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Gauss sum for the adjoint representation of $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$

by

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1. Introduction. Let λ be a nontrivial additive character of $\mathbb{F}_q$, the finite field of q elements, and χ be a multiplicative character of $\mathbb{F}_q$. For a finite group of Lie type G defined over $\mathbb{F}_q$ (see [2]) and its finite-dimensional (rational) representation ϕ over $\mathbb{F}_q$, we define the *Gauss sum* $\mathcal{G}(G,\phi,\chi,\lambda)$ as follows:

$$\mathcal{G}(G,\phi,\chi,\lambda)=\sum_{x\in G}\chi(\det(\phi(x)))\cdot\lambda(\mathrm{tr}(\phi(x))).$$

The explicit expression of the above sum has been obtained in [5]–[12] for a finite classical group with respect to its natural representation and in [13] for the finite simple group of exceptional type $G=\mathbf{G}_2(q)$ with respect to its 7-dimensional faithful representation ϕ over $\mathbb{F}_q$.

When G are various finite classical groups and ϕ are the natural representations, the Gauss sums have turned out to be polynomials in q with coefficients involving mostly well-known exponential sums over $\mathbb{F}_q$. (See [5]–[12].) We also refer to [5]–[12] for motivations and applications of the Gauss sum $\mathcal{G}$.

These results for the classical groups and $\mathbf{G}_2(q)$ can be rephrased in the following conjectural statement: Let $G=G_l$ be a finite group of Lie type of rank l. Let S be a maximal $\mathbb{F}_q$-split torus of G. Then the centralizer $H=H_l=C_G(S)$ of S is the Levi subgroup of a minimal parabolic subgroup of G. Note that a minimal parabolic subgroup is a Borel subgroup of G and H is a maximal torus in G. (See [1, §20] and [4, §34] for details.) For $r\le l$,

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we denote by G_r a finite group of rank r defined over $\mathbb{F}_q$ and assume that $G = G_l$ and G_r are of "the same type" (see [2, p. 38]). Similarly, we denote by H_r the Levi subgroup of G_r contained in H_l. Let

$$\mathcal{H}(G,\phi) = \sum_{t\in H} \chi(\det(\phi(t))) \cdot \lambda(\mathrm{tr}(\phi(t)))$$

be the Gauss sum restricted to H. Then it is very likely that the Gauss sum $\mathcal{G}(G_l,\phi,\chi,\lambda)$ is a polynomial in q with coefficients involving some $\mathcal{H}(G_r,\phi)$ for $r \le l$. To be more precise, we need a slight modification of the above statement when G is a twisted group. Indeed, the Gauss sum of twisted group G_l involves not only $\mathcal{H}(G_r,\phi)$ but also "twisted" $\mathcal{H}(G_r,\phi)$. (Although results in [5]–[13] are not stated in the above form, it is not difficult to translate them into the above. See [14] for details.) We also note that there is an analogous result for classical Lie groups (see, for example, [3, 26.19]).

The purpose of this paper is to add more evidence for the above conjecture.

When G is the finite general linear group $\mathrm{GL}_n(q)$ and ϕ is the adjoint representation $\mathrm{Ad} : \mathrm{GL}_n(q) \to \mathrm{GL}(\mathfrak{gl}_n(q))$, using the "parabolic induction", we show that the Gauss sum is

$$\mathcal{G}(\mathrm{GL}_n(q), \mathrm{Ad}, \chi, \lambda) = L_{n,0} + q^{\binom{n}{2}} \sum_{k=0}^{[n/2]} c_k \mathcal{H}(\mathrm{GL}_{n-2k}(q), \mathrm{Ad})$$

where c_k and $L_{n,0}$ are polynomials in q (see Corollary 3.8 for details). In this case,

$$\mathcal{H}(\mathrm{GL}_m(q), \mathrm{Ad}) = \sum_{x_1,\dots,x_m\in\mathbb{F}_q^\times} \lambda\biggl((x_1+\ldots+x_m)\Bigl(\frac{1}{x_1}+\ldots+\frac{1}{x_m}\Bigr)\biggr).$$

Identifying the finite projective general linear group $\mathrm{PGL}_n(q)$ with the image of Ad, we thus obtain:

$$\mathcal{G}(\mathrm{PGL}_n(q), \mathrm{id}, \chi, \lambda) = \frac{1}{q-1} L_{n,0} + q^{\binom{n}{2}} \sum_{k=0}^{[n/2]} c_k \mathcal{H}(\mathrm{PGL}_{n-2k}(q), \mathrm{id}),$$

where

$$\mathcal{H}(\mathrm{PGL}_m(q), \mathrm{id}) = \frac{1}{q-1} \mathcal{H}(\mathrm{GL}_m(q), \mathrm{Ad}).$$

We note that $\frac{1}{q-1} L_{n,0}$ are polynomials in q.

