

NUMERICAL ANALYSIS AND SIMULATIONS OF QUASISTATIC FRICTIONLESS CONTACT PROBLEMS

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A summary of recent results concerning the modelling as well as the variational and numerical analysis of frictionless contact problems for viscoplastic materials are presented. The contact is modelled with the Signorini or normal compliance conditions. Error estimates for the fully discrete numerical scheme are described, and numerical simulations based on these schemes are reported.

Keywords: quasistatic contact, Signorini condition, normal compliance, viscoplasticity, variational inequality, error estimate, finite element method (FEM), numerical approximation

1. Introduction

Contact phenomena among deformable bodies abound in industry and everyday life, and play an important role in structural and mechanical systems. The complicated surface structure, physics and chemistry involved in contact processes make it necessary to model them with highly complex and nonlinear initial-boundary value problems. Indeed, the now famous Signorini problem was formulated as an idealized model of unilateral frictionless contact between an elastic body and a rigid foundation. The mathematical analysis of this problem was first provided by Fichera (1964). Duvaut and Lions, in their monograph (Duvaut and Lions, 1976), systematically modelled and analyzed many important contact problems within the framework of the theory of variational inequalities. Numerical approximations of variational inequalities arising from contact problems were described in detail in (Hlaváček *et al.*, 1988), and in (Kikuchi and Oden, 1988). The mathematical, mechanical and numerical state of

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the art can be found in the proceedings (Raous *et al.*, 1995), and in the special issue (Shillor, 1998).

In earlier mathematical publications it was invariably assumed that the deformable bodies were linearly elastic. However, numerous recent publications are dedicated to the modelling, variational analysis and numerical approximations of contact problems involving viscoelastic and viscoplastic materials. Moreover, a variety of new and modified contact conditions have been employed, reflecting a variety of possible physical contact settings and conditions. The problems have been studied for a unilateral or a bilateral contact, with friction or frictionless. Investigation of these problems led to new variational inequalities, the well-posedness of which was established. Moreover, two types of numerical approximations were analyzed and error estimates were derived. These were semi-discrete schemes, where only the spatial variables were discretized, and fully discrete schemes, where both the time and spatial variables were discretized.

Here, we summarize some of our main recent results on viscoplastic contact problems and present a few numerical examples. More details and other problems can be found in (Fernández-García *et al.*, 2000).

The paper is structured as follows: In Section 2 we introduce the notation and preliminary material. We also discuss the mechanical problems and their classical formulation. In Section 3 we present their weak formulations and well-posedness results, and then give error estimates for their numerical approximations. We consider only the frictionless contact involving viscoplastic materials and give numerical results only for the fully discrete scheme. Finally, in Section 4 we provide numerical simulations which depict possible types of behaviour of the models.

2. Preliminaries

We consider mathematical models for the quasistatic process of frictionless contact between a viscoplastic body and a rigid foundation. The physical setting is as follows. A viscoplastic body occupies an open, bounded and connected set $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3 . The boundary $\Gamma = \partial\Omega$ is assumed to be Lipschitz continuous and has the decomposition $\Gamma = \cup_{i=1}^3 \bar{\Gamma}_i$ into mutually disjoint, relatively open sets Γ_1 , Γ_2 and Γ_3 , with Lipschitz relative boundaries if $d = 3$. The set Γ_3 represents the potential contact surface, and we assume $\text{meas}(\Gamma_1) > 0$, which is essential in quasistatic problems, since it ensures the coercivity of the problem. As the boundary is Lipschitz continuous, the unit outward normal vector ν exists a.e. on Γ .

We are interested in the evolution of the body's mechanical state over the time interval $[0, T]$ ($T > 0$). The body is clamped on Γ_1 and so the displacement field vanishes there. A surface traction of density \mathbf{f}_2 acts on Γ_2 and a volume force of density \mathbf{f}_0 acts in Ω (each may vary in time). We assume that they change slowly in time so that the accelerations in the system are negligible, which means that the process is quasistatic.

We denote by \mathbb{S}^d the space of second-order symmetric tensors on \mathbb{R}^d , or equivalently, the space of symmetric matrices of order d . The inner products and the

corresponding norms on \mathbb{R}^d and \mathbb{S}^d are, respectively,

$$\begin{aligned} \mathbf{u} \mathbf{v} &= u_i v_i, & \|\mathbf{v}\| &= (\mathbf{v} \mathbf{v})^{1/2} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \\ \boldsymbol{\sigma} \boldsymbol{\tau} &= \sigma_{ij} \tau_{ij}, & \|\boldsymbol{\tau}\| &= (\boldsymbol{\tau} \boldsymbol{\tau})^{1/2} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d. \end{aligned}$$

Here and below, $i, j = 1, 2, \dots, d$, and the summation convention over repeated indices is adopted. Moreover, an index which follows a comma indicates a partial derivative.

Let ε and Div be the *deformation* and *divergence* operators defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j}),$$

respectively. Denoting by \mathbf{u} the displacement and $\boldsymbol{\sigma}$ the stress fields in the body, we have

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\varepsilon(\dot{\mathbf{u}}) + G(\boldsymbol{\sigma}, \varepsilon(\mathbf{u})) \quad \text{in } \Omega \times (0, T), \quad (1)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega \times (0, T), \quad (2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0, T), \quad (3)$$

$$\boldsymbol{\sigma} \boldsymbol{\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \quad (4)$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_3 \times (0, T), \quad (5)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in } \Omega. \quad (6)$$

We assume a rate-type viscoplastic constitutive law (1), where \mathcal{E} and G are given functions. Here (2) are the mechanical equilibrium equations. The boundary conditions (3)–(5) are the displacement, traction and frictionless contact conditions on Γ_1 , Γ_2 and Γ_3 , respectively. Finally, (6) are the initial conditions. In (1) and below, a dot above a variable represents its derivative with respect to time. To complete the model, we need to supplement these relations with the contact conditions in the normal direction on $\Gamma_3 \times (0, T)$.

Rate-type viscoplastic models of the form (1) have been used to describe the behaviour of rubbers, metals, pastes, and rocks, see, e.g., (Cristescu and Suliciu, 1982) and references therein. Existence and uniqueness results for initial-boundary value problems involving (1) were obtained in (Ionescu and Sofonea, 1993) for displacement-traction conditions. The Perzyna law is such an elastic-viscoplastic constitutive law,

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}^{-1} \dot{\boldsymbol{\sigma}} + \frac{1}{\mu^*} (\boldsymbol{\sigma} - P_K \boldsymbol{\sigma}), \quad (7)$$

where $\mu^* > 0$ is the viscosity constant, K is a nonempty, closed, convex set in the space of symmetric tensors, and P_K denotes the projection mapping onto K . A simple one-dimensional example of a constitutive law of the form (1), when a full coupling between the stress and the strain is assumed in G , can be found in (Cristescu and Suliciu, 1982, p. 35).

