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Quartic power series in $\mathbb{F}_3((T^{-1}))$ with bounded partial quotients

by

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1. Introduction. We are concerned with diophantine approximation and continued fractions in function fields. The rôles of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ in the classical theory are played by $K[T]$, $K(T)$ and $K((T^{-1}))$, where K is an arbitrary given field. An element of the field $K((T^{-1}))$ of power series will be denoted by $\alpha = a_kT^k + a_{k-1}T^{k-1} + \ldots$ where $k \in \mathbb{Z}$, $a_i \in K$ and $a_k \neq 0$. The rational k is called the *degree* of α, denoted by $\deg \alpha$. An ultrametric absolute value is defined by $|\alpha| = |T|^{\deg \alpha}$ and $|0| = 0$, where $|T|$ is a fixed real number greater than 1. Thus the field $K((T^{-1}))$ should be viewed as a completion of the field $K(T)$ for this absolute value.

We are considering the case when the base field K is finite. Let p be a prime number and q a positive power of p. Let K be a field of characteristic p. We consider the following algebraic equation with coefficients A, B, C and D in $K[T]$ and $\Delta = AD - BC \neq 0$:

$$x = \frac{Ax^q + B}{Cx^q + D}. \tag{1}$$

If α is an irrational solution in $K((T^{-1}))$ of such an equation, we say that α is *algebraic of class I*. The subset of algebraic elements of class I has different important properties concerning diophantine approximation. One of these properties, proved by Voloch [11] and de Mathan [7], implies the following:

If α is algebraic of class I, and $P/Q \in K(T)$, either we have

$$\liminf_{|Q|\to\infty} |Q|^2|\alpha - P/Q| > 0 \tag{2}$$

or there exists a real number $\mu > 2$ such that

$$\liminf_{|Q|\to\infty} |Q|^{\mu}|\alpha - P/Q| < \infty. \tag{3}$$

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Notice that there exist in $K((T^{-1}))$ algebraic elements over $K(T)$ which satisfy none of these two conditions and thus are not of class I. For such an algebraic element α we have $\liminf_{|Q|\to\infty} |Q|^2|\alpha - P/Q| = 0$ and

$$\liminf_{|Q|\to\infty} |Q|^{\mu}|\alpha - P/Q| = \infty \qquad \text{for all } \mu > 2.$$

An example of such an algebraic element is given in [3].

We will now use the continued fractions in the field $K((T^{-1}))$. The reader is referred to [9] for a good study on this subject. If $\alpha \in K((T^{-1}))$ we write $\alpha = [a_0, a_1, \ldots]$ for its continued fraction expansion, where the $a_i \in K[T]$ are the partial quotients and $\deg a_i > 0$ for $i > 0$. We denote by $(p_n/q_n)_{n\ge 0}$ the sequence of the convergents to α such that $p_n/q_n = [a_0, a_1, \ldots, a_n]$ for $n \ge 0$. We have the following important equation:

$$|\alpha - p_n/q_n| = |a_{n+1}|^{-1}|q_n|^{-2} \qquad \text{for } n \ge 0.$$

Then the two conditions above can be translated into the following:

(2) *The sequence of partial quotients in the continued fraction expansion for α is bounded*, *i.e.* $\limsup_n |a_n| < \infty$.

(3) *The sequence of partial quotients in the continued fraction expansion for α is unbounded and moreover there exists a positive number μ' such that* $\limsup_n |a_{n+1}| \cdot |q_n|^{-\mu'} > 0$.

If an element in $K((T^{-1}))$ of class I satisfies condition (2) we will say that it is an element of class I^{b} and if it satisfies condition (3) we will say that it is of class I^{ub}.

It is easy to remark that an element in $K((T^{-1}))$ which is algebraic over $K(T)$ of degree at most 3 is of class I. By Liouville's theorem [6] we know that the quadratic elements in $K((T^{-1}))$ are of class I^{b}. Moreover the continued fraction for quadratic power series has been studied [9] and it is periodic when the base field K is finite. This is another argument to see that a quadratic element in $K((T^{-1}))$ is of class I^{b}.

