology for Model Reduction and Control

The main obstacles in developing a general model reduction method for systems of the form of eqn. (1) are: (a) the spatial differential operator is nonlinear, and (b) the domain of definition of the process is generally time-varying. These issues do not allow the computation of analytic expressions for the eigenvalues and eigenfunctions of the spatial differential operator of the system, and thus, they prohibit the direct use of Galerkin's methods or orthogonal collocation methods with the eigenfunctions as basis functions to derive finite-dimensional approximations of the PDE system.

To overcome the above problems, we employ the following methodology for the derivation of finite-dimensional approximations and the synthesis of low-dimensional nonlinear output feedback controllers for systems of the form of eqn. (1):

1. Initially, the nonlinear parabolic PDE system is expressed with respect to an appropriate time-invariant spatial coordinate, and a representative (with respect to different initial conditions and input perturbations) ensemble of solutions of the resulting time-varying PDE system is constructed either by computing and solving a high-order discretization of the PDE, or alternatively, using available process data (see Remark 5).

2. Then, Karhunen-Loève expansion (and in particular the method of snapshots (Sirovich, 1987)) is directly applied to the ensemble of solutions to derive a small set of empirical eigenfunctions (dominant spatial patterns) that capture almost all the energy of the ensemble of solutions.
3. The empirical eigenfunctions are subsequently used as basis functions within a Galerkin's model reduction framework to derive low-order ODE systems that accurately describe the dominant dynamics of the PDE system.
4. These ODE systems are used as a basis for the synthesis of low dimensional nonlinear controllers, which use on-line measurements of process outputs to stabilize the closed-loop infinite-dimensional system and force its outputs to follow their respective set-points.

3. Order Reduction

3.1. Computation of Empirical Eigenfunctions

In this section, we review the K-L expansion in the context of nonlinear one-dimensional parabolic PDE systems of the form of eqn. (1) with $n = 1$ (see (Fukunaga, 1990; Holmes *et al.*, 1996) for a general presentation and analysis of the K-L expansion). Introducing the time-invariant spatial coordinate $\zeta = z/l(t)$, the system of eqns. (1)–(3) can be written in the following form:

$$\begin{aligned} \frac{\partial \bar{x}}{\partial t} &= L(\bar{x}) + \frac{\dot{l}}{l(t)}\zeta\frac{\partial \bar{x}}{\partial \zeta} + wb(\zeta,t)u + f(t,\bar{x}), \\ y_{ci} &= l(t)\int_0^1 c_i(\zeta,t)k\bar{x}\,\mathrm{d}\zeta, \quad i = 1,\dots,l, \\ y_{m\kappa} &= l(t)\int_0^1 s_\kappa(\zeta,t)\omega\bar{x}\,\mathrm{d}\zeta, \quad \kappa = 1,\dots,q, \end{aligned} \tag{11}$$

with boundary conditions

$$\begin{aligned} C_1\bar{x}(0,t) + \frac{D_1}{l(t)}\frac{\partial \bar{x}}{\partial \zeta}(0,t) &= R_1, \\ C_2\bar{x}(1,t) + \frac{D_2}{l(t)}\frac{\partial \bar{x}}{\partial \zeta}(1,t) &= R_2 \end{aligned} \tag{12}$$

and the initial condition

$$\bar{x}(\zeta,0) = \bar{x}_0\big(\zeta l(t)\big). \tag{13}$$

We assume that the solution of the system of eqn. (11) is known and consider a sufficiently large set (called ensemble), $\{\bar{x}_\kappa\}$, consisting of N sampled states, $\bar{x}_\kappa(\zeta)$ (typically called "snapshots") of the solution of eqn. (1). To simplify our presentation, we assume uniform in time sampling of $\bar{x}_\kappa(\zeta)$, (i.e., the time interval between any two successive sampled states is the same), while we define the ensemble average of snapshots as $\langle\bar{x}_\kappa\rangle := (1/N)\sum_{\kappa=1}^{N}\bar{x}_\kappa(\zeta)$ (we note that non-uniform sampling of the snapshots and weighted ensemble average can be also considered; see, for example, (Graham and Kevrekidis, 1996)). Furthermore, the ensemble average of snapshots $\langle\bar{x}_\kappa\rangle$ is subtracted out from the snapshots, i.e.,

$$x_\kappa = \bar{x}_\kappa - \langle\bar{x}_\kappa\rangle, \tag{14}$$

so that only fluctuations are analyzed. The issue is how to obtain the most typical or characteristic structure $\phi(\zeta)$ among these snapshots $\{x_\kappa\}$. Mathematically, this problem can be posed as the one of obtaining a function $\phi(\zeta)$ that maximizes the following objective function:

$$\begin{aligned} &\text{Maximize } \frac{\langle(\phi, x_\kappa)^2\rangle}{(\phi,\phi)}, \\ &\text{s.t. } (\phi,\phi) = 1, \quad \phi \in L^2\big([0,1]\big). \end{aligned} \tag{15}$$

The constraint $(\phi,\phi) = 1$ is imposed to ensure that the function $\phi(\zeta)$, computed as a solution to the above maximization problem, is unique. The Lagrangian functional corresponding to this constrained optimization problem is

$$\bar{L} = \big\langle(\phi, x_\kappa)^2\big\rangle - \lambda\big((\phi,\phi) - 1\big) \tag{16}$$

and a necessary condition for extrema is that the functional derivative vanishes for all variations $\phi + \delta\psi \in L^2[0,1]$, where δ is a real number:

$$\left.\frac{\mathrm{d}\bar{L}(\phi+\delta\psi)}{\mathrm{d}\delta}\right|_{\delta=0} = 0, \quad (\phi,\phi) = 1. \tag{17}$$

Using the definitions of inner product and ensemble average, $\left.\frac{\mathrm{d}\bar{L}(\phi+\delta\psi)}{\mathrm{d}\delta}\right|_{\delta=0}$ can be computed as follows:

$$\begin{aligned} \left.\frac{\mathrm{d}\bar{L}(\phi+\delta\psi)}{\mathrm{d}\delta}\right|_{\delta=0} &= \frac{\mathrm{d}}{\mathrm{d}\delta}\big[\big\langle(x_\kappa, \phi+\delta\psi)(\phi+\delta\psi, x_\kappa)\big\rangle - \lambda(\phi+\delta\psi, \phi+\delta\psi)\big]_{\delta=0} \\ &= 2\mathrm{Re}\big[\big\langle(x_\kappa,\psi)(\phi,x_\kappa)\big\rangle - \lambda(\phi,\psi)\big] \\ &= \left\langle\int_0^1 \psi(\zeta)x_\kappa(\zeta)\,\mathrm{d}\zeta\int_0^1\phi(\xi)x_\kappa(\xi)\,\mathrm{d}\xi\right\rangle - \lambda\int_0^1\phi(\xi)\psi(\xi)\,\mathrm{d}\xi \\ &= \int_0^1\int_0^1\big\langle x_\kappa(\zeta)x_\kappa(\xi)\big\rangle\phi(\zeta)\,\mathrm{d}\zeta\psi(\xi)\,\mathrm{d}\xi - \int_0^1\lambda\phi(\xi)\psi(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{18}$$

Since $\psi(\xi)$ is an arbitrary function, the necessary conditions for optimality take the form

$$\int_0^1\big\langle x_\kappa(\zeta)x_\kappa(\xi)\big\rangle\phi(\zeta)\,\mathrm{d}\zeta = \lambda\phi(\xi), \quad (\phi,\phi) = 1. \tag{19}$$

Introducing the two-point correlation function:

$$K(\zeta,\xi) := \langle x_\kappa(\zeta)x_\kappa(\xi)\rangle = \frac{1}{N}\sum_{\kappa=1}^{N} x_\kappa(\zeta)x_\kappa(\xi) \tag{20}$$

and the linear operator:

$$R := \int_0^1 K(\zeta,\xi)\,\mathrm{d}\xi, \tag{21}$$

the optimality condition of eqn. (19) reduces to the following eigenvalue problem of the integral equation:

$$R\phi = \lambda\phi \Longrightarrow \int_0^1 K(\zeta,\xi)\phi(\xi)\,\mathrm{d}\xi = \lambda\phi(\zeta). \tag{22}$$

The computation of the solution to the above integral eigenvalue problem is, in general, a very computationally expensive task. To circumvent this problem, Sirovich introduced in 1987 (Sirovich, 1987) the method of snapshots. The central idea of this technique is to assume that the required eigenfunction, $\phi(\zeta)$, can be expressed as a linear combination of the snapshots, i.e.,

$$\phi(\zeta) = \sum_k c_k x_k(\zeta). \tag{23}$$

Substituting the above expression for $\phi(\zeta)$ into eqn. (22), we obtain the following eigenvalue problem:

$$\int_0^1 \frac{1}{N}\sum_{\kappa=1}^{N} x_\kappa(\zeta)x_\kappa(\xi)\sum_{k=1}^{N} c_k x_k(\xi)\,\mathrm{d}\xi = \lambda\sum_{k=1}^{N} c_k x_k(\zeta). \tag{24}$$

Defining

$$B^{\kappa k} := \frac{1}{N}\int_0^1 x_\kappa(\xi)x_k(\xi)\,\mathrm{d}\xi \tag{25}$$

the eigenvalue problem of eqn. (24) can be equivalently written as

$$Bc = \lambda c. \tag{26}$$

The solution of the above eigenvalue problem (which can be obtained by utilizing standard methods from matrix theory) yields the eigenvectors $c = [c_1 \cdots c_N]$ which can be used in eqn. (23) to construct the eigenfunction $\phi(\zeta)$. Due to its structure, it follows that the matrix B is symmetric and positive semi-definite, and thus, its eigenvalues, λ_κ, $\kappa = 1,\dots,N$, are real and non-negative. Furthermore,

$$\int_0^1 \phi_\kappa(\zeta)\phi_k(\zeta)\,\mathrm{d}\zeta = 0, \quad \kappa \neq k. \tag{27}$$

Remark 2. The calculated eigenvalues, once normalized, represent the percentage of energy, or equivalently, time that the solution of the PDE system spends along the spatial structure of the empirical eigenfunction.