To proceed with the variational formulation, we need the following function spaces:

$$H = \{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\}, \quad Q = \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$

$$H_1 = \{\mathbf{u} = (u_i) \mid u_i \in H^1(\Omega)\}, \quad Q_1 = \{\boldsymbol{\sigma} \in Q \mid \sigma_{ij,j} \in H\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\mathbf{u}, \mathbf{v})_H = \int_{\Omega} u_i v_i \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx,$$

$$(\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\text{Div } \boldsymbol{\sigma}, \text{Div } \boldsymbol{\tau})_H,$$

and the associated norms $\|\cdot\|_H$, $\|\cdot\|_Q$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{Q_1}$, respectively.

Below, V stands for the space $V = \{\mathbf{v} \in (H^1(\Omega))^d \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}$ equipped with the inner product $(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q$ and the associated norm $\|\cdot\|_V$. Since $\text{meas}(\Gamma_1) > 0$, it follows from Korn's inequality that $\|\cdot\|_{(H^1(\Omega))^d}$ and $\|\cdot\|_V$ are equivalent norms on V .

For an element $\mathbf{v} \in H_1$, we also denote by $\boldsymbol{\gamma}\mathbf{v}$ on Γ ; v_ν and \mathbf{v}_τ denote the *normal* and *tangential* components of \mathbf{v} on Γ given by $v_\nu = \mathbf{v} \boldsymbol{\nu}$ and $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$, respectively. For an element $\boldsymbol{\sigma} \in Q_1$, we denote by $\boldsymbol{\sigma}\boldsymbol{\nu}$ its trace on Γ . If $\boldsymbol{\sigma}$ is a smooth function, then the *normal* and *tangential* components of $\boldsymbol{\sigma}$ are given by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, respectively. Given a real normed space $(X, \|\cdot\|_X)$, we use the standard notation $C([0, T]; X)$, $C^1([0, T]; X)$ and $W^{k,p}(0, T; X)$ for function spaces.

In the numerical approximations of the problems, we use the finite-element method (FEM) for spatial discretization, and finite differences for the time derivative. We now describe briefly a finite-dimensional space H_1^h which approximates H_1 via the FEM. The details can be found, e.g., in (Ciarlet, 1978). For simplicity, we assume that $\bar{\Omega}$ is a polygon or a polyhedron. Then $\bar{\Gamma}_3 = \cup_{i=1}^I \bar{\Gamma}_{3,i}$, and each piece $\bar{\Gamma}_{3,i}$ is represented by an affine function. Let \mathcal{T}^h be a regular finite-element partition of $\bar{\Omega}$ such that whenever a side of an element lies on the boundary, it belongs entirely to one of the subsets $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and $\bar{\Gamma}_{3,i}$, $1 \leq i \leq I$. Let h be the maximal diameter of the elements. We define $H_1^h \subset H_1$ to be the finite-element space consisting of piecewise linear functions corresponding to the partition \mathcal{T}^h . If the solution \mathbf{u} is known to have higher regularity, we may use higher-order elements (our error analysis can be easily extended to such cases).

We employ a partition of the time interval $[0, T] : 0 = t_0 < t_1 < \dots < t_N = T$. We denote the step size by $k_n = t_n - t_{n-1}$ for $n = 1, \dots, N$, and let $k = \max_n k_n$ be the maximal step size. For a function $w(t)$ we let $w_n = w(t_n)$. Given a sequence $\{w_n\}_{n=0}^N$ for $n = 1, \dots, N$, we write $\Delta w_n = w_n - w_{n-1}$ and let $\delta w_n = \Delta w_n / k_n$ be the corresponding divided differences (no summation is implied over the index n).

In what follows, the symbol c represents a positive constant which may change its value from place to place and may depend on the input data, but is independent of the discretization parameters h and k .

3. Frictionless Contact Problems

We present two contact problems involving viscoplastic materials of the type (1). We assume that $\mathcal{E}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ and $G: \Omega \times \mathbb{S}^d \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ satisfy the assumptions:

$$\left. \begin{array}{l} \text{(a) } \mathcal{E}_{ijkl} \in L^\infty(\Omega), \quad 1 \leq i, j, k, l \leq d. \\ \text{(b) } \mathcal{E}\boldsymbol{\sigma}\boldsymbol{\tau} = \boldsymbol{\sigma}\mathcal{E}\boldsymbol{\tau}, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d \quad \text{a.e. in } \Omega. \\ \text{(c) There exists an } \alpha_0 > 0 \text{ such that} \\ \quad \mathcal{E}\boldsymbol{\tau}\boldsymbol{\tau} \geq \alpha_0|\boldsymbol{\tau}|^2 \quad \forall \boldsymbol{\tau} \in \mathbb{S}^d \quad \text{a.e. in } \Omega. \end{array} \right\} \quad (8)$$

$$\left. \begin{array}{l} \text{(a) There exists an } \mathcal{L} > 0 \text{ such that} \\ \quad \|G(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - G(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \leq \mathcal{L} (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|) \\ \quad \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \quad \text{a.e. in } \Omega. \\ \text{(b) For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \boldsymbol{x} \mapsto G(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable.} \\ \text{(c) The mapping } \boldsymbol{x} \mapsto G(\boldsymbol{x}, \mathbf{0}, \mathbf{0}) \text{ lies in } H. \end{array} \right\} \quad (9)$$

The forces and tractions are assumed to satisfy

$$\boldsymbol{f}_0 \in W^{1,\infty}(0, T; H), \quad \boldsymbol{f}_2 \in W^{1,\infty}\left(0, T; (L^2(\Gamma_2))^d\right), \quad (10)$$

and $\boldsymbol{f}(t)$ denotes the element of V given by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = (\boldsymbol{f}_0(t), \boldsymbol{v})_H + (\boldsymbol{f}_2(t), \boldsymbol{v})_{(L^2(\Gamma_2))^d}$$

for all $\boldsymbol{v} \in V$ and $t \in [0, T]$.