Most of the elements of class I are of class I^{ub}. Actually it is possible to show that if an element is of class I^{b} then the degree of its partial quotients is bounded by $\deg \Delta/(q-1)$, except for the first ones ([4] and [8]). Consequently, if $q > 1 + \deg \Delta$ in equation (1), then the corresponding solution is of class I^{ub}. Of course, this condition is only sufficient. The case where $\Delta \in K^*$ is special and has been studied in [9]. In this case the above condition, i.e. $q > 1 + \deg \Delta$, is true for all p and q, thus such an element is always of class I^{ub}. Moreover the continued fraction expansion for these elements can be explicitly described ([9] and [10]).

When $K = \mathbb{F}_2$, Baum and Sweet [1] were the first to prove that the class I^{b} is larger than the class of quadratic elements. They gave a famous example of a cubic power series with partial quotients of degree two or less. Their approach has been generalized and we have obtained other examples when the base field is $\mathbb{F}_2$ [4]. Furthermore, Baum and Sweet [2] have also described the power series in $\mathbb{F}_2((T^{-1}))$ which have all partial quotients of degree one in their continued fraction expansion. We have given a different characterization of these elements [5]. It follows from this new characterization that if an algebraic element in $\mathbb{F}_2((T^{-1}))$ is not quadratic and has all its partial quotients of degree one, then it is not an element of class I. Thakur [10] has also given examples of non-quadratic elements of class I^{b} when the base field K is a finite extension of $\mathbb{F}_2$.

In characteristic $p \geq 3$ less examples are known. Nevertheless Mills and Robbins [8] have described an algorithm to compute the continued fraction expansion for elements of class I. This enabled them to give an example of a non-quadratic element of class I^{b} with all partial quotients of degree one when the base field is $\mathbb{F}_p$ for all prime $p \geq 3$.

As was first observed by Baum and Sweet [1] and [2], each of the three classes I, I^{b} and I^{ub}, is stable under a Möbius transformation, under the Frobenius isomorphism and also under the substitution of T into a polynomial in T. These two last transformations induce an evident transformation on the corresponding continued fractions. But it is not easy to say in general what the partial quotients become after a Möbius transformation. Nevertheless, the case when the determinant of this map is in the base field K is special and in this situation the element and its image have almost the same expansion [9]. It is also interesting to notice that each of these three transformations preserves the degree of an algebraic element.

2. The case $K = \mathbb{F}_3$ and $q = p = 3$. We have investigated the case when the base field is $\mathbb{F}_3$. Non-quadratic elements of class I^{b} seem to be rare in class I. We have searched for examples with all partial quotients of degree one. According to what we have said above, if an element in $\mathbb{F}_3((T^{-1}))$ is of class I^{b} and if $\deg \Delta = 2$ then all its partial quotients are of degree one, except for a finite number. Thus we have checked up all possible equations (1) having a unique solution α in $\mathbb{F}_3((T^{-1}))$ with $|\alpha| = |T|^{-1}$ and where the polynomials A, B, C and $D \in \mathbb{F}_3[T]$ are of low degree satisfying $\deg(AD - BC) = 2$.

From the results obtained by computer, we think that after some of the transformations mentioned above—i.e. a Möbius transformation of determinant in $\mathbb{F}_3^*$ and the change of T into $uT + v$—the non-quadratic elements of class I^{b} that have all partial quotients of degree one reduce to a set of exceptional cases. These elements have a very peculiar continued fraction expansion and this is what we want to illustrate with the following theorem.

Theorem. *Let k be a non-negative integer. Define the sequence of integers $(u_n)_{n\ge 0}$ by*

$$u_0 = k \quad \text{and} \quad u_{n+1} = 3u_n + 4 \quad \text{for } n \ge 0.$$

If $a \in \mathbb{F}_3[T]$ and $n \ge 0$ is an integer, $a^{[n]}$ denotes the sequence $a, a, \ldots, a$ where a is repeated n times and $a^{[0]}$ is the empty sequence. Then define a finite sequence $H_n(T)$ of elements of $\mathbb{F}_3[T]$, for $n \ge 0$, by

$$H_n(T) = T+1, T^{[u_n]}, T+1.$$

Let $H_\infty(k)$ be the infinite sequence defined by juxtaposition

$$H_\infty(k) = H_0(T), H_1(-T), H_2(T), \ldots, H_n((-1)^n T), \ldots$$

Let $\omega(k)$ be the element of $\mathbb{F}_3((T^{-1}))$ defined by its continued fraction expansion

$$\omega(k) = [0, H_\infty(k)].$$

Let $(p_n)_{n\ge 0}$ and $(q_n)_{n\ge 0}$ be the usual sequences for the numerators and the denominators of the convergents of $\omega(k)$.