If n and $q-1$ are relatively prime, then we also get the Gauss sum for the adjoint representation $\mathrm{Ad} : \mathrm{SL}_n(q) \to \mathrm{GL}(\mathfrak{sl}_n(q))$ of $\mathrm{SL}_n(q)$ using the results for $\mathrm{GL}_n(q)$. In this case the Gauss sum is

$$\mathcal{G}(\mathrm{SL}_n(q),\mathrm{Ad},\chi,\lambda)=\lambda(-1)\Big\{\frac{1}{q-1}L_{n,0}+q^{\binom{n}{2}}\sum_{k=0}^{[n/2]}c_k\mathcal{H}(\mathrm{SL}_{n-2k}(q),\mathrm{Ad})\Big\}$$

(see Proposition 6.5 for details) and the Gauss sum restricted to H is

$$\mathcal{H}(\mathrm{SL}_m(q),\mathrm{Ad})=\sum_{\substack{x_1,\dots,x_m\in\mathbb{F}_q^\times\\ x_1\dots x_m=1}}\lambda\bigg((x_1+\dots+x_m)\Big(\frac{1}{x_1}+\dots+\frac{1}{x_m}\Big)\bigg).$$

2. Preliminaries and notations. The main tool of this paper may be called *parabolic induction*. Thus we describe the Bruhat decomposition of $\mathrm{GL}_n(q)$ with respect to its parabolic subgroups.

Let $P=P_{l,m}$ (with $l,m\ge 1$ and $l+m=n$) be the parabolic subgroup of $\mathrm{GL}_n(q)$ given by

$$P_{l,m}=\left\{\begin{pmatrix}A_l & B\\ 0 & A_m\end{pmatrix}\ \middle|\ A_l\in\mathrm{GL}_l(q),\ A_m\in\mathrm{GL}_m(q),\ B\in\mathrm{Mat}_{l\times m}(q)\right\}$$

and let

$$\sigma_r=\begin{pmatrix}0&0&1_r&0\\0&1_{l-r}&0&0\\-1_r&0&0&0\\0&0&0&1_{m-r}\end{pmatrix}$$

where $0\le r\le\min\{l,m\}$ and 1_k is the $k\times k$ identity matrix.

Let

$$Q_r=\{x\in P\mid \sigma_r x\sigma_r^{-1}\in P\}$$

and let $Q_r\setminus P$ be a complete set of representatives for the right cosets of Q_r in P. Then the following decomposition of $\mathrm{GL}_n(q)$ into a disjoint union of right cosets of P is well known. (Our decomposition is slightly modified from that of [2, §2.8].)

Lemma 2.1. *We have*

$$\mathrm{GL}_n(q)=\coprod_{r=0}^{t}P\cdot\sigma_r\cdot(Q_r\setminus P)$$

where $t=\min\{l,m\}$.

The case $P=P_{n-1,1}$ will be particularly useful for our purpose. In this case

$$\mathrm{GL}_n(q)=P\coprod PwN$$

where $w = \sigma_1$ and $N = Q_1 \setminus P$. We recall that

$$|\mathrm{GL}_n(q)| = \prod_{k=0}^{n-1}(q^n - q^k)$$

and thus we have

$$|N| = \frac{q(q^{n-1}-1)}{q-1}$$

for $n \geq 2$.

Now we introduce some notation which will be used throughout this paper. We assume $P = P_{n-1,1}$ and $w = \sigma_1$. For

$$x = \begin{pmatrix} A & B \\ 0 & b_{nn} \end{pmatrix} \in P,$$

let

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & \cdots \\ \vdots & A' \end{pmatrix}, \qquad B = {}^t(b_{1n}, b_{2n}, \ldots, b_{n-1,n})$$

and

$$A' = \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{pmatrix}.$$

In this paper we use many equations with summations. For simplicity we use the following notations:

$$\sum_X = \sum_{x\in X} \quad \text{and} \quad \sum t_i = \sum_{i=1}^n t_i,$$

if $x \in X$ and n are explicit in those equations.

Finally, we consider a 0×0 matrix group as the trivial group, for example, $\mathrm{GL}_0(q) = \mathrm{SL}_0(q) = \{1\}$. But the trace of an element of such a group is defined to be zero.

3. Gauss sum for the adjoint representation of $\mathrm{GL}_n(q)$**.** The adjoint representation $\mathrm{Ad}_{\mathrm{GL}_n(q)} = \mathrm{Ad} : \mathrm{GL}_n(q) \to \mathrm{GL}(\mathfrak{gl}_n(q))$ of $\mathrm{GL}_n(q)$ over $\mathbb{F}_q$ is defined as

$$\mathrm{Ad}(x).X = xXx^{-1}$$

for $x \in \mathrm{GL}_n(q)$ and $X \in \mathfrak{gl}_n(q)$, where $\mathfrak{gl}_n(q)$ is the general linear Lie algebra over $\mathbb{F}_q$.

The following lemma is supposed to be well known.

LEMMA 3.1. *For a given* $g \in \mathrm{GL}_n(q)$, *we have*

(a) $\mathrm{tr}(\mathrm{Ad}(g)) = \mathrm{tr}(g)\,\mathrm{tr}(g^{-1})$,
(b) $\det(\mathrm{Ad}(g)) = 1$.