3.1. The Signorini Problem

We assume that there is no interpenetration between the body's surface and the foundation, which represents a highly idealized case of a perfectly rigid foundation. The classical formulation is: Find a displacement field $\boldsymbol{u}: \Omega \times [0, T] \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}: \Omega \times [0, T] \rightarrow \mathbb{S}^d$ satisfying (1)–(6) and

$$u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu u_\nu = 0 \quad \text{on } \Gamma_3 \times (0, T). \quad (11)$$

Here (11) is the Signorini condition. Let $U = \{\boldsymbol{v} \in V \mid v_\nu \leq 0 \text{ on } \Gamma_3\}$ and assume that

$$\boldsymbol{u}_0 \in U, \quad \boldsymbol{\sigma}_0 \in Q, \quad (12)$$

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_0))_Q \geq (\boldsymbol{f}(0), \boldsymbol{v} - \boldsymbol{u}_0)_V \quad \forall \boldsymbol{v} \in U. \quad (13)$$

The weak formulation of this contact problem is as follows:

Problem 1. Find a displacement $\boldsymbol{u}: [0, T] \rightarrow U$ and a stress tensor $\boldsymbol{\sigma}: [0, T] \rightarrow Q$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ and, for a.e. $t \in (0, T)$,

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + G(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))),$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall \boldsymbol{v} \in U.$$

The following result was established in (Sofonea, 1997).

Theorem 1. *Assume that (8)–(10), (12) and (13) hold. Then Problem 1 has a unique solution such that $\mathbf{u} \in W^{1,\infty}(0, T; U)$ and $\boldsymbol{\sigma} \in W^{1,\infty}(0, T; Q_1)$.*

For numerical approximations, let $V^h \subset V$ be a finite-dimensional subspace of V and define

$$U^h = \{\mathbf{v}^h \in V^h \mid v_\nu^h \leq 0 \text{ on } \Gamma_3\}.$$

Let $Q^h \subset Q$ be a finite-dimensional subspace of Q . Let $\mathcal{P}_{Q^h} : Q \rightarrow Q^h$ be the orthogonal projection defined by

$$(\mathcal{P}_{Q^h} \mathbf{q}, \mathbf{q}^h)_Q = (\mathbf{q}, \mathbf{q}^h)_Q \quad \forall \mathbf{q} \in Q, \quad \mathbf{q}^h \in Q^h.$$

Then a fully discrete approximation of Problem 1 is:

Problem 2. Find $\mathbf{u}^{hk} = \{\mathbf{u}_n^{hk}\}_{n=1}^N \subset U^h$ and $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}_n^{hk}\}_{n=1}^N \subset Q^h$ such that $\mathbf{u}_0^{hk} = \mathbf{u}_0^h$, $\boldsymbol{\sigma}_0^{hk} = \boldsymbol{\sigma}_0^h$ and, for $n = 1, \dots, N$,

$$\begin{aligned} \delta \boldsymbol{\sigma}_n^{hk} &= \mathcal{P}_{Q^h} \mathcal{E} \boldsymbol{\varepsilon} (\delta \mathbf{u}_n^{hk}) + \mathcal{P}_{Q^h} G (\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{u}_n^{hk})), \\ (\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{u}_n^{hk}))_Q &\geq (\mathbf{f}_n, \mathbf{v}^h - \mathbf{u}_n^{hk})_V \quad \forall \mathbf{v}^h \in U^h. \end{aligned}$$

Problem 2 has a unique solution. We obtain the following error estimates by modifying the results in (Chen *et al.*, 1999a):

Theorem 2. *Assume that the conditions of Theorem 1 hold. Then*

$$\begin{aligned} \max_{1 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q) &\leq c(\|\mathbf{u}_0 - \mathbf{u}_0^h\|_V + \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q) \\ &+ ck(1 + \|\dot{\mathbf{u}}\|_{L^\infty(0, T; V)} + \|\dot{\boldsymbol{\sigma}}\|_{L^\infty(0, T; Q)}) \\ &+ c \max_{1 \leq n \leq N} \left(\inf_{\mathbf{v}_n^h \in U^h} \{\|\mathbf{u}_n - \mathbf{v}_n^h\|_V \right. \\ &\left. + |R_n(\mathbf{u}_n, \mathbf{v}_n^h)|^{1/2} \} + \|(I_Q - \mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0)\|_Q \right), \end{aligned}$$

where

$$R_n(\mathbf{u}_n, \mathbf{v}_n^h) = (\boldsymbol{\sigma}_n, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{u}_n))_Q - (\mathbf{f}_n, \mathbf{v}_n^h - \mathbf{u}_n)_V.$$

Theorem 2 is a basis for order error estimates. Let $V^h = H_1^h \cap V$ and let Q^h be the corresponding space of piecewise constants. Assume that

$$\mathbf{u} \in L^\infty(0, T; H^2(\Omega)^d), \quad \boldsymbol{\sigma} \in L^\infty(0, T; H^1(\Omega)^{d \times d}),$$

$$\mathbf{u}_0 \in H^2(\Omega)^d, \quad \boldsymbol{\sigma}_0 \in H^1(\Omega)^{d \times d}.$$

If we choose $\mathbf{u}_0^h \in U^h$ and $\boldsymbol{\sigma}_0^h \in Q^h$ to be finite-element projections of \mathbf{u}_0 and $\boldsymbol{\sigma}_0$, respectively, then

$$\max_{1 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q) = O(k + h^{3/4}).$$

If we further assume that $\dot{\mathbf{u}}_\tau \in C([0, T]; H^2(\Gamma_{3,i})^d)$, $1 \leq i \leq I$, then we obtain an optimal order error estimate

$$\max_{1 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q) = O(k + h). \quad (14)$$

3.2. Frictionless Contact Problems with Normal Compliance

We next consider a frictionless contact of the body with a deformable foundation which we model by the normal compliance condition,

$$\sigma_\nu = -\frac{1}{\mu}(u_\nu - g)_+^\alpha \quad \text{on } \Gamma_3 \times (0, T). \quad (15)$$

Here g is the initial gap between the elastic-viscoplastic body and the foundation, and μ may be interpreted as the *coefficient of deformability* of the foundation. The expression $u_\nu - g$, when positive, represents the interpenetration of the body's surface asperities into the foundation. Signorini's nonpenetration condition is obtained from (15) as $\mu \rightarrow 0$, i.e., when the coefficient of deformability of the foundation tends to zero.

We assume that

$$\alpha \in (0, 1], \quad \mu \in L^\infty(\Gamma_3), \quad g \in L^2(\Gamma_3), \quad \mu \geq \mu_0 > 0, \quad g \geq 0 \quad \text{a.e. on } \Gamma_3. \quad (16)$$

The classical formulation of the problem is as follows: Find a displacement field $\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \rightarrow \mathbb{S}^d$ satisfying (1)–(6) and (15).