Then $\omega(k)$ is the unique solution in $\mathbb{F}_3((T^{-1}))$ of the irreducible quartic equation

$$x = \frac{p_k x^3 + p_{k+3}}{q_k x^3 + q_{k+3}}. \tag{1}$$

Remark. For example, if $k = 0$ then

$$\omega(0) = [0, T+1, T+1, -T+1, -T^{[4]}, -T+1, T+1, T^{[16]}, T+1, -T+1, \ldots]$$

and this element satisfies the algebraic equation

$$x = \frac{T^2+1}{T^3+T^2-T-x^3}.$$

Moreover, it is easy to show that equation (1) has a unique solution in $\mathbb{F}_3((T^{-1}))$. Therefore if we prove that this solution is $\omega(k)$, then since its continued fraction expansion is neither finite nor periodic, it will follow that $\omega(k)$ is algebraic of degree 4 over $\mathbb{F}_3(T)$.

Proof (of the Theorem). Let k be a non-negative integer. Let $\omega(k) \in \mathbb{F}_3((T^{-1}))$ be defined by the continued fraction expansion described in the Theorem. We write $\omega(k) = [a_0, a_1, \ldots]$ where the $a_i \in \mathbb{F}_3[T]$ are the partial quotients. We recall that if $(p_n/q_n)_{n\ge 0}$ is the sequence of convergents to $\omega(k)$ defined by $p_n/q_n = [a_0, a_1, \ldots, a_n]$, we have

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2} \tag{2}$$

for $n \ge 0$ with $p_{-2} = 0$, $q_{-2} = 1$, $p_{-1} = 1$, $q_{-1} = 0$. Since $a_0 = 0$ and $|a_n| = |T|$ for $n \ge 1$, it is clear that $|q_n| = |T|^n$ for $n \ge 0$. Moreover we have

$$|\omega(k) - p_n/q_n| = |a_{n+1}|^{-1}|q_n|^{-2} = |T|^{-1}|q_n|^{-2} \quad \text{for } n \ge 0. \tag{3}$$

The first step will be to prove that the Theorem is equivalent to a property satisfied by the two sequences $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$. Let f_k be the Möbius transformation involved in equation (1), such that (1) can be written $x = f_k(x^3)$. Hence we must prove that $\omega(k) = f_k(\omega(k)^3)$.

We put

$$\begin{cases} A_n = p_k p_n^3 + p_{k+3} q_n^3, \\ B_n = q_k p_n^3 + q_{k+3} q_n^3 \end{cases} \tag{4}$$

for $n \geq 0$. Thus we have

$$\frac{A_n}{B_n} = f_k\left(\left(\frac{p_n}{q_n}\right)^3\right).$$

Suppose now that $\omega(k) = f_k(\omega(k)^3)$. Then

$$|\omega(k) - A_n/B_n| = |f_k(\omega(k)^3) - f_k((p_n/q_n)^3)|. \tag{5}$$

By straightforward calculation we obtain

$$f_k(\omega(k)^3) - f_k((p_n/q_n)^3) = \frac{(q_{k+3}p_k - p_{k+3}q_k)(\omega(k) - p_n/q_n)^3}{(q_k\omega(k)^3 + q_{k+3})(q_k(p_n/q_n)^3 + q_{k+3})}. \tag{6}$$

Since $|p_n/q_n| = |\omega(k)| = |T|^{-1}$, we have

$$|q_k\omega(k)^3 + q_{k+3}| = |q_k(p_n/q_n)^3 + q_{k+3}| = |q_{k+3}|. \tag{7}$$

By (5)–(7), we can write

$$|\omega(k) - A_n/B_n| = |q_{k+3}p_k - p_{k+3}q_k| \cdot |q_{k+3}|^{-2}|\omega(k) - p_n/q_n|^3. \tag{8}$$