Remark 3. Note that the empirical eigenfunctions that were derived using the above methodology can be used directly as basis functions of the Hilbert space defined in Subsection 2.2 since the two spaces are completely analogous, and the basis functions can be interchanged through the coordinate transformation $z = \zeta\ l(t)$ (note that in order for the basis functions to be orthonormal instead of just orthogonal to each other, the multiplication $\phi_j(z) = \sqrt{l(t)}\ \phi_j(\zeta)$ must also take place).

Remark 4. The K-L expansion can also be used to compute a set of empirical eigenfunctions when the optimization objective of eqn. (15) is modified to $\langle(\dot{\phi}, \dot{x}_\kappa)^2\rangle$ under the same constraints. The modified objective focuses on the acceleration energy of the fluctuations towards the average value, rather than the energy of the fluctuations which is used in our development. It has been shown that the empirical eigenfunctions that are generated using the modified objective function also form a set of orthonormal basis functions (the reader may refer to (Sirovich *et al.*, 1990) for a detailed discussion on this issue). A variety of optimization objectives have been formulated in the literature to address the specific characteristics of processes, such as periodic phenomena, etc.

Remark 5. The ensemble of solutions can be derived from process data as well as through the solution of a high-order discretization of the PDE system of eqn. (11). Process data-based construction may be more suitable in controller design problems for which there is an abundance of measured data and the model of the process is too complex to be solved in a time-effective manner with reasonable computing power (note that numerous model solutions are needed to obtain a representative ensemble).

3.2. Galerkin's Method

In this section, we use Galerkin's method to derive low-dimensional dynamical systems of nonlinear ordinary differential equations that accurately reproduce the dynamics and solutions of the nonlinear parabolic PDE system of eqn. (1). To this end, we assume that we have available an orthogonal and complete set of global (in the sense that they span the entire spatial domain of definition of the process) basis functions, $\phi_i(\zeta)$, that satisfy the boundary conditions of eqn. (2). In practice, $\phi_i(\zeta)$ may be the set of empirical eigenfunctions obtained through K-L expansion.

Let $\mathcal{H}_s$, $\mathcal{H}_f$ be two subspaces of $\mathcal{H}$, defined as $\mathcal{H}_s = \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathcal{H}_f = \text{span}\{\phi_{m+1}, \phi_{m+2}, \ldots, \}$. The basis functions ϕ_j can be obtained through K-L expansion. Defining the orthogonal projection operators P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of eqn. (9) can be decomposed as

$$x = x_s + x_f = P_s x + P_f x. \tag{28}$$

Applying P_s and P_f to the system of eqn. (9) and using the above decomposition for x, the system of eqn. (9) can be equivalently written in the following form:

$$\begin{aligned}\frac{\mathrm{d}x_s}{\mathrm{d}t} &= \mathcal{L}_s(t,x_s,x_f)+\mathcal{B}_s(t)u+f_s(t,x_s,x_f),\\ \frac{\partial x_f}{\partial t} &= \mathcal{L}_f(t,x_s,x_f)+\mathcal{B}_f(t)u+f_f(t,x_s,x_f),\\ y_c &= \mathcal{C}(t)x_s+\mathcal{C}(t)x_f, \quad y_m=\mathcal{S}(t)x_s+\mathcal{S}(t)x_f,\\ x_s(0) &= P_sx(0)=P_sx_0, \quad x_f(0)=P_fx(0)=P_fx_0,\end{aligned} \tag{29}$$

where $\mathcal{L}_s(t,x_s,x_f)=P_s\mathcal{L}(t,x_s+x_f)$, $\mathcal{B}_s(t)=P_s\mathcal{B}(t)$, $f_s(t,x_s,x_f)=P_sf(t,x_s+x_f)$, $\mathcal{L}_f(t,x_s,x_f)=P_f\mathcal{L}(t,x_s+x_f)$, $\mathcal{B}_f(t)=P_f\mathcal{B}(t)$, and $f_f(t,x_s,x_f)=P_ff(t,x_s+x_f)$.

Owing to the parabolic nature of the spatial differential operator, the nonlinear vector $\mathcal{L}_f(t,x_s,x_f)$ satisfies $\mathcal{L}_f(t,x_s,x_f)=L_{fs}(t)x_s+(1/\epsilon)L_f(t)x_f+\tilde{f}_f(t,x_s,x_f)$, where ϵ is a small positive parameter quantifying the separation between the slow (dominant) and fast (negligible) eigenmodes of the spatial operator, and $L_{fs}(t)$, $L_f(t)$ are matrices with $L_f(t)$ being stable (in the sense that the state of the system $\partial x_f/\partial t=L_f(t)x_f$ tends exponentially to zero), and $\tilde{f}_f(t,x_s,x_f)$ is a nonlinear vector function which does not include linear terms. Note that the partial derivative in the term $\partial x_f/\partial t$ is used to denote that x_f belongs to an infinite dimensional Hilbert space. Neglecting the infinite dimensional x_f-subsystem in the system of eqn. (29) (this is equivalent to assuming that $\epsilon=0$ when the system of eqn. (29) is formulated within a singular perturbations framework), the following m-dimensional slow system is obtained:

$$\begin{aligned}\frac{\mathrm{d}x_s}{\mathrm{d}t} &= \mathcal{L}_s(t,x_s)+\mathcal{B}_s(t)u+f_s(t,x_s),\\ y_c &= \mathcal{C}(t)x_s, \ \ y_m=\mathcal{S}(t)x_s,\\ x_s(0) &= P_sx(0)=P_sx_0,\end{aligned} \tag{30}$$

where $\mathcal{L}_s(t,x_s)=P_s\mathcal{L}(t,x_s)$, $\mathcal{B}_s(t)=P_s\mathcal{B}(t)$, $f_s(t,x_s)=P_sf(t,x_s)$. Moreover, $\mathcal{L}_s(t,x_s)$, $f_s(t,x_s)$ are Lipschitz vector functions.

Remark 6. We note that when the approximate ODE model of eqn. (30) is obtained through Galerkin's method with empirical eigenfunctions, it provides a valid approximation of the parabolic PDE model in a broad region of the state space and not only in the region that was used for the computation of the snapshots, provided that the ensemble of snapshots is sufficiently large and contains sufficient information on the global dynamics of the PDE system. This property is a consequence of the fact that the empirical eigenfunctions form an orthogonal set of functions whose dimension is equal to the number of snapshots, and thus, it can be made arbitrarily large (even though completeness of this set cannot be guaranteed). Therefore, the use of empirical eigenfunctions for discretization of the PDE system is not fundamentally different from the use of other standard basis functions sets (sine and cosine functions, Legendre polynomials, etc.) for discretization with Galerkin's method, and thus, the finite-dimensional approximation obtained through Galerkin's method with empirical eigenfunctions is valid in a broad region of the state space, as long as the dimension of the ensemble is sufficiently large.

4. Nonlinear Output Feedback Control

In this section, we synthesize nonlinear finite-dimensional output feedback controllers that guarantee local exponential stability and force the controlled output of the closed-loop PDE system to follow the reference input, provided that ϵ is sufficiently small. The output feedback controllers are constructed through combination of state feedback controllers with state observers.

More specifically, we use the system of eqn. (30) to synthesize nonlinear state feedback controllers of the following general form:

$$u = p(t, x_s) + Q(t, x_s)v, \tag{31}$$

where $p(t, x_s)$ is a smooth vector function, $Q(t, x_s)$ is a smooth matrix, and $v \in \mathbb{R}^l$ is the constant reference input vector. The synthesis of $[p(t, x_s), Q(t, x_s)]$ so that a nonlinear controller of the form of eqn. (31) guarantees local exponential stability and forces the output of the system of eqn. (30) to follow a desired linear response is performed by utilizing geometric control methods for nonlinear ODEs (the details of the controller synthesis can be found in (Palanki and Kravaris, 1997), and are omitted for brevity).