To obtain a variational formulation of the problem, let

$$j(\mathbf{v}) = \frac{1}{\alpha + 1} \int_{\Gamma_3} \frac{1}{\mu} (v_\nu - g)_+^{\alpha+1} \, da.$$

Assume that the initial data satisfy,

$$\mathbf{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q, \quad (17)$$

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}_0))_Q + j(\mathbf{v}) - j(\mathbf{u}_0) \geq (\mathbf{f}(0), \mathbf{v} - \mathbf{u}_0)_V \quad \forall \mathbf{v} \in V. \quad (18)$$

The weak formulation of the problem is as follows:

Problem 3. Find a displacement $\mathbf{u} : [0, T] \rightarrow V$ and a stress tensor $\boldsymbol{\sigma} : [0, T] \rightarrow Q$ satisfying $\mathbf{u}(0) = \mathbf{u}_0$, $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ and, for a.e. $t \in (0, T)$,

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + G(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t))),$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q + j(\mathbf{v}) - j(\mathbf{u}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_V \quad \forall \mathbf{v} \in V.$$

The well-posedness of Problem 3 was studied in (Sofonea, 1997), where the following result was established:

Theorem 3. *Under the conditions (8)–(10), (17) and (18), Problem 3 has a unique solution such that $\mathbf{u} \in W^{1,\infty}(0, T; V)$ and $\boldsymbol{\sigma} \in W^{1,\infty}(0, T; Q_1)$.*

For numerical approximations, let $V^h \subset V$ and $Q^h \subset Q$ be finite-dimensional spaces. We assume that these spaces satisfy $\boldsymbol{\varepsilon}(V^h) \subset Q^h$. This assumption is very natural and holds for finite-element approximations when the polynomial degree for the space V^h is at most one higher than that for the space Q^h . Then a fully discrete approximation of Problem 3 is as follows:

Problem 4. Find $\mathbf{u}^{hk} = \{\mathbf{u}_n^{hk}\}_{n=1}^N \subset V^h$ and $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}_n^{hk}\}_{n=1}^N \subset Q^h$ such that $\mathbf{u}_0^{hk} = \mathbf{u}_0^h$, $\boldsymbol{\sigma}_0^{hk} = \boldsymbol{\sigma}_0^h$ and, for $n = 1, \dots, N$,

$$\begin{aligned} \delta \boldsymbol{\sigma}_n^{hk} &= \mathcal{P}_{Q^h} \mathcal{E} \boldsymbol{\varepsilon}(\delta \mathbf{u}_n^{hk}) + \mathcal{P}_{Q^h} G(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk})), \\ (\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{u}_n^{hk}))_Q + j(\mathbf{v}^h) - j(\mathbf{u}_n^{hk}) &\geq (\mathbf{f}_n, \mathbf{v}^h - \mathbf{u}_n^{hk})_V \quad \forall \mathbf{v}^h \in U^h. \end{aligned}$$

Problem 4 has a unique solution. We have the following error estimates, obtained by modifying a similar result in (Chen et al., 1999b):

Theorem 4. *Assume that the conditions of Theorem 3 hold. Then*

$$\begin{aligned} \max_{1 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q) &\leq c (\|\mathbf{u}_0 - \mathbf{u}_0^h\|_V + \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q) \\ &+ ck (1 + \|\dot{\mathbf{u}}\|_{L^\infty(0, T; V)} + \|\dot{\boldsymbol{\sigma}}\|_{L^\infty(0, T; Q)}) \\ &+ c \max_{1 \leq n \leq N} \left(\inf_{\mathbf{v}_n^h \in V^h} \{ \|\mathbf{u}_n - \mathbf{v}_n^h\|_V + |R_n(\mathbf{u}_n, \mathbf{v}_n^h)|^{1/2} \} \right) \\ &+ \|(I_Q - \mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0)\|_Q, \end{aligned}$$

where

$$R_n(\mathbf{u}_n, \mathbf{v}_n^h) = (\boldsymbol{\sigma}_n, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{u}_n))_Q + j(\mathbf{v}^h) - j(\mathbf{u}_n^{hk}) - (\mathbf{f}_n, \mathbf{v}_n^h - \mathbf{u}_n)_V.$$

The theorem is a basis for order error estimates. Let $V^h = H_1^h \cap V$, where H_1^h is defined in Section 2, and let H^h be the corresponding space of piecewise constants. Assume that

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, T; H^2(\Omega)^d), \quad \boldsymbol{\sigma} \in L^\infty(0, T; H^1(\Omega)^{d \times d}), \\ \mathbf{u}_0 &\in H^2(\Omega)^d, \quad \boldsymbol{\sigma}_0 \in H^1(\Omega)^{d \times d}. \end{aligned}$$

If we choose $\mathbf{u}_0^h \in U^h$ and $\boldsymbol{\sigma}_0^h \in Q^h$ to be finite-element projections of \mathbf{u}_0 and $\boldsymbol{\sigma}_0$, then we have

$$\max_{1 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q) = O(k + h^{3/4}).$$

If we further assume that

$$\dot{\mathbf{u}}_\tau \in C([0, T]; H^2(\Gamma_3)^d),$$

then we have an optimal order error estimate

$$\max_{1 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q) = O(k + h). \tag{19}$$

A variational and numerical analysis of a quasistatic frictionless contact problem for viscoplastic materials with a general normal compliance contact condition was made in (Fernández-García *et al.*, 1999).

4. Numerical Examples

To verify the accuracy of the numerical methods described in Section 3, a number of numerical experiments have been performed on test problems in one, two and three dimensions. We describe in this section some of the numerical results.

4.1. The Signorini Contact Problem

4.1.1. A Two-Dimensional Test Problem

We used the data:

$$\begin{aligned} \Omega &= (0, 1) \times (0, 1), \quad T = 1 \text{ sec}, \quad \Gamma_1 = [0, 1] \times \{1\}, \quad \Gamma_2 = \{0, 1\} \times (0, 1), \\ \Gamma_3 &= [0, 1] \times \{0\}, \quad \mathbf{f}_0 = (0, -10t) \text{ N/m}^2, \quad \mathbf{f}_2 = (0, 0) \text{ N/m}, \\ \boldsymbol{\sigma}_0 &= \mathbf{0} \text{ N/m}^2, \quad \mathbf{u}_0 = \mathbf{0} \text{ m}. \end{aligned}$$