P r o o f. Let V be an n-dimensional vector space over $\mathbb{F}_q$. Then we may identify $\mathrm{GL}_n(q)$ with $\mathrm{GL}(V)$ and $\mathfrak{gl}_n(q)$ with $\mathfrak{gl}(V) = \mathrm{End}(V)$. Since $\mathrm{GL}(V)$ acts naturally on V, $V \otimes V^*$ is a $\mathrm{GL}(V)$-module, where V^* is the dual $\mathrm{GL}(V)$-module of V. Identifying $V \otimes V^*$ with $\mathrm{End}(V) = \mathfrak{gl}(V)$, we can easily see that the adjoint action of $\mathrm{GL}(V)$ on $\mathfrak{gl}(V)$ is equivalent to the $\mathrm{GL}(V)$-action on $V \otimes V^*$. Thus

$$\mathrm{tr}(\mathrm{Ad}(g)) = \mathrm{tr}(g \otimes ({}^t g^{-1})) = \mathrm{tr}(g) \cdot \mathrm{tr}({}^t g^{-1}) = \mathrm{tr}(g) \cdot \mathrm{tr}(g^{-1}),$$

and

$$\det(\mathrm{Ad}(g)) = \det(g \otimes ({}^t g^{-1})) = \det(g)^n \cdot \det({}^t g^{-1})^n = 1. \ \blacksquare$$

From the above lemma, if we want to get the Gauss sum for the adjoint representation of $\mathrm{GL}_n(q)$, it is enough to calculate

$$\sum_{x \in \mathrm{GL}_n(q)} \lambda(\mathrm{tr}(x)\,\mathrm{tr}(x^{-1})).$$

We denote by H_l the standard maximal $\mathbb{F}_q$-split torus in $\mathrm{GL}_l(q)$, that is,

$$H_l = C_{\mathrm{GL}_l(q)}(H_l) = \{\mathrm{diag}(t_1, \ldots, t_l) \mid t_1, \ldots, t_l \in \mathbb{F}_q^\times\}.$$

We recall that

$$\mathcal{H}(\mathrm{GL}_l(q), \mathrm{Ad}) = \sum_{t \in H_l} \chi(\det(\mathrm{Ad}(t))) \cdot \lambda(\mathrm{tr}(\mathrm{Ad}(t)))$$

is the Gauss sum restricted to H_l. Therefore, we have

$$\mathcal{H}(\mathrm{GL}_l(q), \mathrm{Ad}) = \sum_{x_1, \ldots, x_l \in \mathbb{F}_q^\times} \lambda\bigg((x_1 + \ldots + x_l)\bigg(\frac{1}{x_1} + \ldots + \frac{1}{x_l}\bigg)\bigg).$$

The integers $D_{n,l}$ given in the following definition, which appear in our main result (Theorem 3.6), are interesting by themselves.

DEFINITION 3.2. We set

$$D_{n,l} = \sum_{\mathrm{diag}(t_1, \ldots, t_n) \in H_l} \ \sum_{\substack{x \in \mathrm{GL}_n(q) \\ \mathrm{tr}(x) + \sum t_i = 0}} 1$$

for $n \geq 2$ and $l > 0$.

Using "parabolic induction" of $\mathrm{GL}_n(q)$ we obtain:

LEMMA 3.3. *Let $n \geq 2$ and $l \geq 0$. Then*

(a) $D_{0,l+1} = (q-1)((q-1)^l - (-1)^l)/q$,
(b) $D_{1,l} = D_{0,l+1}$,
(c) $D_{n,l} = q^{n-1} D_{n-1,l+1} + q^{n-1}(q-1)^l (q^{n-1} - 1)|\mathrm{GL}_{n-1}(q)|$.

Proof. (a) Since $t_i \neq 0$, it is clear that $\sum_{i=1}^{l+1} t_i = 0$ implies $\sum_{i=1}^{l} t_i \neq 0$. Thus

$$D_{0,l+1} = |H_l| - D_{0,l}.$$

(b) Clear.

(c) Using the fact that

$$\sum_{p\in P} \mathrm{tr}(pwp') = \sum_{p\in P} \mathrm{tr}(pw)$$

for $p' \in N$, we have (see the notation in Section 2)

$$\begin{aligned} D_{n,l} &= \sum_{H_l} \sum_{\substack{x\in \mathrm{GL}_n(q)\\ \mathrm{tr}(x)+\sum t_i=0}} 1 \\ &= \sum_{H_l} \sum_{\substack{x\in P\\ \mathrm{tr}(x)+\sum t_i=0}} 1 + \sum_{H_l} \sum_{\substack{x\in PwN\\ \mathrm{tr}(x)+\sum t_i=0}} 1 \\ &= q^{n-1} \sum_{H_l} \sum_{\substack{A\in \mathrm{GL}_{n-1}(q),\, b_{nn}\in \mathbb{F}_q^\times\\ \mathrm{tr}(A)+b_{nn}+\sum t_i=0}} 1 + |N| \sum_{H_l} \sum_{\substack{x\in P\\ -b_{1n}+\mathrm{tr}(A')+\sum t_i=0}} 1. \end{aligned}$$

Thus, we may assume $b_{1n} = \mathrm{tr}(A') + \sum t_i$, and hence

$$D_{n,l} = q^{n-1} D_{n-1,l+1} + |N|(q-1)^l q^{n-2}(q-1)|\mathrm{GL}_{n-1}(q)|. \quad ■$$

Now for a given nonnegative integer k, let $[0]_q = 1$,

$$[k]_q = \frac{q^k - 1}{q-1} \quad \text{and} \quad [k]_q^! = [k]_q[k-1]_q \dots [1]_q.$$