Since measurements of $\bar{x}(z, t)$ (and thus, $x_s(t)$) are usually not available in practice, we assume that there exists a matrix L so that the nonlinear dynamical system

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \mathcal{L}_s(t, \eta) + \mathcal{B}_s(t)u + f_s(t, \eta) + L\big[y_m - \mathcal{S}(t)\eta\big], \tag{32}$$

where η denotes an m-dimensional state vector, is a local exponential observer for the system of eqn. (30) (i.e., the discrepancy $|\eta(t) - x_s(t)|$ tends exponentially to zero).

Theorem 1 that follows provides the synthesis formula of the output feedback controller and conditions that guarantee closed-loop stability and output tracking. To state our result, we need to use the Lie derivative notation and the concepts of relative order and characteristic matrix (which are defined in the appendix) for the system of eqn. (30). First, the Lie derivative of the scalar function $h_i(t, x_s)$ with respect to the vector function $f(t, x_s)$ is defined as $L_f h_i(t, x_s) = (\partial h_i / \partial x_s) f(t, x_s) + \partial h_i / \partial t$ (this definition of Lie derivative was introduced in (Palanki and Kravaris, 1997) and is different from the standard one used in (Isidori, 1989) for the case of time-invariant h_i, f_0), $L_f^k h_i(t, x_s)$ denotes the k-th order Lie derivative and $L_{g_l} L_f^k h_l(t, x_s)$ denotes the mixed Lie derivative. Now, referring to the system of eqn. (30), we set $\mathcal{L}_s(t, x_s) +$

$f_s(t, x_s) = f(t, x_s)$, $\mathcal{B}_s(t) = g(t, x_s)$, $\mathcal{C}_i(t)x_s = h_i(t, x_s)$ to obtain

$$\begin{aligned} \frac{\mathrm{d}x_s}{\mathrm{d}t} &= f(t, x_s) + g(t, x_s)u, \\ y_{cs\,i} &= h_i(t, x_s). \end{aligned} \tag{33}$$

For the above system, the relative order of the output $y_{cs\,i}$ with respect to the vector of manipulated inputs u is defined as the smallest integer r_i for which we have

$$\left[L_{g_1} L_f^{r_i-1} h_i(t, x_s) \ \cdots \ L_{g_l} L_f^{r_i-1} h_i(t, x_s)\right] \not\equiv [0 \ \cdots \ 0] \tag{34}$$

or $r_i = \infty$ if such an integer does not exist. Furthermore, the matrix

$$C(t, x_s) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(t, x_s) & \cdots & L_{g_l} L_f^{r_1-1} h_1(t, x_s) \\ L_{g_1} L_f^{r_2-1} h_2(t, x_s) & \cdots & L_{g_l} L_f^{r_2-1} h_2(t, x_s) \\ \vdots & & \\ L_{g_1} L_f^{r_l-1} h_l(t, x_s) & \cdots & L_{g_l} L_f^{r_l-1} h_l(t, x_s) \end{bmatrix} \tag{35}$$

is the characteristic matrix of the system of eqn. (33).

Theorem 1. *Suppose that the following conditions hold:*

1. *The roots of the equation*

$$\det\big(B(s)\big) = 0, \tag{36}$$

where $B(s)$ is an $l \times l$ matrix whose (i, j)-th element is of the form $\sum_{k=0}^{r_i} \beta_{jk}^i s^k$, lie in the open left-half of the complex plane, where β_{jk}^i are adjustable controller parameters.

2. *The zero dynamics of the system of eqn. (33) are locally exponentially stable.*

Then, there exist positive real numbers $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\epsilon}^*$ such that if $|x_s(0)| \le \tilde{\mu}_1$, $\|x_f(0)\|_2 \le \tilde{\mu}_2$ and $\epsilon \in (0, \tilde{\epsilon}^*]$, and $\eta(0) = x_s(0)$, the dynamic output feedback controller

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \mathcal{L}_s(t, \eta) + \mathcal{B}_s(t) \left\{ \left[\beta_{1r_1} \cdots \beta_{lr_l}\right] C(t, \eta) \right\}^{-1} \left\{ v - \sum_{i=1}^{l} \sum_{k=0}^{r_i} \beta_{ik} L_f^k h_i(t, \eta) \right\} \\ &\quad + f_s(t, \eta) + L\big(y_m - \mathcal{S}(t)\eta\big), \\ u &= \left\{ \left[\beta_{1r_1} \cdots \beta_{lr_l}\right] C(t, \eta) \right\}^{-1} \left\{ v - \sum_{i=1}^{l} \sum_{k=0}^{r_i} \beta_{ik} L_f^k h_i(t, \eta) \right\} \end{aligned} \tag{37}$$

(a) guarantees local exponential stability of the closed-loop system, and
(b) ensures that the outputs of the closed-loop system satisfy

$$\lim_{t \to \infty} |y_{ci}(t) - v_i| = O(\epsilon), \quad i = 1, \ldots, l, \tag{38}$$

where v_i is the set-point for the i-th controlled output.

Proof. Substituting the controller of eqn. (38) into the infinite-dimensional system of eqn. (9), we obtain

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \mathcal{L}_s(t,\eta) + \mathcal{B}_s(t)u(t,\eta) + f_s(t,\eta) + L\big(y_m - \mathcal{S}(t)\eta\big), \\ \dot{x} &= \mathcal{L}(t,x) + \mathcal{B}(t)\big[p(t,\eta) + Q(t,\eta)v\big] + f(t,x), \;\; x(0) = x_0, \\ y_c &= \mathcal{C}(t)x, \quad y_m = \mathcal{S}(t)x. \end{aligned} \tag{39}$$

A direct application of Galerkin's method to the above system with $\mathcal{H}_s = \mathrm{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathcal{H}_f = \mathrm{span}\{\phi_{m+1}, \phi_{m+2}, \ldots, \}$, and P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, yields

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \mathcal{L}_s(t,\eta) + \mathcal{B}_s(t)u(t,\eta) + f_s(t,\eta) + L\big(y_m - \mathcal{S}(t)\eta\big), \\ \frac{\mathrm{d}x_s}{\mathrm{d}t} &= \mathcal{L}_s(t,x_s,x_f) + \mathcal{B}_s(t)\big[p(t,\eta) + Q(t,\eta)v\big] + f_s(t,x_s,x_f), \\ \frac{\partial x_f}{\partial t} &= \mathcal{L}_f(t,x_s,x_f) + \mathcal{B}_f(t)\big[p(t,\eta) + Q(t,\eta)v\big] + f_f(t,x_s,x_f), \end{aligned} \tag{40}$$

or

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \mathcal{L}_s(t,\eta) + \mathcal{B}_s(t)\big[p(t,\eta) + Q(t,\eta)v\big] + f_s(t,\eta) + L\big(y_m - \mathcal{S}(t)\eta\big), \\ \frac{\mathrm{d}x_s}{\mathrm{d}t} &= \mathcal{L}_s(t,x_s,x_f) + \mathcal{B}_s(t)\big[p(t,\eta) + Q(t,\eta)v\big] + f_s(t,x_s,x_f), \\ \epsilon\frac{\partial x_f}{\partial t} &= \epsilon L_{fs}(t)x_s + L_f(t)x_f + \epsilon\tilde{f}_f(t,x_s,x_f) \\ &\quad + \epsilon\mathcal{B}_f(t)\big[p(t,\eta) + Q(t,\eta)v\big] + \epsilon f_f(t,x_s,x_f). \end{aligned} \tag{41}$$

The system of eqn. (41) is in the standard singularly perturbed form (see (Kokotovic *et al.*, 1986) for a precise definition of standard form), with x_s being the slow states and x_f being the fast states. Introducing the fast time-scale $\tau = t/\epsilon$ and setting $\epsilon = 0$, we obtain the following infinite-dimensional fast subsystem from the system of eqn. (41):

$$\frac{\partial x_f}{\partial \tau} = L_f(t)x_f. \tag{42}$$

Since $L_f(t)$ is a stable matrix, we have that the above system is globally exponentially stable. Setting $\epsilon = 0$ in the system of eqn. (41), we have $x_f = 0$ and thus, the finite-dimensional slow system takes the form

$$\begin{aligned}\frac{\mathrm{d}\eta}{\mathrm{d}t} &= \mathcal{L}_s(t,\eta) + \mathcal{B}_s(t)\big[p(t,\eta) + Q(t,\eta)v\big] + f_s(t,\eta) + L\big(y_m - \mathcal{S}(t)\eta\big),\\ \frac{\mathrm{d}x_s}{\mathrm{d}t} &= \mathcal{L}_s(t,x_s,0) + f_s(t,x_s,0) + \mathcal{B}_s(t)\big[p(t,\eta) + Q(t,\eta)v\big]\\ &=: f(t,x_s) + g(t,x_s)\big[p(t,\eta) + Q(t,\eta)v\big],\\ y_{csi} &= \mathcal{C}_i(t)x_s =: h_i(t,x_s).\end{aligned} \tag{43}$$