\mathcal{E} is the plane stress elasticity tensor, i.e.,

$$(\mathcal{E}\boldsymbol{\tau})_{\alpha\beta} = \frac{E\kappa}{1 - \kappa^2}(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \frac{E}{1 + \kappa}\tau_{\alpha\beta},$$

for $\alpha, \beta = 1, 2$, where E is Young’s modulus and κ is Poisson’s ratio. In this example, $E = 10^8 \text{ N/m}^2$ and $\kappa = 0.3$. We consider an obstacle defined implicitly by

$$\frac{(x_1 - 3)^2}{900} + \frac{(x_2 + 3)^2}{9} - 1 = 0.$$

We used the 2-D version of Perzyna’s viscoplastic law (7), i.e.,

$$G(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = -\frac{1}{\mu^*}\mathcal{E}(\boldsymbol{\sigma} - P_K\boldsymbol{\sigma}), \tag{20}$$

where P_K is the orthogonal projection operator (with respect to the norm $\|\boldsymbol{\tau}\| = (\mathcal{E}\boldsymbol{\tau}, \boldsymbol{\tau})^{1/2}$) onto the convex subset $K \subset \mathbb{S}^2$ defined by

$$K = \{ \boldsymbol{\tau} \in \mathbb{S}^2 \mid \tau_{11}^2 + \tau_{22}^2 - \tau_{11}\tau_{22} + 3\tau_{12}^2 \leq \sigma_Y^2 \},$$

where σ_Y is the uniaxial yield stress. We used $\sigma_Y = \sqrt{10} \text{ N/m}^2$ and $\mu^* = 200 \text{ N/m}^2$.

In Fig. 1 the deformed configuration and the Von-Mises norm for the stress at time $t = 1 \text{ sec}$ are shown, and in Fig. 2 the evolution of the contact boundary is illustrated.

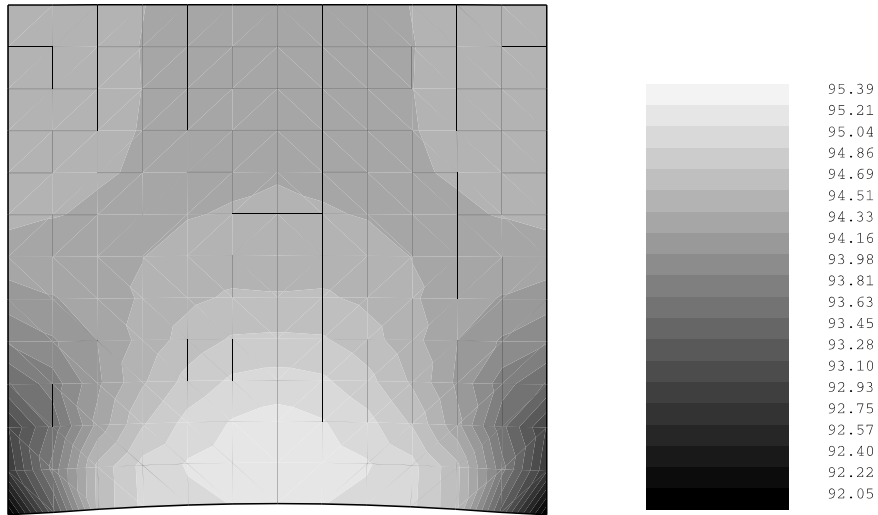


Fig. 1. The deformed configuration and the Von-Mises norm; the 2-D Signorini problem.

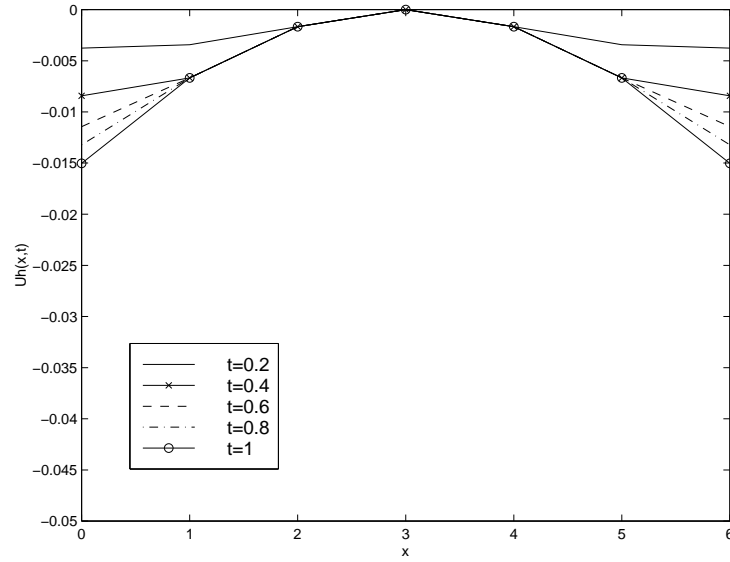


Fig. 2. Evolution of the U_2 component; the 2-D Signorini problem.

4.1.2. A Three-Dimensional Test Problem

In the three-dimensional case we used the data:

$$\Omega = (0, 3) \times (0, 1) \times (0, 1), \quad T = 1 \text{ sec},$$

$$\Gamma_1 = \{0\} \times [0, 1] \times [0, 1], \quad \Gamma_3 = (0, 3) \times (0, 1) \times \{0\}, \quad \Gamma_2 = \partial\Omega - (\Gamma_1 \cup \Gamma_3),$$

$$\mathbf{f}_2(x_1, x_2, x_3, t) = \begin{cases} (0, 0, -10t) \text{ N/m}^2 & \text{if } x_1 = 0, \\ (0, 0, 0) \text{ N/m}^2 & \text{otherwise,} \end{cases}$$

$$\mathbf{f}_0 = (0, 0, 0) \text{ N/m}^3, \quad \boldsymbol{\sigma}_0 = \mathbf{0} \text{ N/m}^3, \quad \mathbf{u}_0 = \mathbf{0} \text{ m}.$$

Here, \mathcal{E} is the three-dimensional elasticity tensor,

$$(\mathcal{E}\boldsymbol{\tau})_{ij} = \frac{E\kappa}{(1+\kappa)(1-2\kappa)} \left(\sum_{k=1}^3 \tau_{kk} \right) \delta_{ij} + \frac{E}{1+\kappa} \tau_{ij},$$

for $i, j = 1, 2, 3$, where Young's modulus E and Poisson's ratio κ are 10^8 N/m^2 and 0.3 , respectively.

The constitutive function $G(\boldsymbol{\sigma}, \boldsymbol{\varepsilon})$ is the 3-D Perzyna's (20),

$$K = \left\{ \boldsymbol{\tau} \in \mathbb{S}^3 \mid (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2) \leq \sigma_Y^2 \right\}.$$

Here, we used $\sigma_Y = \sqrt{10} \text{ N/m}^2$ and $\mu^* = 200 \text{ N/m}^2$.

In Fig. 3 the displacements and the Von-Mises norm for the stress are shown at the final time moment T . The evolution of the second component of the displacement field of the contact nodes on the surface $x_1 = 0$ is illustrated in Fig. 4.