Then, from Lemma 3.3(c), we obtain

$$D_{n,l} = q^{\binom{n}{2}} \Big\{ D_{0,n+l} + (q-1)^n \sum_{j=1}^{n-1} [j]_q [j]_q^! \Big\}$$

for $n \geq 2$ and $l \geq 0$. Also from the direct calculation we have the identity

$$\sum_{j=1}^{n-1} [j]_q [j]_q^! = \frac{[n]_q^! - 1}{q}.$$

Thus we have shown:

Proposition 3.4. *Let $n \geq 2$ and $l \geq 0$. Then*

$$D_{n,l} = q^{\binom{n}{2}} \frac{(q-1)^{n+l} - (q-1)^n + (q-1)(-1)^{n+l}}{q} + \frac{|\mathrm{GL}_n(q)|}{q}.$$

Remark. In particular, for $n \geq 2$, we have

$$|\{x \in \mathrm{GL}_n(q) \mid \mathrm{tr}(x) = 0\}| = D_{n,0} = q^{\binom{n}{2}} \frac{(q-1)(-1)^n}{q} + \frac{|\mathrm{GL}_n(q)|}{q}.$$

Definition 3.5. For $n, l \geq 0$, we define

$$\mathcal{G}_{n,l} = \sum_{H_l} \sum_{x \in \mathrm{GL}_n(q)} \lambda\Big((\mathrm{tr}(x) + t_1 + \ldots + t_l)\Big(\mathrm{tr}(x^{-1}) + \frac{1}{t_1} + \ldots + \frac{1}{t_l}\Big)\Big).$$

Now we state the main results of this paper.

Theorem 3.6. *Let $n \geq 2$ and $l \geq 0$ (if $n = 2$ we assume $l \neq 0$). Then*

(a) $\mathcal{G}_{1,l} = \mathcal{H}(\mathrm{GL}_{l+1}(q), \mathrm{Ad}) = \mathcal{G}_{0,l+1}$,
(b) $\mathcal{G}_{2,0} = q\mathcal{H}(\mathrm{GL}_2(q), \mathrm{Ad})$,
(c) $\mathcal{G}_{n,l} = q^{n-1}\mathcal{G}_{n-1,l+1} + q^{2n-2}(q^{n-1} - 1)\mathcal{G}_{n-2,l} + q^{2n-2}\{(q-1)^l |\mathrm{GL}_{n-1}(q)| - 2(q^{n-1} - 1)D_{n-2,l}\}$.

Theorem 3.6 is proved in Section 4. Using Theorem 3.6, we can compute the Gauss sum for the adjoint representation of $\mathrm{GL}_n(q)$. To state the result, we define $L_{m,i}$ inductively as follows.

Definition 3.7. We define

$$L_{2,0} = L_{1,i} = L_{0,i+1} = 0,$$
$$L_{m,i} = q^{m-1}L_{m-1,i+1} + q^{2m-2}(q^{m-1} - 1)L_{m-2,i} + q^{2m-2}\{(q-1)^i |\mathrm{GL}_{m-1}(q)| - 2(q^{m-1} - 1)D_{m-2,i}\},$$

where $m \geq 2$ and $i \geq 0$ (if $m = 2$ then $i \neq 0$). Clearly $L_{m,i}$ are polynomials in q.

Corollary 3.8. *Let $n \geq 2$. Then*

$$\mathcal{G}(\mathrm{GL}_n(q), \mathrm{Ad}, \chi, \lambda) = \mathcal{G}_{n,0} = L_{n,0} + q^{\binom{n}{2}} \sum_{k=0}^{[n/2]} c_k \mathcal{H}(\mathrm{GL}_{n-2k}(q), \mathrm{Ad}),$$

where

$$c_k = \begin{cases} 1 & \text{if } k = 0, \\ q \displaystyle\sum_{\substack{n_1 \in \mathbb{N} \\ 0 < n_1 < n}} (q^{n_1} - 1) & \text{if } k = 1, \\ q^k \displaystyle\sum_{\substack{(n_1, \ldots, n_k) \in \mathbb{N}^k \\ 0 < n_i + 1 < n_{i+1} < n}} (q^{n_1} - 1)(q^{n_2} - 1)\ldots(q^{n_k} - 1) & \text{if } k \geq 2. \end{cases}$$

Proof. This is a sheer computation and is omitted. ■

Since the kernel of Ad is the scalar matrix in $\mathrm{GL}_n(q)$, we can identify the finite projective general linear group $\mathrm{PGL}_n(q)$ with the image of Ad. Let id be the identification map from $\mathrm{PGL}_n(q)$ onto the image of Ad. If H_l is the standard maximal torus in $\mathrm{GL}_l(q)$, then $\mathrm{Ad}(H_l) = C_{\mathrm{PGL}_l(q)}(\mathrm{Ad}(H_l))$

is a maximal $\mathbb{F}_q$-split torus in $\mathrm{PGL}_l(q)$. Hence the Gauss sum restricted to $\mathrm{Ad}(H_l)$ is

$$\mathcal{H}(\mathrm{PGL}_l(q),\mathrm{id}) = \frac{1}{q-1}\mathcal{H}(\mathrm{GL}_l(q),\mathrm{Ad})$$

and the Gauss sum for $\mathrm{PGL}_n(q)$ is

$$\mathcal{G}(\mathrm{PGL}_n(q),\mathrm{id},\chi,\lambda) = \frac{1}{q-1}\mathcal{G}(\mathrm{GL}_n(q),\mathrm{Ad},\chi,\lambda).$$

Therefore we have:

COROLLARY 3.9. *Let $n \geq 2$. Then*

$$\mathcal{G}(\mathrm{PGL}_n(q),\mathrm{id},\chi,\lambda) = \frac{1}{q-1}L_{n,0} + q^{\binom{n}{2}}\sum_{k=0}^{[n/2]} c_k \mathcal{H}(\mathrm{PGL}_{n-2k}(q),\mathrm{id}).$$

Note that $\frac{1}{q-1}L_{n,0}$ are polynomials in q.