For the above system, one can show (see (Isidori, 1989; Palanki and Kravaris, 1997) for details) that it is locally exponentially stable provided that Assumptions 1 and 2 of the theorem hold and that the output follows the reference input. Finally, since the infinite-dimensional fast subsystem of eqn. (42) is exponentially stable, an application of Proposition 1 in (Armaou and Christofides, 1999) (a singular perturbation stability result for infinite dimensional systems) yields that there exists an ϵ^* such that if $\epsilon \in (0,\epsilon^*]$, $\max\{|x_s(0)|, \|x_f(0)\|_2\} \leq \delta$, then the state of the closed-loop parabolic PDE system is exponentially stable and that its outputs satisfy the relation of eqn. (38). ■

Remark 7. We note that the approach followed here for the synthesis of output feedback controllers is not applicable to hyperbolic PDE systems (i.e., convection-reaction processes) where the eigenvalues cluster along vertical or nearly vertical asymptotes in the complex plane and thus, the controller synthesis problem has to be addressed directly on the basis of the hyperbolic PDE system (see, for example, (Kowalewski, 1998; 2000) for results on optimal control and (Christofides, 2001) for results on geometric and Lyapunov-based control).

Remark 8. It is important to point out that the result of Theorem 1 can be generalized to the case where the ODE systems used for controller design are obtained from combination of Galerkin's method with approximate inertial manifolds (see (Armaou and Christofides, 1999; Christofides, 2001) for details on the design of output feedback controllers on the basis of such ODE systems and (Byrnes *et al.*, 1994; 1995; Sano and Kunimatsu, 1995) for other applications of inertial manifold theory to control of nonlinear parabolic PDEs).

5. Application to a Diffusion-Reaction Process

In this section, we present an application of the proposed method to a diffusion-reaction process with nonlinearities, spatially-varying coefficients and a time-dependent spatial domain. Specifically, we consider a diffusion-reaction process which is described by the following parabolic PDE in dimensionless form:

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial}{\partial z}\left(k(z)\frac{\partial \bar{x}}{\partial z}\right) + \beta_T(z)\left(e^{-\gamma/(1+\bar{x})} - e^{-\gamma}\right) + \beta_U\big(b(z,t)u(t) - \bar{x}\big) \tag{44}$$

subject to the Dirichlet boundary conditions

$$\bar{x}(0,t) = 0, \quad \bar{x}(l(t),t) = 0, \tag{45}$$

and the initial condition

$$\bar{x}(z,0) = 0.5, \tag{46}$$

where $\bar{x} \in \mathbb{R}$ is the state of the system, γ, β_u are constant dimensionless process parameters, $\beta_T(z), k(z)$ are dimensionless process parameters that are explicit functions of the spatial coordinate z, $u(t) = [u_1(t)\ u_2(t)]^T$ is the vector of the inputs (which will be used in the construction of the ensemble of solutions), and $b(z,t) = [b_1(z,t)\ b_2(z,t)]$ is a vector function which determines how the inputs $u_1(t), u_2(t)$ are distributed in space. The values and expressions of the process parameters that were used in our calculations are: $\beta_U = 2.0$, $\gamma = 4.0$, $\beta_T(z) = 45(1.5 - e^{-0.5\,z})$, $k(z) = e^{-0.5\,z}$ and $l(t) = \pi[1.4 - 0.4\exp(-0.02t^{2.7})]$. Note that $l(t)$ is a smooth function of time (i.e., $\mathrm{d}l/\mathrm{d}t$ exists and is continuous) as requested per Assumption 1.

An accurate high-order discretization of the PDE of eqn. (44) was constructed using Galerkin's method with the following set of basis functions (which is the set of eigenfunctions resulting form the solution of the eigenvalue problem of the spatial operator for the constant value of $k = 1$):

$$\phi_j(z,t) = \sqrt{\frac{2}{l(t)}} \sin\left(j\,\pi\,\frac{z}{l(t)}\right), \quad j = 1, \ldots, \infty. \tag{47}$$

It was found that a 30-th order Galerkin truncation of the system of eqn. (44) using the above basis functions leads to an accurate solution of the PDE (it was verified that a further increase in the order of the Galerkin model as well as reduction in the temporal discretization step provide no substantial improvement on the accuracy of the simulation results). Figure 2 shows the evolution of the state of the PDE for $u(t) = 0$ starting from initial conditions which are very close to the steady-state $\bar{x}(z,t) = 0$. We observe that the system moves to another steady-state which is characterized by a maximum at $z = 0.375\,l(t)$. This implies that the steady state $\bar{x}(z,t) = 0$ is an unstable one, and thus, the system moves to a stable spatially non-uniform steady state.

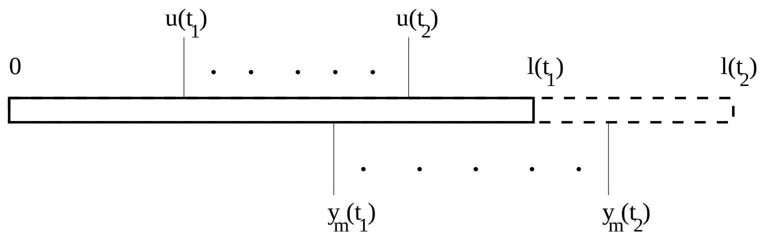

Fig. 1. Schematic of a process with moving boundaries.

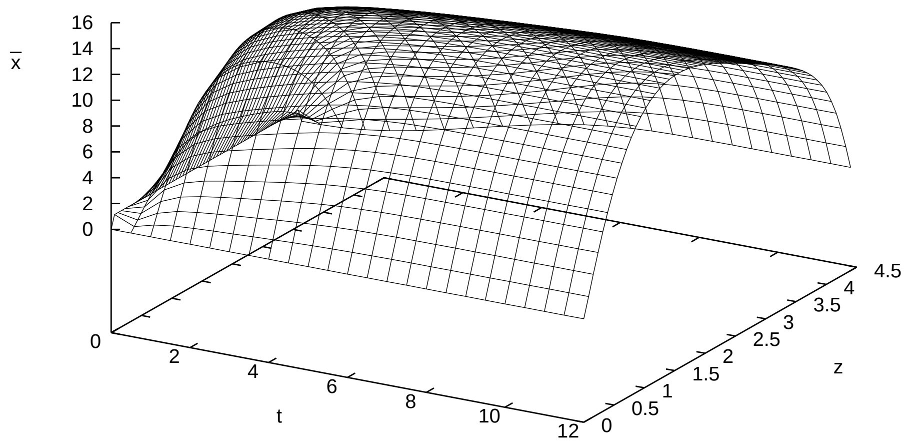

Fig. 2. Profile of $\bar{x}$ for spatially varying β_T, k; $u(t) = 0$.

We now continue with the computation of the set of empirical eigenfunctions. We initially constructed an ensemble of solutions. This was accomplished by solving the high-order discretization of eqn. (44) for four different initial conditions (including the initial condition in the examined problem and the unstable steady-state as an initial condition) and for five different time-profiles of the manipulated variables. This led to a total of 20 spatiotemporal solution profiles. Subsequently, 51 “snapshots” of the profile of the state of eqn. (44) as a function of the spatial coordinate, z, for 51 fixed time instants (during the process time and domain size) were taken from each solution data set and were combined to generate an ensemble of 1020 solutions. To successfully apply Karhunen-Loève expansion, we first expressed the developed ensemble of solutions into an appropriate spatial coordinate $\zeta = z/l(t)$ whose domain of definition is time-invariant. The Karhunen-Loève expansion was then applied to the developed ensemble of solutions to compute seven empirical eigenfunctions that describe the dominant spatial solution patterns embedded in the ensemble (they account for more than 99.9% of the energy included in the entire ensemble). The first three of these empirical eigenfunctions are presented in Figure 3. Note that they are not symmetric with respect to the center of the system, $\zeta = 0.5$, owing to the spatial nonuniformity of $\beta_T = 45(1.5 - e^{-0.5\,z})$ and $k = e^{-0.5\,z}$.