4.2. Contact Problems with Normal Compliance

4.2.1. A One-Dimensional Test Problem

The contact problem with a deformable foundation described in Section 3.2 is considered with the data:

$$\Omega = (0, 1), \quad T = 10 \text{ sec}, \quad \Gamma_1 = \{0\}, \quad \Gamma_2 = \emptyset, \quad \Gamma_3 = \{1\}, \quad f_0(x, t) = 10 \text{ N/m},$$

$$g = 0.25 \text{ m}, \quad \alpha = 1, \quad u_0(x) = 0 \text{ m}, \quad \sigma_0(x) = 10 - 10x \text{ N/m},$$

$$\mathcal{E}(x) = 10 \text{ N}, \quad G(\sigma, \varepsilon) = -\sigma + 10\varepsilon, \quad \mu = 100 \text{ N/m}.$$

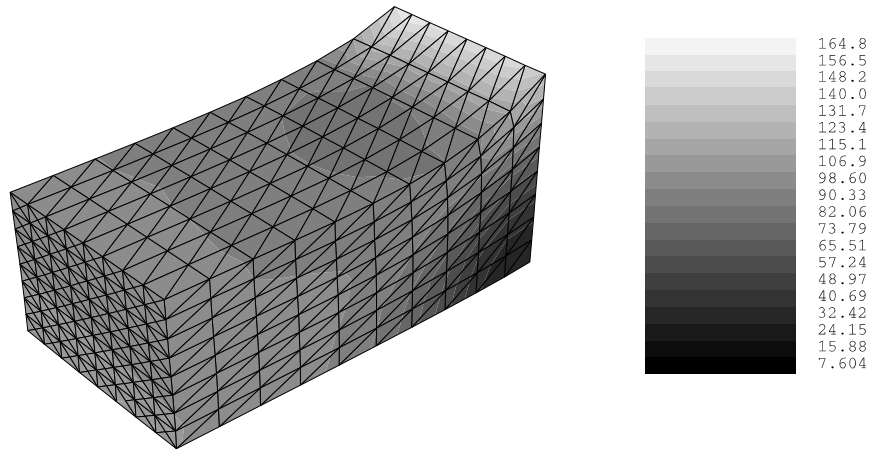


Fig. 3. The displacements and the Von-Mises norm; the 3-D Signorini problem.

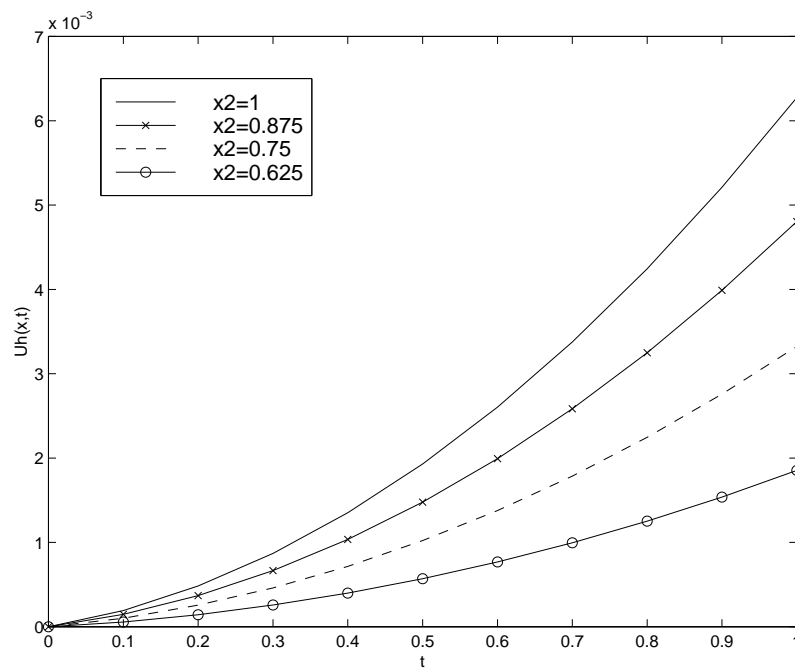


Fig. 4. Evolution of the U_2 component; the 3-D Signorini problem.

The exact solution to this problem is as follows:

- For $0 \leq t \leq \ln 2$ there is no contact,

$$\begin{cases} \sigma_\mu(t, x) = 10 - 10x, \\ u_\mu(t, x) = (1 - e^{-t})\left(x - \frac{x^2}{2}\right). \end{cases} \quad (21)$$

- For $t > \ln 2$ the body is in contact,

$$\begin{cases} \sigma_\mu(t, x) = \frac{5(2e^{-t} + 3 + 40\mu)}{2(10\mu + 1)} - 10x, \\ u_\mu(t, x) = \frac{x^2}{2}(e^{-t} - 1) + x \left[\frac{2e^{-t} + 3 + 40\mu}{4(10\mu + 1)} - e^{-t} \right]. \end{cases} \quad (22)$$

We used the discretized scheme for Problem 4 of Section 3.2 and implemented the numerical method on a standard workstation. Figure 5 depicts the displacements at the times $t = 0.5, 1, 2, 4, 8$ sec, calculated with parameters $h = 0.01$ and $k = 0.01$. We also show the difference from the exact solution (21)–(22) scaled by a factor of 10^3 . In Fig. 6 we illustrate the evolution of the points $x = 0.25, 0.5, 1$, and the corresponding error between the numerical and exact solutions. From the error values, the asymptotic behaviour (19) is obtained with $O(r) = Cr$, where $C = 0.9557 \times 10^{-1}$, which is independent of h and k (see Fig. 7).

4.2.2. A Two-Dimensional Test Problem

We test a two-dimensional contact problem with a deformable foundation described in Section 3.2 using the data:

$$\Omega = (0, 6) \times (0, 3), \quad T = 1 \text{ sec}, \quad \Gamma_1 = 6 \times [0, 3], \quad \Gamma_2 = (0, 6) \times \{2\},$$