Now we have

$$|p_{k+3}/q_{k+3} - p_k/q_k| = |\omega(k) - p_k/q_k| = |T|^{-1}|q_k|^{-2}$$

and, since $|q_n| = |T|^n$ for $n \geq 0$, we get

$$|q_{k+3}p_k - p_{k+3}q_k| = |T|^{-1}|q_k|^{-1}|q_{k+3}| = |T|^2.$$

Finally, using (3) and observing that $|B_n| = |q_{k+3}q_n^3|$, (8) becomes

$$|\omega(k) - A_n/B_n| = |T|^{-1}|B_n|^{-2}. \tag{9}$$

Consequently, by (9) we have $|B_n|^2|\omega(k) - A_n/B_n| < 1$, and this proves that A_n/B_n is a convergent to $\omega(k)$. Put $A_n/B_n = p_m/q_m$. Comparing equality (3) for $n = m$ to (9) we obtain $|B_n| = |q_m|$. Since $|q_m| = |T|^m$ and $|B_n| = |T|^{3n+k+3}$, we get $m = 3n + k + 3$ and thus $A_n/B_n = p_{3n+k+3}/q_{3n+k+3}$.

Conversely, if $A_n/B_n = p_{3n+k+3}/q_{3n+k+3}$, then it follows from (4) that $p_{3n+k+3}/q_{3n+k+3} = f_k((p_n/q_n)^3)$ for $n \geq 0$. Letting now n go to infinity, we obtain $\omega(k) = f_k(\omega(k)^3)$.

This shows that the theorem is equivalent to the following: There exists $\varepsilon_n \in \mathbb{F}_3^*$, for $n \geq 0$, such that

$$\begin{cases} p_{3n+k+3} = \varepsilon_n(p_k p_n^3 + p_{k+3} q_n^3), \\ q_{3n+k+3} = \varepsilon_n(q_k p_n^3 + q_{k+3} q_n^3). \end{cases} \tag{10}$$

We now introduce the following notation: for $a \in \mathbb{F}_3[T]$, we denote by $\varepsilon(a)$ the leading coefficient of the polynomial a. Then $\varepsilon(a) = \pm 1$. By (2), we have $\varepsilon(q_n) = \varepsilon(a_n)\varepsilon(q_{n-1})$ for $n \geq 1$. Thus $\varepsilon(q_n) = \prod_{1 \leq i \leq n} \varepsilon(a_i)$. This allows us to determine ε_n, assuming that (10) holds. We see indeed that

$$\varepsilon(q_{3n+k+3}) = \varepsilon_n \varepsilon(q_k p_n^3 + q_{k+3} q_n^3) = \varepsilon_n \varepsilon(q_{k+3} q_n^3) = \varepsilon_n \varepsilon(q_{k+3}) \varepsilon(q_n).$$

We observe that $\varepsilon(a_i) = 1$ for $1 \leq i \leq k+2$ and $\varepsilon(a_{k+3}) = -1$. Hence $\varepsilon(q_{k+3}) = -1$ and $\varepsilon_n = -\varepsilon(q_{3n+k+3})\varepsilon(q_n)$. Consequently, we have

$$\varepsilon_n = -\prod_{n+1 \leq i \leq 3n+k+3} \varepsilon(a_i) \quad \text{for } n \geq 0. \tag{11}$$

The last step will be to prove (10). For this we shall use induction on n. Clearly (10) is true for $n = 0$ with $\varepsilon_0 = 1$. Next, it follows from (2) and (4) that we can write

$$A_n = p_k p_n^3 + p_{k+3} q_n^3 = p_k(a_n p_{n-1} + p_{n-2})^3 + p_{k+3}(a_n q_{n-1} + q_{n-2})^3.$$

Using the Frobenius isomorphism, this equality becomes

$$A_n = a_n^3 A_{n-1} + A_{n-2} \quad \text{for } n \geq 2. \tag{12}$$

Because of the same recursive definition for the two sequences $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$, the same recurrence relation holds clearly for the sequence $(B_n)_{n\geq 0}$.