We now proceed with the use of the computed empirical eigenfunctions as basis functions in Galerkin's method to construct accurate low-dimensional ODE approximations of the PDE. To accomplish this, the parabolic PDE system of eqns. (44)–(45) is equivalently expressed in terms of $\zeta = z/l(t)$ as follows:

$$\begin{aligned}\frac{\partial \bar{x}}{\partial t} &= \frac{1}{l(t)^2}\frac{\partial}{\partial \zeta}\left(k(\zeta)\frac{\partial \bar{x}}{\partial \zeta}\right) + \frac{\dot{l}}{l(t)}\zeta\frac{\partial \bar{x}}{\partial \zeta} + \beta_T(\zeta)(e^{-\gamma/(1+\bar{x})} - e^{-\gamma}) \\ &\quad + \beta_U\big(b(\zeta,t)u(t) - \bar{x}\big),\end{aligned} \tag{48}$$

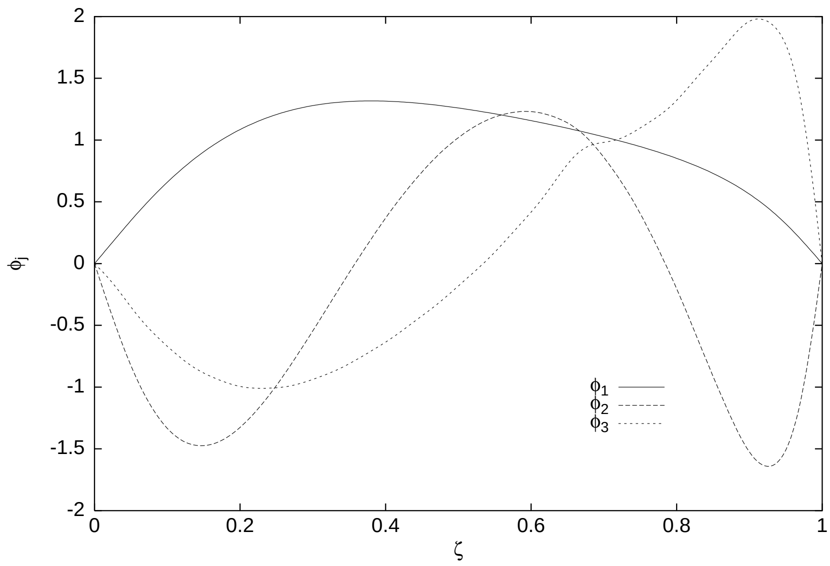

Fig. 3. First three empirical eigenfunctions.

subject to the boundary and initial conditions

$$\bar{x}(0,t) = 0, \quad \bar{x}(1,t) = 0, \quad \bar{x}(\zeta,0) = 0.5. \tag{49}$$

We initially applied Galerkin's method with the first two of the seven empirical eigenfunctions as basis functions to the PDE of eqn. (48) to construct a second-order model. Figure 4 shows the deviation between the spatiotemporal profiles of the state of the system computed by the second-order ODE model and the high-order discretization of the PDE; we observe a very good agreement between the two models for all times, with the maximum deviation being less than 3.4% (the deviation is computed to be the maximum error in the solution profile divided by the value of the state at that point; for accuracy results in the sense of L_2 norms the reader may refer to Table 1).

To further improve the accuracy of the second-order ODE model, we noted that the spatial pattern of the error in Fig. 4 resembles the shape of the third empirical eigenfunction (see Fig. 3), and thus, we applied Galerkin's method with the first three empirical eigenfunctions as basis functions to the PDE of eqn. (48) to construct a third-order model. Figure 5 shows the discrepancy between the third-order model and the high-order discretization of the PDE; it is clear that the addition of the third eigenfunction has greatly improved the accuracy of the low-order approximation (the maximum deviation in this case is less than 0.5%).

For the sake of comparison, we also constructed a third-order approximation of the PDE of eqn. (44) using Galerkin's method with the first three eigenfunctions of eqn. (47) as basis functions. We observe that the error (Fig. 6) between this third-

Table 1. Model error for different process conditions

Basis functions	Model order	Process conditions	Error [%]*	Figure
Empirical	2	nominal	3.300	4
Empirical	3	nominal	0.352	5
Sinusoids	3	nominal	4.939	6
Sinusoids	8	nominal	0.483	
Empirical	3	+20% change in β_T	0.358	8
Empirical	3	−20% change in β_T	0.353	9
Empirical	3	+20% change in k	0.588	10
Empirical	3	−20% change in k	0.459	11
Empirical	3	arbitrary initial condition	0.515	12
Empirical	3	arbitrary initial condition	2.905	13
Empirical	5	constant process parameters	0.037	16
Analytical	5	constant process parameters	0.710	

* The error is calculated using eqn. (50).

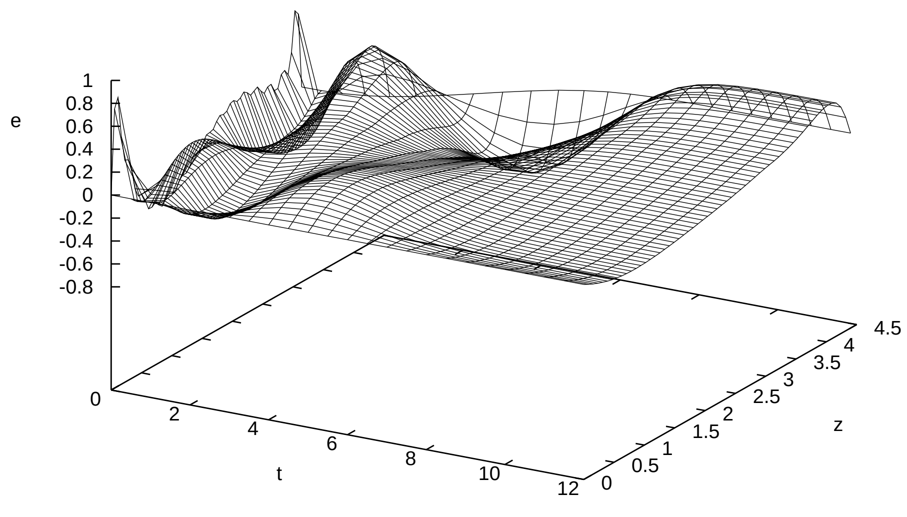

Fig. 4. Deviation between the second-order model and the high-order discretization of the PDE (nominal case).

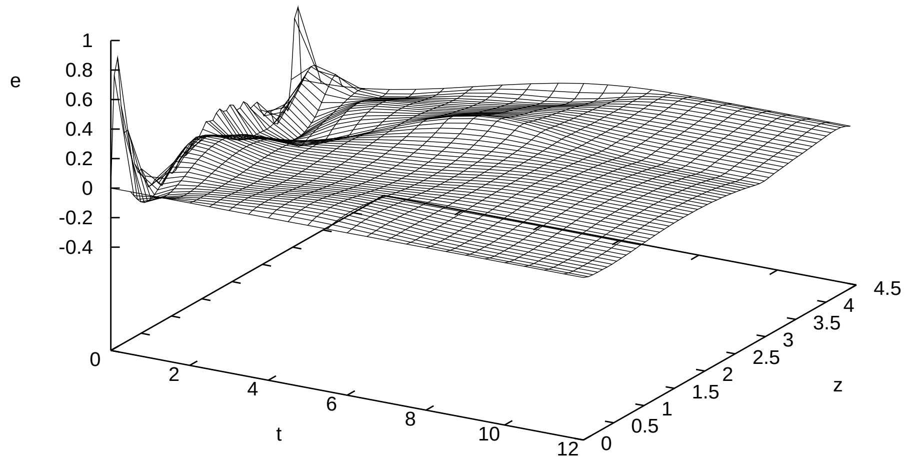

Fig. 5. Deviation between the third-order model and the high-order discretization of the PDE (nominal case).

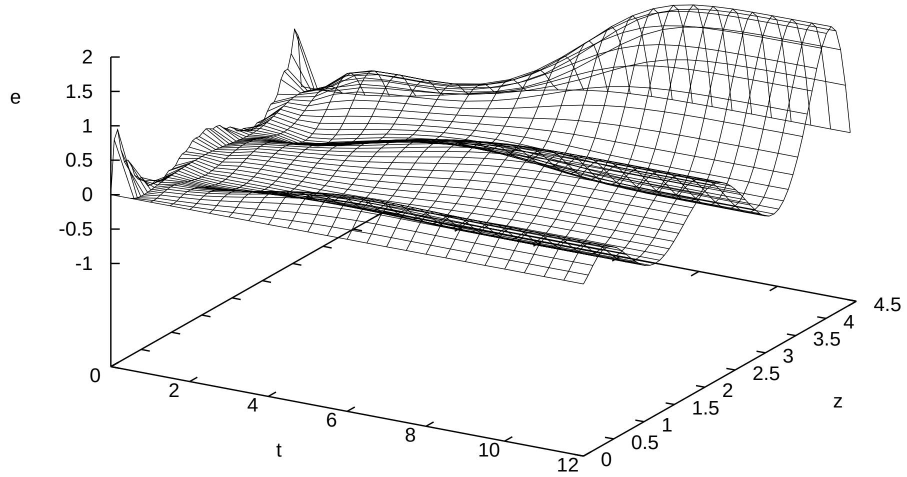

Fig. 6. Deviation between the third-order model and the high-order discretization of the PDE (sinusoidal functions used as basis functions).

order model and the high-order discretization of the PDE is significant (about 11%), which is mainly a result of the fact that this approach does not account for the spatial variation of the coefficient $k(z)$. We note that higher-order approximations computed using this approach result in more accurate models; specifically, we found that a 7-th order model was needed in order to produce the same error of 3.4% as the second-order model based on empirical eigenfunctions, and a 10-th order model was needed to have the same error of 0.5% as the third-order model based on empirical eigenfunctions. This comparison is also presented in Fig. 7, where the relative error between the norm of the reduced-order model and the norm of the high-order discretization of the PDE, $\left|\frac{\|\bar{x}_{\rm low}\|_2-\|\bar{x}_{\rm high}\|_2}{\|\bar{x}_{\rm high}\|_2}\right|$, is shown as a function of time for the cases of a third-order model derived using empirical eigenfunctions, a third-order model derived using sinusoids, and an eighth-order model based on sinusoids. We observe that initially the model based on empirical eigenfunctions gives the highest error, but as time progresses the error becomes insignificant. This is due to the fact that the empirical eigenfunctions capture the dominant spatial structures of the PDE and as a result, the third-order model follows the process dynamics closely after the fast dynamics of the PDE system die out.