4. Proof of Theorem 3.6. We begin with some lemmas.

LEMMA 4.1. *For any $a,b\in\mathbb{F}_q$, $b\neq 0$, we have*

$$\sum_{t\in\mathbb{F}_q}\lambda(a+bt) = \sum_{t\in\mathbb{F}_q}\lambda(t) = 0.$$

Proof. This is obvious. (Recall that λ is nontrivial.) ∎

LEMMA 4.2. *We have*

$$\sum_{x\in\mathbb{F}_q}\sum_{y,z\in\mathbb{F}_q^\times}\lambda(x^2yz) = 0.$$

Proof. Dividing the above sum into the sum when $x=0$ and the sum when $x\neq 0$, we get

$$\begin{aligned}\sum_{x\in\mathbb{F}_q}\sum_{y,z\in\mathbb{F}_q^\times}\lambda(x^2yz) &= (q-1)^2\lambda(0) + \sum_{x,y,z\in\mathbb{F}_q^\times}\lambda(x^2yz)\\ &= (q-1)^2 + \Big\{\sum_{y\in\mathbb{F}_q}\sum_{x,z\in\mathbb{F}_q^\times}\lambda(x^2yz) - (q-1)^2\lambda(0)\Big\}\\ &= 0. \ \blacksquare\end{aligned}$$

LEMMA 4.3. *Let $a,b\in\mathbb{F}_q$ and $c\in\mathbb{F}_q^\times$. Then*

$$\sum_{x,y\in\mathbb{F}_q^\times}\lambda((a+x)(b+cxy)) = \begin{cases} -(q-1) & \text{if } a=0,\ b=0,\\ 1 & \text{if } a=0,\ b\neq 0,\\ 1 & \text{if } a\neq 0,\ b=0,\\ \lambda(ab)+q & \text{if } a\neq 0,\ b\neq 0.\end{cases}$$

Proof. For each fixed x, replacing y by $(cx)^{-1}y$, we have

$$\begin{aligned}
&\sum_{x,y\in\mathbb{F}_q^\times} \lambda((a+x)(b+cxy))\\
&\quad= \sum_{x,y\in\mathbb{F}_q^\times} \lambda((a+x)(b+y))\\
&\quad= \lambda(ab) + \sum_{x,y\in\mathbb{F}_q} \lambda((a+x)(b+y)) - \sum_{y\in\mathbb{F}_q} \lambda(a(b+y)) - \sum_{x\in\mathbb{F}_q} \lambda((a+x)b)\\
&\quad= \lambda(ab) + \sum_{x,y\in\mathbb{F}_q} \lambda(xy) - \sum_{y\in\mathbb{F}_q} \lambda(ay) - \sum_{x\in\mathbb{F}_q} \lambda(xb)\\
&\quad= \lambda(ab) + q - \sum_{y\in\mathbb{F}_q} \lambda(ay) - \sum_{x\in\mathbb{F}_q} \lambda(xb).
\end{aligned}$$

Now the result follows from this. ■

Now we prove Theorem 3.6. Part (a) is obvious. For part (b), using the Bruhat decomposition of $\mathrm{GL}_2(q)$ with respect to the parabolic subgroup $P = P_{1,1}$ (see Section 2), we have

$$\begin{aligned}
\mathcal{G}_{2,0} &= \sum_{x\in\mathrm{GL}_2(q)} \lambda(\mathrm{tr}(x)\,\mathrm{tr}(x^{-1}))\\
&= \sum_{x\in P} \lambda(\mathrm{tr}(x)\,\mathrm{tr}(x^{-1})) + \sum_{x\in PwN} \lambda(\mathrm{tr}(x)\,\mathrm{tr}(x^{-1}))\\
&= q\mathcal{H}(\mathrm{GL}_2(q),\mathrm{Ad}) + \sum_{x\in PwN} \lambda(\mathrm{tr}(x)\,\mathrm{tr}(x^{-1}))\\
&= q\mathcal{H}(\mathrm{GL}_2(q),\mathrm{Ad}) + \sum_{x\in P}\sum_{y\in N} \lambda(\mathrm{tr}(xwy)\,\mathrm{tr}((xwy)^{-1}))\\
&= q\mathcal{H}(\mathrm{GL}_2(q),\mathrm{Ad}) + \sum_{x\in P}\sum_{y\in N} \lambda(\mathrm{tr}(yxw)\,\mathrm{tr}((yxw)^{-1}))\\
&= q\mathcal{H}(\mathrm{GL}_2(q),\mathrm{Ad}) + |N|\sum_{x\in P} \lambda(\mathrm{tr}(xw)\,\mathrm{tr}((xw)^{-1})).
\end{aligned}$$