In order to prove (10) by induction, we will show that the sequences $(p_{3n+k+3})_{n\geq 0}$ and $(q_{3n+k+3})_{n\geq 0}$ satisfy a recurrence relation similar to (12). For this we first express p_n in terms of p_{n-3}, p_{n-5} and p_{n-6}. Applying (2) successively, we can write

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} = a_n(a_{n-1} p_{n-2} + p_{n-3}) + p_{n-2}, \\ p_n &= a_n[a_{n-1}(a_{n-2} p_{n-3} + p_{n-4}) + p_{n-3}] + a_{n-2} p_{n-3} + p_{n-4}, \\ p_n &= (a_n a_{n-1} a_{n-2} + a_n + a_{n-2}) p_{n-3} + (a_n a_{n-1} + 1) p_{n-4} \end{aligned}$$

for $n \geq 6$. We now introduce some new notations. Since a_i and a_j are polynomials of degree 1 for $i, j \geq 1$, there exist $\lambda_{i,j} \in \mathbb{F}_3^*$ and $\mu_{i,j} \in \mathbb{F}_3$, such that we can write $a_i = \lambda_{i,j} a_j + \mu_{i,j}$. Thus we obtain

$$\begin{aligned} (a_n a_{n-1} + 1) p_{n-4} &= a_n(\lambda_{n-1,n-3} a_{n-3} + \mu_{n-1,n-3}) p_{n-4} + p_{n-4} \\ &= a_n \lambda_{n-1,n-3}(p_{n-3} - p_{n-5}) \\ &\quad + (\lambda_{n,n-3} a_{n-3} + \mu_{n,n-3}) \mu_{n-1,n-3} p_{n-4} + p_{n-4} \\ &= (a_n \lambda_{n-1,n-3} + \lambda_{n,n-3} \mu_{n-1,n-3})(p_{n-3} - p_{n-5}) \\ &\quad + (1 + \mu_{n,n-3} \mu_{n-1,n-3}) p_{n-4}. \end{aligned}$$

Finally, combining these equalities and using again $p_{n-4} = a_{n-4}p_{n-5} + p_{n-6}$, we can write

$$p_n = x_n p_{n-3} + y_n p_{n-6} + z_n p_{n-5} \tag{13}$$

for $n \geq 6$ with

$$\begin{cases} x_n = a_n a_{n-1} a_{n-2} + a_n(1 + \lambda_{n-1,n-3}) + a_{n-2} + \lambda_{n,n-3}\mu_{n-1,n-3}, \\ y_n = 1 + \mu_{n,n-3}\mu_{n-1,n-3}, \\ z_n = a_{n-4}(1 + \mu_{n,n-3}\mu_{n-1,n-3}) - \lambda_{n-1,n-3}a_n - \lambda_{n,n-3}\mu_{n-1,n-3}. \end{cases} \tag{14}$$

Once again, the same recursive definition for the two sequences $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ shows that (13) holds with p changed to q.

We want to apply (13) with n replaced by $3n + k + 3$. Since (13) holds for $n \geq 6$, the same relation with $3n+k+3$ instead of n will hold for $n \geq 1$. We need to express x_{3n+k+3}, y_{3n+k+3} and z_{3n+k+3} for $n \geq 1$. It is clear that this will be possible if we know the five consecutive partial quotients from a_{3n+k+3} to a_{3n+k-1}. For this reason, we return to the description of the sequence of partial quotients $(a_n)_{n\geq 0}$ given in the Theorem. We introduce the sequence of integers $(\pi_i)_{i\geq 0}$ in the following way. The finite subsequence of partial quotients represented by $H_i((-1)^i T)$ will be denoted by

$$H_i((-1)^i T) = a_{\pi_i}, a_{\pi_i+1}, \ldots, a_{\pi_{i+1}-1}.$$

From the definition of the sequence $(a_n)_{n\geq 0}$ it is easy to remark that we have $\pi_{i+1} - \pi_i = u_i + 2$ for $i \geq 0$ and therefore $\pi_{i+1} - \pi_i = 3u_{i-1} + 6 = 3(\pi_i - \pi_{i-1} - 2) + 6 = 3(\pi_i - \pi_{i-1})$ for $i \geq 1$. Thus $\pi_{i+1} - 3\pi_i = \pi_i - 3\pi_{i-1}$, and we obtain $\pi_{i+1} - 3\pi_i = \pi_1 - 3\pi_0 = (k + 3) - 3 = k$ for $i \geq 0$. Thus the sequence $(\pi_i)_{i\geq 0}$ is defined by