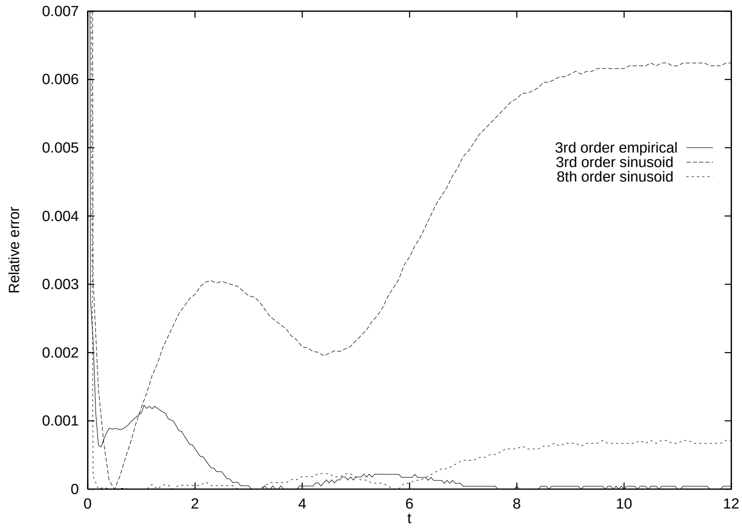

Fig. 7. Relative error profiles (defined as $\left|[\|\bar{x}_{\rm low}\|_2 - \|\bar{x}_{\rm high}\|_2]/\|\bar{x}_{\rm high}\|_2\right|$) between the reduced-order models, based on empirical or analytical basis functions, and the high-order discretization of the PDE (nominal case).

Having numerically established the ability of the computed reduced-order models to describe the PDE, we then tested the ability of the third-order model to give accurate predictions when the process parameters have values which are different from the ones used in the construction of the ensemble of solutions for the computation of the empirical eigenfunctions. Specifically, for $\beta_T = 55(1.5 - e^{-0.5\,z})$ (which corresponds to of variation about $+20\%$ with respect to the nominal value of β_T), the maximum error between the third-order model and the high-order discretization of the PDE is less than 0.5% for all times as can be seen in Fig. 8. On the other hand, for $\beta_T = 45(1.5 - e^{-0.4\,z})$ (which corresponds to of variation about -20% with respect to the nominal value of β_T), the error between the two models remains under 0.7% for all times as can be seen in Fig. 9. We also tested the robustness of the third-order model for a $+20\%$ ($k = e^{-0.4\,z}$) and -20% ($k = e^{-0.6\,z}$) variation in the spatial dependence of k. The corresponding errors between the third-order model and the high-order discretization of the PDE are shown in Figs. 10 and 11, respectively; they remain small for all times, under 0.6% in the first and 1.6% in the second case.

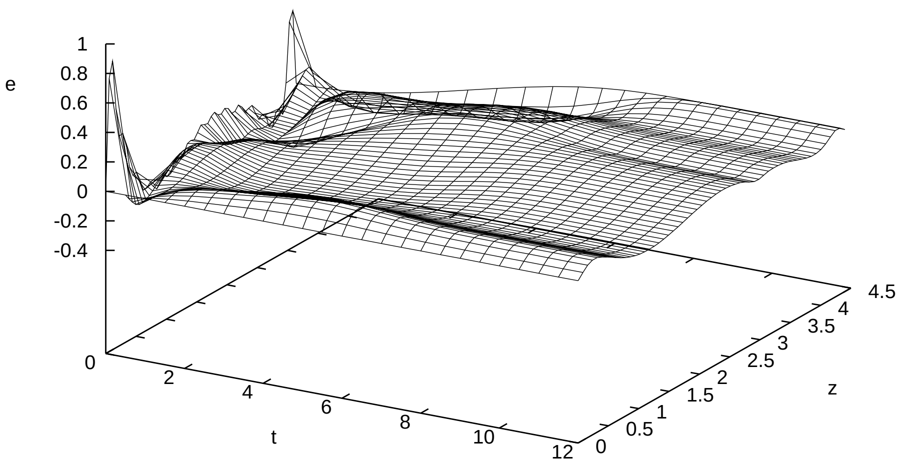

Fig. 8. Deviation between the third-order model and the high-order discretization of the PDE ($\beta_T(z) = 55(1.5 - e^{-0.5\,z})$).

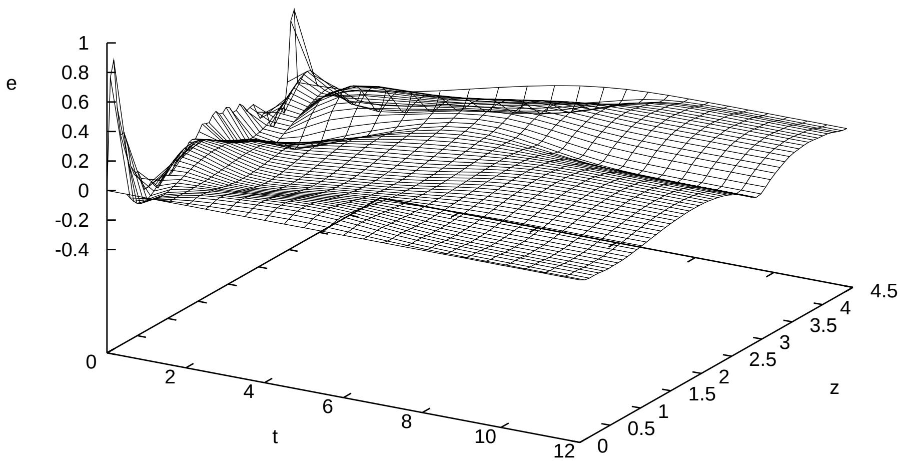

Fig. 9. Deviation between the third-order model and the high-order discretization of the PDE ($\beta_T(z) = 45(1.5 - e^{-0.4\,z})$).

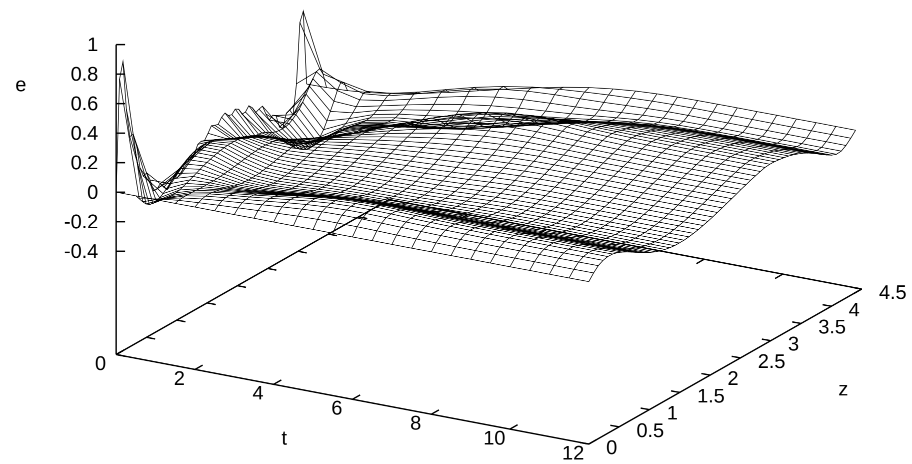

Fig. 10. Deviation between the third-order model and the high-order discretization of the PDE ($k(z) = e^{-0.4\ z}$).