Since

$$xw = \begin{pmatrix} a_{11} & b_{12}\\ 0 & b_{22}\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix} -b_{12} & a_{11}\\ -b_{22} & 0\end{pmatrix},$$

we obtain

$$\mathcal{G}_{2,0} = q\mathcal{H}(\mathrm{GL}_2(q),\mathrm{Ad}) + q\sum_{b_{12}\in\mathbb{F}_q}\ \sum_{a_{11},b_{22}\in\mathbb{F}_q^\times} \lambda\Big(-b_{12}\cdot\frac{-b_{12}}{a_{11}b_{22}}\Big)$$

and thus

$$\mathcal{G}_{2,0} = q\mathcal{H}(\mathrm{GL}_2(q),\mathrm{Ad})$$

by Lemma 4.2.

Now we calculate $\mathcal{G}_{n,l}$ in part (c) using the Bruhat decomposition of $\mathrm{GL}_n(q)$ with respect to the parabolic subgroup $P = P_{n-1,1}$ (see Section 2). First, we observe that

$$\begin{aligned}\mathcal{G}_{n,l} &= \sum_{H_l} \sum_{x\in \mathrm{GL}_n(q)} \lambda\Big(\Big(\mathrm{tr}(x) + \sum t_i\Big)\Big(\mathrm{tr}(x^{-1}) + \sum \frac{1}{t_i}\Big)\Big) \\ &= \sum_{H_l} \sum_{x\in P} \lambda\Big(\Big(\mathrm{tr}(x) + \sum t_i\Big)\Big(\mathrm{tr}(x^{-1}) + \sum \frac{1}{t_i}\Big)\Big) \\ &\quad + \sum_{H_l} \sum_{x\in PwN} \lambda\Big(\Big(\mathrm{tr}(x) + \sum t_i\Big)\Big(\mathrm{tr}(x^{-1}) + \sum \frac{1}{t_i}\Big)\Big).\end{aligned}$$

One can easily see that

$$\sum_{H_l} \sum_{x\in P} \lambda\Big(\Big(\mathrm{tr}(x) + \sum t_i\Big)\Big(\mathrm{tr}(x^{-1}) + \sum \frac{1}{t_i}\Big)\Big) = q^{n-1}\mathcal{G}_{n-1,l+1}. \tag{4.1}$$

Therefore it is enough to compute

$$\sum_{H_l} \sum_{x\in PwN} \lambda\Big(\Big(\mathrm{tr}(x) + \sum t_i\Big)\Big(\mathrm{tr}(x^{-1}) + \sum \frac{1}{t_i}\Big)\Big). \tag{4.2}$$

Let A_{ij} be the cofactor of a_{ij} in A and let A' be the submatrix of A obtained by deleting the first row and the first column (see Section 2 for the notation). Then

$$\begin{aligned}&\sum_{H_l} \sum_{x\in PwN} \lambda\Big(\Big(\mathrm{tr}(x) + \sum t_i\Big)\Big(\mathrm{tr}(x^{-1}) + \sum \frac{1}{t_i}\Big)\Big) \\ &= \sum_{H_l} \sum_{x\in P} \sum_{y\in N} \lambda\Big(\Big(\mathrm{tr}(xwy) + \sum t_i\Big)\Big(\mathrm{tr}(y^{-1}w^{-1}x^{-1}) + \sum \frac{1}{t_i}\Big)\Big) \\ &= |N| \sum_{H_l} \sum_{x\in P} \lambda\Big(\Big(\mathrm{tr}(xw) + \sum t_i\Big)\Big(\mathrm{tr}(w^{-1}x^{-1}) + \sum \frac{1}{t_i}\Big)\Big) \\ &= |N| \sum_{H_l} \sum_{B\in(\mathbb{F}_q)^{n-1}} \sum_{b_{nn}\in\mathbb{F}_q^\times} \sum_{A\in\mathrm{GL}_{n-1}(q)} \lambda\Big(\Big(\mathrm{tr}(A') - b_{1n} + \sum t_i\Big) \\ &\quad \times \Big(\frac{A_{22} + \ldots + A_{n-1,n-1}}{\det(A)} + \frac{\pm b_{1n}A_{11} \pm \ldots \pm b_{n-1,n}A_{n-1,1}}{b_{nn}\det(A)} + \sum \frac{1}{t_i}\Big)\Big) \\ &= |N| \sum_{H_l} \sum_{B\in(\mathbb{F}_q)^{n-1}} \sum_{b_{nn}\in\mathbb{F}_q^\times} \sum_{A\in\mathrm{GL}_{n-1}(q)} \lambda\Big(\Big(\mathrm{tr}(A') - b_{1n} + \sum t_i\Big) \\ &\quad \times \Big(\frac{A_{22} + \ldots + A_{n-1,n-1}}{\det(A)} + \frac{b_{1n}A_{11} + \ldots + b_{n-1,n}A_{n-1,1}}{b_{nn}\det(A)} + \sum \frac{1}{t_i}\Big)\Big).\end{aligned}$$