$$\pi_0 = 1 \quad \text{and} \quad \pi_{i+1} = 3\pi_i + k. \tag{15}$$

We now use a partition of the set $\mathbb{N}^* = \mathbb{N} - \{0\}$ into three classes defined by

$$E_1 = \{n \in \mathbb{N}^* : \text{there exists } i \geq 0 \text{ such that } n = \pi_i\},$$
$$E_2 = \{n \in \mathbb{N}^* : \text{there exists } i \geq 1 \text{ such that } n = \pi_i - 1\},$$
$$E_3 = \{n \in \mathbb{N}^* : \text{there exists } i \geq 1 \text{ such that } \pi_{i-1} < n < \pi_i - 1\}.$$

The expression of x_{3n+k+3}, y_{3n+k+3} and z_{3n+k+3} will depend on the class to which the integer n belongs.

• Assume that $n \in E_1$. By (15), there is $i \geq 0$ such that $3n + k = \pi_{i+1}$. Therefore we have

$$a_{3n+k} = (-1)^{i+1}T + 1, \quad a_{3n+k-1} = (-1)^i T + 1$$

and

$$a_{3n+k+1} = a_{3n+k+2} = a_{3n+k+3} = (-1)^{i+1}T.$$

Hence

$$\lambda_{3n+k+2,3n+k} = \lambda_{3n+k+3,3n+k} = 1, \quad \mu_{3n+k+3,3n+k} = \mu_{3n+k+2,3n+k} = -1.$$

Then, by (14), a simple calculation shows that

$$\begin{cases} x_{3n+k+3} = (-1)^{i+1}T^3 - 1, \\ y_{3n+k+3} = -1, \\ z_{3n+k+3} = 0. \end{cases}$$

Furthermore, as $n = \pi_i$ we have $a_n = (-1)^i T + 1$, and so (13) becomes

$$p_{3n+k+3} = -a_n^3 p_{3n+k} - p_{3n+k-3}. \tag{16}$$

• Assume that $n \in E_2$. By (15), there is $i \geq 1$ such that $3n+k+3 = \pi_{i+1}$. Therefore we have

$$a_{3n+k+3} = (-1)^{i+1}T + 1, \qquad a_{3n+k+2} = (-1)^i T + 1$$

and

$$a_{3n+k+1} = a_{3n+k} = a_{3n+k-1} = (-1)^i T.$$

Hence

$$\lambda_{3n+k+2,3n+k} = 1, \quad \lambda_{3n+k+3,3n+k} = -1,$$
$$\mu_{3n+k+3,3n+k} = \mu_{3n+k+2,3n+k} = 1.$$

Then, by (14), a simple calculation shows that

$$\begin{cases} x_{3n+k+3} = (-1)^{i+1}T^3 + 1, \\ y_{3n+k+3} = -1, \\ z_{3n+k+3} = 0. \end{cases}$$

Furthermore, as $n = \pi_i - 1$ we have $a_n = (-1)^{i-1}T + 1$, and so (13) becomes

$$p_{3n+k+3} = a_n^3 p_{3n+k} - p_{3n+k-3}. \tag{17}$$

• Assume that $n \in E_3$. By (15), there is $i \geq 1$ such that $\pi_i < 3n + k < \pi_{i+1} - 3$. Since $\pi_i - k$ is a multiple of 3 for $i \geq 1$, we have $\pi_i + 3 \leq 3n + k \leq \pi_{i+1} - 6$. Therefore we have $\pi_i + 2 \leq 3n + k - 1$ and $3n + k + 3 \leq \pi_{i+1} - 3$. Thus

$$a_{3n+k+3} = a_{3n+k+2} = a_{3n+k+1} = a_{3n+k} = a_{3n+k-1} = (-1)^i T.$$

Hence

$$\lambda_{3n+k+2,3n+k} = \lambda_{3n+k+3,3n+k} = 1 \quad \text{and} \quad \mu_{3n+k+3,3n+k} = \mu_{3n+k+2,3n+k} = 0.$$