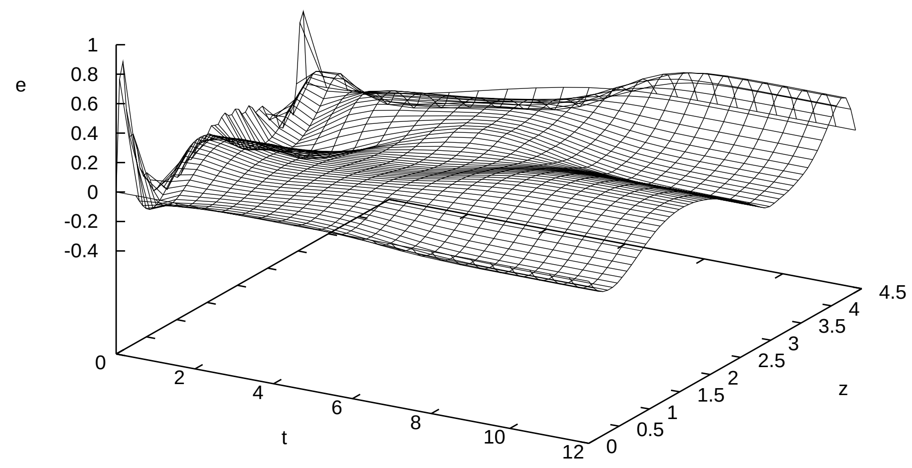

Fig. 11. Deviation between the third-order model and the high-order discretization of the PDE ($k(z) = e^{-0.6\ z}$).

We also tested the robustness of the third-order model for two initial conditions which are different from the ones used for the construction of the empirical eigenfunctions. Specifically, for the initial condition $\bar{x}_0(z) = 0.5 + 0.5\sin(z)$, the deviation between the third-order model and the high-order discretization of the PDE is presented in Fig. 12, while for $\bar{x}_0(z) = 0.4 + 0.6\sin(3\ z)$, the same deviation is shown in Fig. 13. In both the cases, the maximum error between the two models is less than 1.6%, implying the robustness of the proposed model reduction procedure with respect to significant variations in the initial conditions. The aforementioned results are summarized in Table 1 which presents the integral error between the reduced- and high-order models during the process cycle for the different process conditions that we presented in this section. Specifically, the error was calculated using the following formula:

$$\int_0^{t_f} \left| \frac{\|\bar{x}_{\rm low}\|_2 - \|\bar{x}_{\rm high}\|_2}{\|\bar{x}_{\rm high}\|_2} \right| {\rm d}t, \tag{50}$$

where $t_f = 12$ is the final time of the process simulation run. We note that, even though the third-order model closely tracks the solution of the high-discretization for almost all times, in the case of $\bar{x}_0(z) = 0.4 + 0.6\sin(3\ z)$, the integral error is computed to be large compared to the rest of the simulation runs. This is due to the large initial error between the two models as can be seen in Fig. 13, caused by the excitation of the stable and fast-dissipating higher modes that the reduced-order model does not capture. Note that the models that were derived based on the eigenfunctions of eqn. (47) exhibit a larger error in the spatial profile of the solution for all times, as can be seen in Fig. 6.

Finally, we used the proposed method to control the process. Specifically, the controller synthesis formula of Theorem 1 was used to synthesize a second-order controller on the basis of a second-order model obtained using Galerkin's method with the first two empirical eigenfunctions as basis functions. The control objective is to stabilize the system at the unstable steady state $\bar{x}(z,t) = 0$ using one point measurement of the state at $z = l(t)/3$ (i.e., moving sensor with $s(z,t) = \delta(z - (l/3)(t))$, where $\delta(\cdot)$ is the Dirac function). Performing a linearization of the PDE system around the spatially uniform steady-state, we found that the first two modes are unstable. Therefore, the controlled outputs were defined as

$$\begin{aligned} y_{c1}(t) &= \int_0^{l(t)} \phi_1(z,t)\bar{x}(z,t)\,{\rm d}z, \\ y_{c2}(t) &= \int_0^{l(t)} \phi_2(z,t)\bar{x}(z,t)\,{\rm d}z, \end{aligned} \tag{51}$$

where ϕ_i denotes the i-th empirical eigenfunction. The actuator distribution functions were taken to be $b_1(z,t) = 1$ (uniform in space, distributed control action) and $b_2(z,t) = \delta(z - \frac{2}{3}l(t))$ (moving point control actuation).

Figure 14 shows the evolution of the closed-loop rod temperature profile under the nonlinear output feedback controller, while Fig. 15 shows the corresponding manipulated input profiles. Clearly, the proposed controller achieves regulation of the

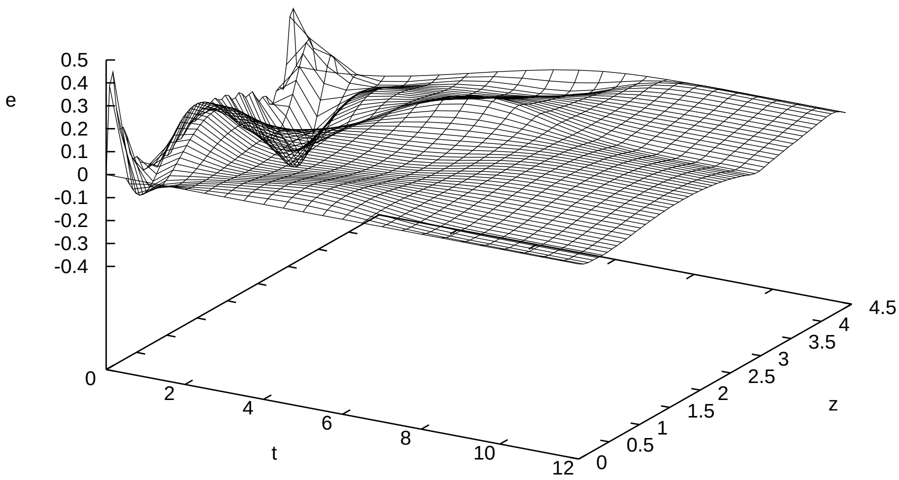

Fig. 12. Deviation between the third-order model and the high-order discretization of the PDE ($\bar{x}_0(z) = 0.5 + 0.5\sin(z)$).

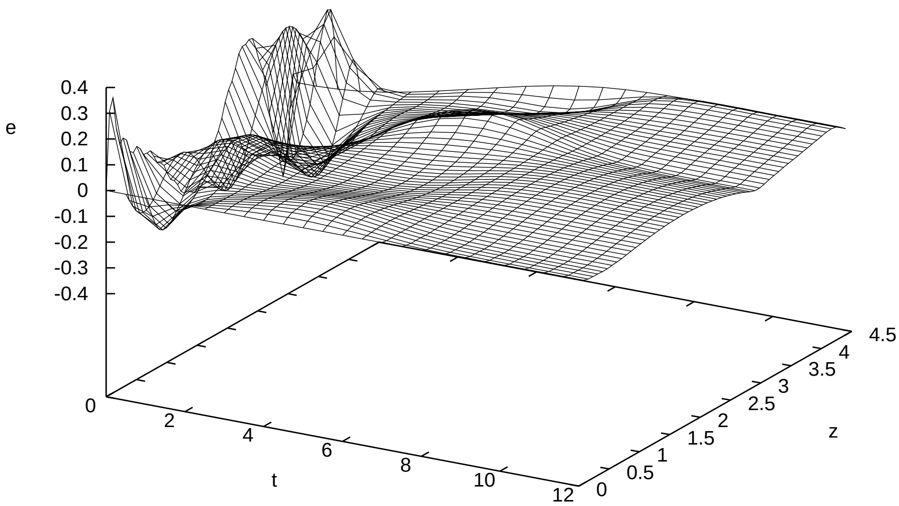

Fig. 13. Deviation between the third-order model and the high-order discretization of the PDE ($\bar{x}_0(z) = 0.4 + 0.6\sin(3\,z)$).

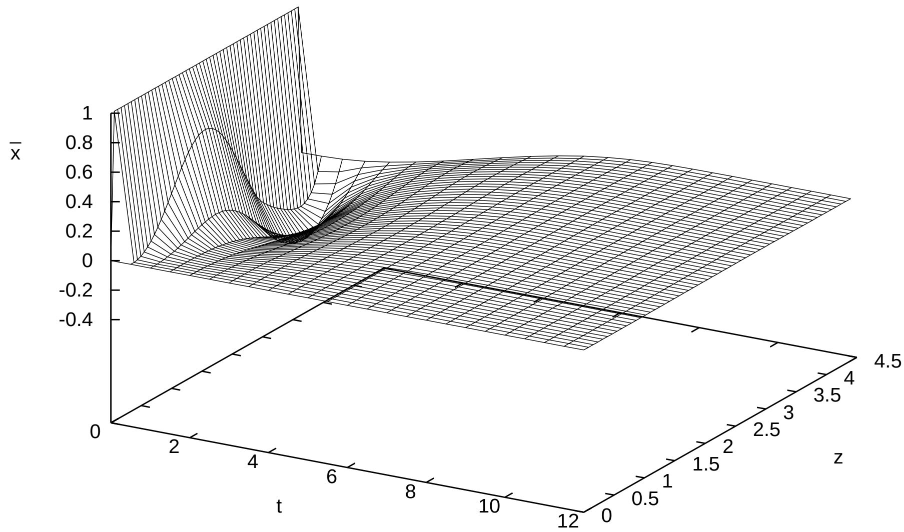

Fig. 14. Closed-loop profile of $\bar{x}$ under nonlinear output feedback control using empirical eigenfunctions ($m = 2$).

temperature profile at the spatially uniform steady state $\bar{x}(z,t) = 0$. For the sake of comparison, we also constructed finite-dimensional models using Galerkin's method with the analytical eigenfunctions of eqn. (47) as basis functions, and we used the resulting models for the synthesis of a nonlinear output feedback controller utilizing the result of Theorem 1. We found that the lowest-order controller that achieves stabilization of the PDE system at $\bar{x}(z,t) = 0$ is of order 4. Therefore, we observe a significant reduction on the order of the controller that stabilizes the system at the spatially uniform steady state when we use empirical eigenfunctions as basis functions. This again happens because the empirical eigenfunctions take into consideration the spatial variation of the coefficients of the process model, while the analytical eigenfunctions were derived on the basis of the spatial operator with spatially uniform coefficients, and thus, they do not capture the spatially varying features of the process.