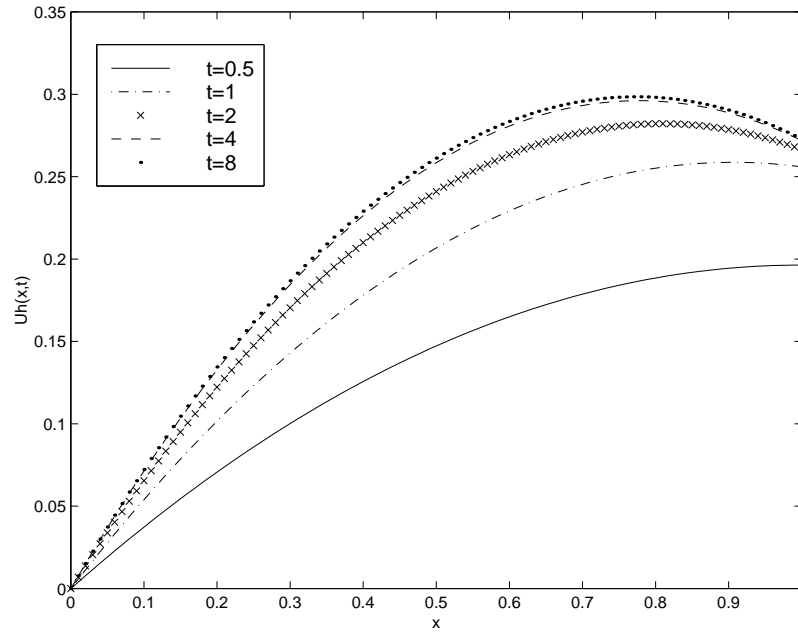
$$\Gamma_3 = (0, 6) \times \{0\}, \quad \mathbf{f}_0 = (0, 0) \text{ N/m}^2,$$

$$\mathbf{f}_2(x_1, x_2, t) = \begin{cases} (0, -10t) \text{ N/m} & \text{if } x_1 = 3, \\ (0, 0) \text{ N/m} & \text{otherwise,} \end{cases}$$

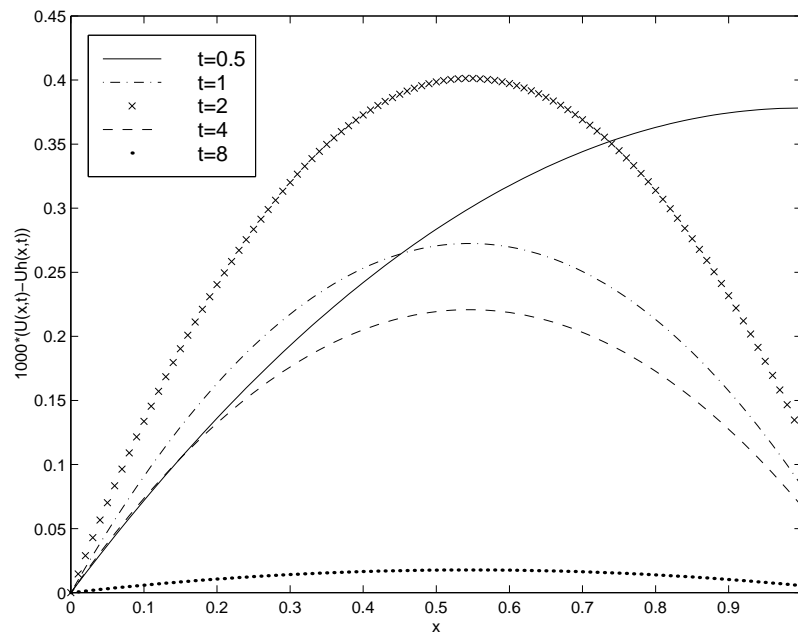
$$g = 0 \text{ m}, \quad \mu = 100 \text{ N/m}^2, \quad \boldsymbol{\sigma}_0 = \mathbf{0} \text{ N/m}^2, \quad \mathbf{u}_0 = \mathbf{0} \text{ m}.$$

\mathcal{E} and $G(\boldsymbol{\sigma}, \boldsymbol{\varepsilon})$ are the plane stress tensor and the Perzyna's function (20), respectively, where $E = 10^8 \text{ N/m}^2$, $\kappa = 0.3$ and $\mu^* = 200 \text{ N/m}^2$. The uniaxial yield stress $\sigma_Y = \sqrt{10} \text{ N/m}^2$ was used.

In Fig. 8 the displacements and the Von-Mises norm for the stress at the final time are shown. In Fig. 9 the evolution of the penetration at a number of contact points are depicted.

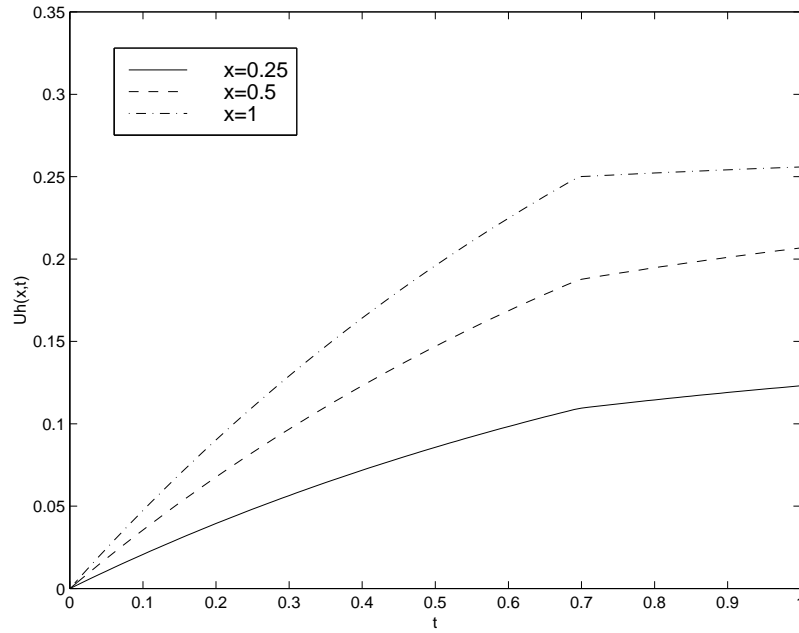


(a)

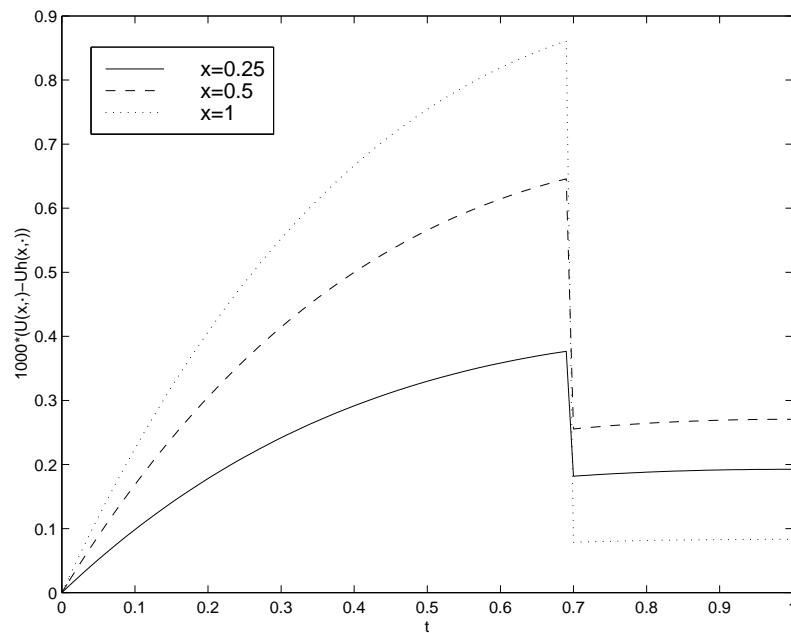


(b)

Fig. 5. Displacements (a) and the error (b) at different times; the 1-D normal compliance problem.