Hence, if $b_{1n} = \mathrm{tr}(A') + \sum t_i$ then the corresponding subsum of (4.2) is equal to

$$|N|(q-1)^l q^{n-2}(q-1)|\mathrm{GL}_{n-1}(q)|. \tag{4.3}$$

If $b_{1n} \neq \mathrm{tr}(A') + \sum t_i$ then, by Lemma 4.1, the corresponding subsum of (4.2) is 0, unless $A_{21} = A_{31} = \ldots = A_{n-1,1} = 0$. So now we assume that $A_{21} = A_{31} = \ldots = A_{n-1,1} = 0$. This is equivalent to saying that

$$A = \begin{pmatrix} a_{11} & 0 \\ A'' & A' \end{pmatrix}$$

where $a_{11} \in \mathbb{F}_q^\times$, $A'' = {}^t(a_{21}, \ldots, a_{n-1,1})$, $a_{i1} \in \mathbb{F}_q$ and $A' \in \mathrm{GL}_{n-2}(q)$. We define

$$\sigma = \mathrm{tr}(A') + \sum t_i \quad \text{and} \quad \tau = \mathrm{tr}(A'^{-1}) + \sum \frac{1}{t_i}.$$

Then the subsum of (4.2) corresponding to $b_{1n} \neq \mathrm{tr}(A') + \sum t_i$ is equal to

$$\begin{aligned}
(4.4)\quad & q^{n-2}|N| \\
&\times \sum_{H_l} \sum_{b_{nn}, a_{11} \in \mathbb{F}_q^\times} \sum_{A' \in \mathrm{GL}_{n-2}(q)} \sum_{\substack{B \in (\mathbb{F}_q)^{n-1} \\ b_{1n} \neq \sigma}} \lambda\Big((\sigma - b_{1n})\Big(\tau + \frac{b_{1n}}{b_{nn}a_{11}}\Big)\Big) \\
&= q^{n-2}|N| \sum_{H_l} \sum_{B \in (\mathbb{F}_q)^{n-1}} \sum_{b_{nn}, a_{11} \in \mathbb{F}_q^\times} \sum_{A' \in \mathrm{GL}_{n-2}(q)} \lambda\Big((\sigma - b_{1n})\Big(\tau + \frac{b_{1n}}{b_{nn}a_{11}}\Big)\Big) \\
&\quad - q^{n-2}|N|(q-1)^l q^{n-2}(q-1)^2|\mathrm{GL}_{n-2}(q)| \\
&= q^{n-2}|N|q^{n-2} \sum_{H_l} \sum_{b_{1n}, b_{nn}, a_{11} \in \mathbb{F}_q^\times} \sum_{A' \in \mathrm{GL}_{n-2}(q)} \lambda\Big((\sigma - b_{1n})\Big(\tau + \frac{b_{1n}}{b_{nn}a_{11}}\Big)\Big) \\
&\quad + q^{n-2}|N|q^{n-2}(q-1)^2\mathcal{G}_{n-2,l} - q^{n-2}|N|(q-1)^{l+2}q^{n-2}|\mathrm{GL}_{n-2}(q)|.
\end{aligned}$$

Therefore it remains to compute

$$\sum_{H_l} \sum_{b_{1n}, b_{nn}, a_{11} \in \mathbb{F}_q^\times} \sum_{A' \in \mathrm{GL}_{n-2}(q)} \lambda\Big((\sigma - b_{1n})\Big(\tau + \frac{b_{1n}}{b_{nn}a_{11}}\Big)\Big). \tag{4.5}$$

By Lemma 4.3, the sum (4.5) is equal to

$$\begin{aligned}
(q-1)\Big\{ & \sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma = 0,\, \tau = 0}} (-q+1) + \sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma = 0,\, \tau \neq 0}} 1 \\
&+ \sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma \neq 0,\, \tau = 0}} 1 + \sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma \neq 0,\, \tau \neq 0}} (q + \lambda(\sigma\tau)) \Big\}.
\end{aligned}$$

Since

$$\sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma=0,\, \tau\neq 0}} 1 = \sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma\neq 0,\, \tau=0}} 1$$

and

$$\begin{aligned}\sum_{H_l} &\sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma\neq 0,\, \tau\neq 0}} \lambda(\sigma\tau) \\ &= \sum_{H_l} \sum_{A' \in \mathrm{GL}_{n-2}(q)} \lambda(\sigma\tau) - 2\sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma=0}} 1 + \sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma=0,\, \tau=0}} 1,\end{aligned}$$

(4.5) becomes

$$\begin{aligned}(q-1)q\Big\{ -\sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma=0,\, \tau=0}} 1 + \sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma\neq 0,\, \tau\neq 0}} 1\Big\} \\ + (q-1)\sum_{H_l} \sum_{A' \in \mathrm{GL}_{n-2}(q)} \lambda(\sigma\tau) \\ = (q-1)q\Big\{ -\sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma=0}} 1 + \sum_{H_l} \sum_{\substack{A' \in \mathrm{GL}_{n-2}(q) \\ \sigma\neq 0}} 1\Big\} \\ + (q-1)\mathcal{G}_{n-2,l}.\end{aligned}$$

Thus we have shown that if $l \neq 0$ and $n \geq 2$ then (4.5) is equal to

$$(q-1)q\{(q-1)^l|\mathrm{GL}_{n-2}(q)| - 2D_{n-2,l}\} + (q-1)\mathcal{G}_{n-2,l}. \tag{4.6}$$