Then, by (14), a simple calculation shows that

$$\begin{cases} x_{3n+k+3} = (-1)^i T^3, \\ y_{3n+k+3} = 1, \\ z_{3n+k+3} = 0. \end{cases}$$

Furthermore, as $\pi_{i-1} < n < \pi_i - 1$ we have $a_n = (-1)^{i-1}T$, and so (13) becomes

$$p_{3n+k+3} = -a_n^3 p_{3n+k} + p_{3n+k-3}. \tag{18}$$

In conclusion we have shown that we can write

$$p_{3n+k+3} = \theta_n a_n^3 p_{3n+k} + \theta'_n p_{3n+k-3} \tag{19}$$

for $n \geq 1$, where $\theta_n = \pm 1$ and $\theta'_n = \pm 1$ are given in (16), (17) or (18). Of course, for the reason given above, we also have the same relation with p changed to q. Now taking $n = 1$, as $1 \in E_1$ by (16), we have

$$p_{k+6} = -a_1^3 p_{k+3} - p_k \quad \text{and} \quad q_{k+6} = -a_1^3 q_{k+3} - q_k.$$

Since $p_1 = 1$ and $q_1 = a_1$, this shows that (10) holds for $n = 1$ with $\varepsilon_1 = -1$.

We can now begin our proof by induction. Let $n \geq 2$ be an integer. We assume that $A_{n-1} = \varepsilon_{n-1} p_{3n+k}$ and $A_{n-2} = \varepsilon_{n-2} p_{3n+k-3}$. Hence from (12) we can write

$$A_n = a_n^3 \varepsilon_{n-1} p_{3n+k} + \varepsilon_{n-2} p_{3n+k-3}$$

and this becomes

$$A_n = \varepsilon_{n-2}\theta'_n(\varepsilon_{n-1}\varepsilon_{n-2}\theta'_n a_n^3 p_{3n+k} + \theta'_n p_{3n+k-3}). \tag{20}$$

Recall that the same relation holds with B instead of A and q instead of p. Comparing (19) and (20), if we prove that

$$\theta_n = \varepsilon_{n-1}\varepsilon_{n-2}\theta'_n \tag{21}$$

for $n \geq 2$, then we will have $A_n = \varepsilon_{n-2}\theta'_n p_{3n+k+3}$ and $B_n = \varepsilon_{n-2}\theta'_n q_{3n+k+3}$. Thus (10) will hold for all $n \geq 2$ with $\varepsilon_n = \varepsilon_{n-2}\theta'_n$. By (11), which is true by induction for $n-1$ and $n-2$, we easily obtain

$$\varepsilon_{n-1}\varepsilon_{n-2} = \varepsilon(a_{n-1})\varepsilon(a_{3n+k-2})\varepsilon(a_{3n+k-1})\varepsilon(a_{3n+k}). \tag{22}$$

Once again we distinguish three cases:

- Assume that $n \in E_1$. By (16), $\theta_n = -1$ and $\theta'_n = -1$. Furthermore, by (15), there is $i \geq 1$ such that $n = \pi_i$ and $3n + k = \pi_{i+1}$. This implies $a_{n-1} = (-1)^{i-1}T + 1$ and

$$a_{3n+k} = (-1)^{i+1}T + 1, \quad a_{3n+k-1} = (-1)^i T + 1, \quad a_{3n+k-2} = (-1)^i T.$$

Hence, by (22), we obtain

$$\varepsilon_{n-1}\varepsilon_{n-2} = (-1)^{i-1}(-1)^i(-1)^i(-1)^{i+1} = 1.$$

Thus we see that (21) is satisfied.

- Assume that $n \in E_2$. By (17), $\theta_n = 1$ and $\theta'_n = -1$. Furthermore, by (15), there is $i \geq 1$ such that $n = \pi_i - 1$ and $3n + k = \pi_{i+1} - 3$. This implies $a_{n-1} = (-1)^{i-1}T$ and

$$a_{3n+k} = a_{3n+k-1} = a_{3n+k-2} = (-1)^i T.$$

Hence, by (22), we obtain

$$\varepsilon_{n-1}\varepsilon_{n-2} = (-1)^{i-1}(-1)^i(-1)^i(-1)^i = -1.$$

Thus we see that (21) is satisfied.