(a)



(b)

Fig. 6. Evolution of displacements at $x = 0.25, 0.5, 1$ (a), and the corresponding scaled errors (b); the 1-D normal compliance problem.

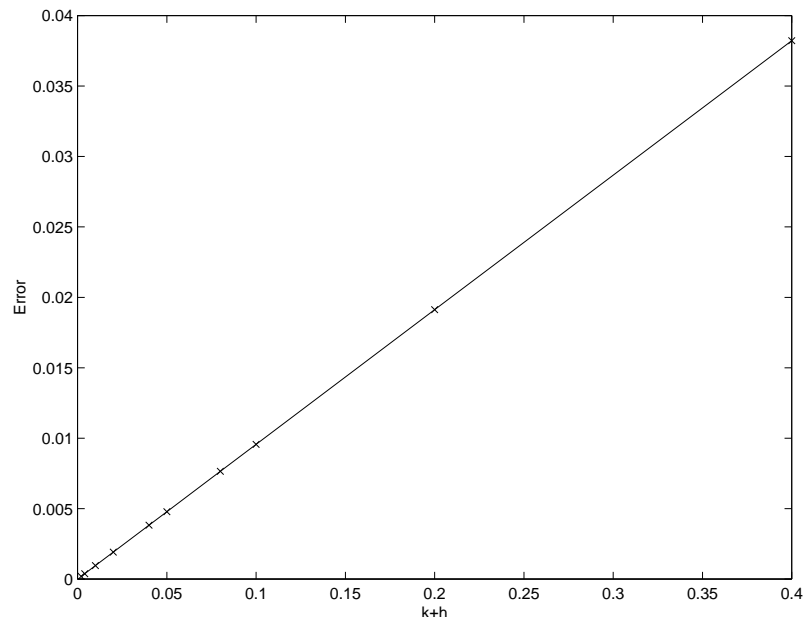


Fig. 7. Asymptotic constant error; the 1-D normal compliance problem.

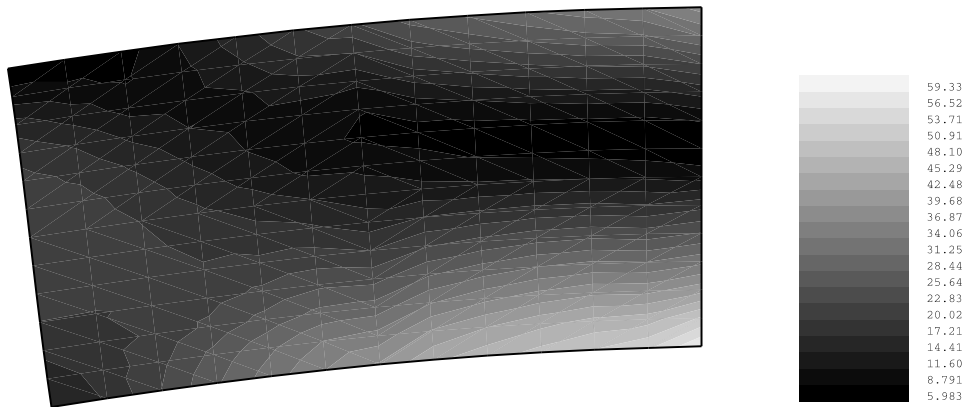


Fig. 8. A deformed configuration and the Von-Mises norm; the 2-D normal compliance problem.

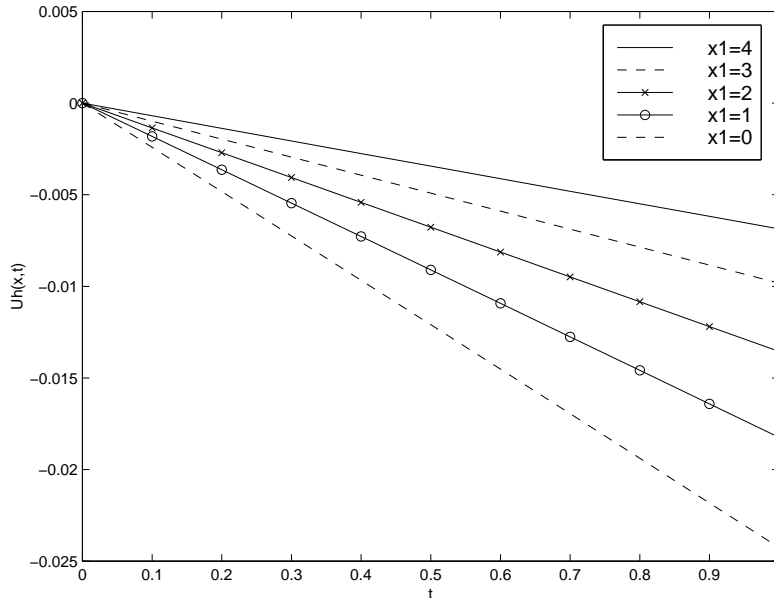


Fig. 9. Evolution of the U2 component; the 2-D normal compliance problem.

5. Conclusion

Considerable progress has been made in the modelling, variational analysis and numerical analysis of quasistatic contact of viscoplastic bodies. Our understanding of the behaviour of the models for these processes has deepened.

However, many open problems still remain. Besides establishing the well-posedness of models for other constitutive laws for plastic behaviour, our analytical tools are insufficient to provide detailed information on the structure of the solutions. Indeed, the most important applied aspects of the problems are the structure of the contact zones and the distribution of the contact stress in them. New mathematical tools need to be developed for this task. Since theoretical investigations are beyond our current capabilities, we must resort to numerical approximations and simulations of the models.

The recent progress in the numerical analysis indicated in this review and the establishing of rigorous error estimates allow for confidence in the results of the computer simulations.

Contact is associated with friction and wear of the contacting surfaces. It is of interest to include friction and wear in the models. As our recent research experience indicates, including additional phenomena leads to new and interesting types of variational inequalities. We conclude that considerable progress has been achieved,

but much remains to be done to construct a comprehensive mathematical theory of quasistatic viscoplastic contact problems.

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