Remark 9. To illustrate the applicability of the proposed approach to parabolic PDE systems with spatially-uniform coefficients, we considered the PDE system of eqn. (44) with

$$\beta_T = 75.0, \quad k = 1.0. \tag{52}$$

For this system, the steady-state $\bar{x}(z,t) = 0$ is also unstable, and the eigenfunctions of the spatial operator can be computed analytically and are given in eqn. (47). Furthermore, for this system, we constructed a representative ensemble of solutions by varying the initial conditions and the inputs, and used it to construct a set of seven empirical eigenfunctions. Then, we employed Galerkin's method to construct two

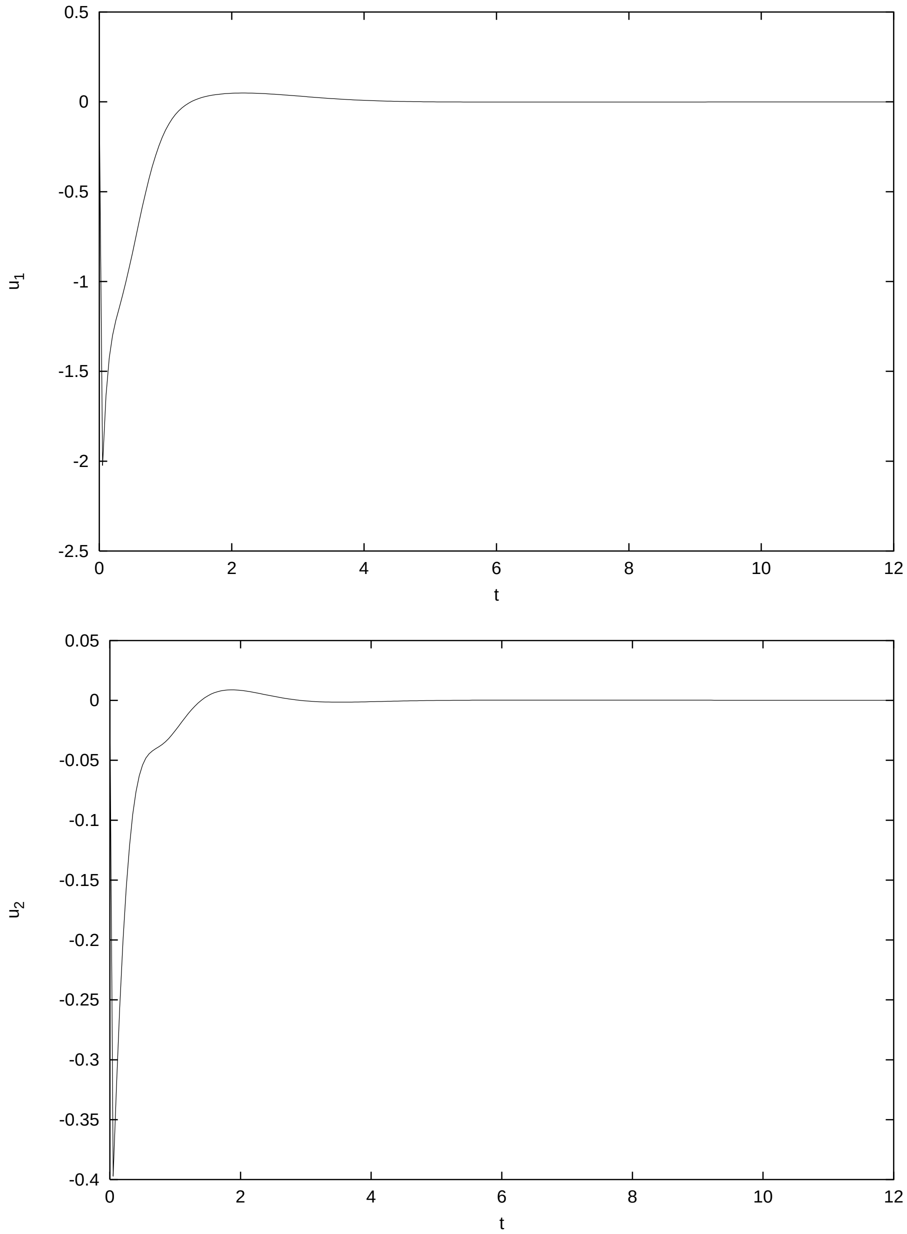

Fig. 15. Manipulated input profiles for the output feedback controller.

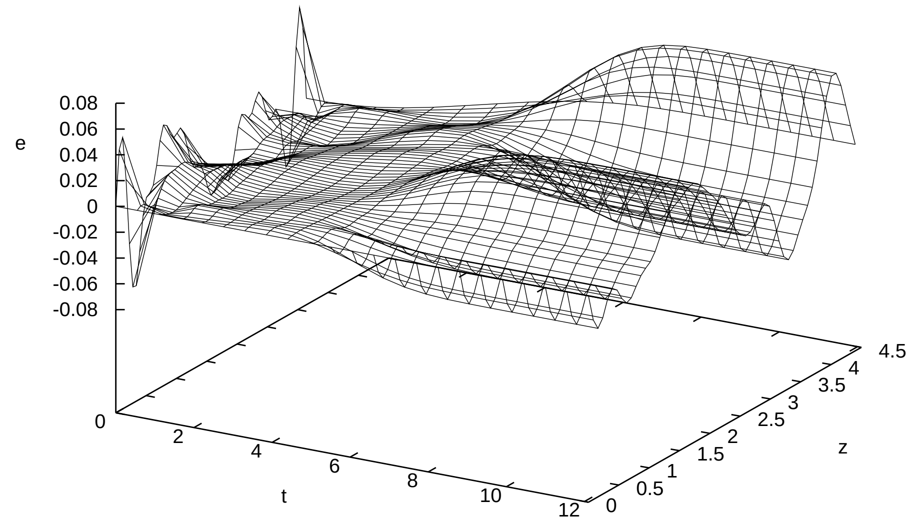

Fig. 16. Deviation between reduced and full-order models for spatially uniform process parameters β_T and k.

fifth-order models corresponding to the two different sets of basis functions (analytical and empirical). Figure 16 shows the discrepancy between the fifth-order model based on empirical eigenfunctions and a high-order discretization of the PDE. We can see that this discrepancy remains small for all times, thereby implying that the use of empirical eigenfunctions leads to accurate low-order models for parabolic PDE systems with spatially-uniform coefficients as well. For comparison purposes, we also present in Table 1 the integral error of eqn. (50) for both the fifth-order models that we derived. We can see that the integral error of the model derived using the proposed methodology is 20 times smaller than the error of the model based on analytical eigenfunctions of the spatial operator. The reason for this difference is that the empirical eigenfunctions take into consideration all the characteristics of the process, including the effect of nonlinearities, while the analytical eigenfunctions account only for the structure of the linear spatial differential operator and of the boundary conditions.

6. Conclusions

In this work, we presented a methodology for the synthesis of finite-dimensional nonlinear output feedback controllers for nonlinear parabolic PDE systems with time-dependent spatial domains. Initially, the nonlinear parabolic PDE system was expressed with respect to an appropriate time-invariant spatial coordinate and a representative (with respect to different initial conditions and input perturbations) ensemble of solutions of the resulting time-varying PDE system was constructed by computing and solving a high-order discretization of the PDE. Then, the Karhunen-Loève expansion was directly applied to the ensemble of solutions to derive a small set of empirical eigenfunctions (dominant spatial patterns) that capture almost all the energy of the ensemble of solutions. The empirical eigenfunctions were subsequently used as basis functions within a Galerkin's model reduction framework to derive low-order ODE systems that accurately describe the dominant dynamics of the PDE system. The ODE systems were used for the synthesis of nonlinear output feedback controllers using geometric control methods. The proposed control method was successfully used to stabilize an unstable steady-state of a diffusion-reaction process with nonlinearities, spatially-varying coefficients and time-dependent spatial domain, and was shown to lead to the construction of accurate low-order models and the synthesis of low-order controllers.

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