• Assume that $n \in E_3$. By (18), $\theta_n = -1$ and $\theta'_n = 1$. Furthermore, by (15), there is $i \geq 1$ such that $\pi_{i-1} < n < \pi_i - 1$ and $\pi_i + 3 \leq 3n+k \leq \pi_{i+1} - 6$. This implies $\varepsilon(a_{n-1}) = (-1)^{i-1}$ and

$$a_{3n+k} = a_{3n+k-1} = a_{3n+k-2} = (-1)^i T.$$

Hence, by (22), we obtain

$$\varepsilon_{n-1}\varepsilon_{n-2} = (-1)^{i-1}(-1)^i(-1)^i(-1)^i = -1.$$

Thus we see again that (21) is satisfied.

In conclusion (21) is satisfied for all $n \geq 2$, and so the proof of the Theorem is complete.

Before concluding, we make a last remark. While searching by computer for promising examples with all partial quotients of degree one, we have observed other types of continued fraction expansions than the one we have described in the Theorem. These have a pattern which is not very far from the previous one, but slightly more complicated. We want to describe here one of these types.

Let $k \geq 0$ and $l \geq 0$ be two integers. Let $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ be two sequences of integers defined recursively by

$$u_0 = k, \quad u_{n+1} = 3u_n + 4 \quad \textit{and} \quad v_0 = l, \quad v_{n+1} = 3v_n + 4.$$

Let H_n and K_n, for $n \geq 0$, be two finite sequences of elements of $\mathbb{F}_3[T]$ befined by

$$H_n = T + (-1)^n, T^{[u_n]}, T + (-1)^{n+1}$$

and

$$K_n = -T + (-1)^{n+1}, -T^{[v_n]}, -T + (-1)^{n+1}.$$

Let $H_\infty(k,l)$ be the infinite sequence defined by juxtaposition

$$H_\infty(k,l) = H_0, K_0, H_1, K_1, H_2, K_2, H_3, K_3, \ldots$$

Let $\Omega(k,l)$ be the element of $\mathbb{F}_3$ defined by its continued fraction expansion

$$\Omega(k,l) = [0, H_\infty(k,l)].$$

Then we conjecture that $\Omega(k,l)$ is an algebraic element of degree 4 over $\mathbb{F}_3(T)$ and that it satisfies an equation of the form $x = f(x^3)$ where f is a Möbius transformation with selected coefficients in $\mathbb{F}_3[T]$.

The case $k = l = 1$ corresponds to the example given by Mills and Robbins [8].

References

[1] L. Baum and M. Sweet, *Continued fractions of algebraic power series in characteristic 2*, Ann. of Math. 103 (1976), 593–610.

[2] —, —, *Badly approximable power series in characteristic 2*, ibid. 105 (1977), 573–580.

[3] A. Lasjaunias, *Diophantine approximation and continued fraction expansions of algebraic power series in positive characteristic*, J. Number Theory 65 (1997), 206–225.

[4] —, *Continued fractions for algebraic formal power series over a finite base field*, Finite Fields Appl. 5 (1999), 46–56.

[5] A. Lasjaunias and B. de Mathan, *Differential equations and diophantine approximation in positive characteristic*, Monatsh. Math. 128 (1999), 1–6.

[6] K. Mahler, *On a theorem of Liouville in fields of positive characteristic*, Canad. J. Math. 1 (1949), 397–400.

[7] B. de Mathan, *Approximation exponents for algebraic functions in positive characteristic*, Acta Arith. 60 (1992), 359–370.

[8] W. Mills and D. Robbins, *Continued fractions for certain algebraic power series*, J. Number Theory 23 (1986), 388–404.

[9] W. M. Schmidt, *On continued fractions and diophantine approximation in power series fields*, Acta Arith., to appear.

[10] D. Thakur, *Diophantine approximation exponents and continued fractions for algebraic power series*, J. Number Theory, to appear.

[11] J. F. Voloch, *Diophantine approximation in positive characteristic*, Period. Math. Hungar. 19 (1988), 217–225.

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