Therefore, if we combine the above results, we get

$$\begin{aligned}\mathcal{G}_{n,l} &= q^{n-1}\mathcal{G}_{n-1,l+1} \\ &\quad + \sum_{H_l} \sum_{x \in PwN} \lambda\Big(\Big(\mathrm{tr}(x) + \sum t_i\Big)\Big(\mathrm{tr}(x^{-1}) + \sum \frac{1}{t_i}\Big)\Big) \qquad \text{(see (4.1))} \\ &= q^{n-1}\mathcal{G}_{n-1,l+1} \\ &\quad + |N|q^{n-2}(q-1)^{l+1}|\mathrm{GL}_{n-1}(q)| \qquad \text{(see (4.3))} \\ &\quad + |N|q^{2n-4}\sum_{H_l} \sum_{b_{1n}, b_{nn}, a_{11} \in \mathbb{F}_q^\times} \sum_{A' \in \mathrm{GL}_{n-2}(q)} \lambda\Big((\sigma - b_{1n})\Big(\tau + \frac{b_{1n}}{b_{nn}a_{11}}\Big)\Big) \\ &\quad + |N|q^{2n-4}(q-1)^2\mathcal{G}_{n-2,l} - |N|q^{2n-4}(q-1)^{l+2}|\mathrm{GL}_{n-2}(q)| \quad \text{(see (4.4))}\end{aligned}$$

$$
\begin{aligned}
&= q^{n-1}\mathcal{G}_{n-1,l+1} + |N|q^{2n-4}(q-1)^2\mathcal{G}_{n-2,l}\\
&\quad + |N|q^{n-2}(q-1)^{l+1}|\mathrm{GL}_{n-1}(q)| - |N|q^{2n-4}(q-1)^{l+2}|\mathrm{GL}_{n-2}(q)|\\
&\quad + |N|q^{2n-4}\sum_{H_l}\ \sum_{b_{1n},b_{nn},a_{11}\in\mathbb{F}_q^\times}\ \sum_{A'\in\mathrm{GL}_{n-2}(q)} \lambda\Big((\sigma-b_{1n})\Big(\tau+\frac{b_{1n}}{b_{nn}a_{11}}\Big)\Big)\\
&= q^{n-1}\mathcal{G}_{n-1,l+1} + |N|q^{2n-4}(q-1)^2\mathcal{G}_{n-2,l}\\
&\quad + |N|q^{n-2}(q-1)^{l+1}|\mathrm{GL}_{n-1}(q)| - |N|q^{2n-4}(q-1)^{l+2}|\mathrm{GL}_{n-2}(q)|\\
&\quad + |N|q^{2n-4}(q-1)\\
&\quad \times \{q((q-1)^l|\mathrm{GL}_{n-2}(q)| - 2D_{n-2,l}) + \mathcal{G}_{n-2,l}\} \qquad \text{(see (4.6))}\\
&= q^{n-1}\mathcal{G}_{n-1,l+1} + q^{2n-2}(q^{n-1}-1)\mathcal{G}_{n-2,l}\\
&\quad + |N|q^{n-2}(q-1)^{l+1}\\
&\quad \times \{(q^{n-1}-1)q^{n-2}|\mathrm{GL}_{n-2}(q)| + q^{n-2}|\mathrm{GL}_{n-2}(q)|\}\\
&\quad + |N|q^{2n-4}(q-1)(-2qD_{n-2,l})\\
&= q^{n-1}\mathcal{G}_{n-1,l+1} + q^{2n-2}(q^{n-1}-1)\mathcal{G}_{n-2,l}\\
&\quad + q^{2n-2}(q-1)^l|\mathrm{GL}_{n-1}(q)| + q^{2n-2}(-2(q^{n-1}-1)D_{n-2,l})\\
&= q^{n-1}\mathcal{G}_{n-1,l+1} + q^{2n-2}(q^{n-1}-1)\mathcal{G}_{n-2,l}\\
&\quad + q^{2n-2}\{(q-1)^l|\mathrm{GL}_{n-1}(q)| - 2(q^{n-1}-1)D_{n-2,l}\}.
\end{aligned}
$$

This completes the proof of Theorem 3.6.

5. Gauss sum for the adjoint representation of $\mathrm{SL}_n(q)$**.** The adjoint representation $\mathrm{Ad}_{\mathrm{SL}_n(q)} = \mathrm{Ad} : \mathrm{SL}_n(q) \to \mathrm{GL}(\mathfrak{sl}_n(q))$ of $\mathrm{SL}_n(q)$ over $\mathbb{F}_q$ is defined as

$$\mathrm{Ad}(x).X = xXx^{-1}$$

for $x \in \mathrm{SL}_n(q)$ and $X \in \mathfrak{sl}_n(q)$, where $\mathfrak{sl}_n(q)$ is the special linear Lie algebra over $\mathbb{F}_q$.

LEMMA 5.1. *For a given* $g \in \mathrm{SL}_n(q)$, *we have*

(a) $\mathrm{tr}(\mathrm{Ad}(g)) = \mathrm{tr}(g)\,\mathrm{tr}(g^{-1}) - 1$,

(b) $\det(\mathrm{Ad}(g)) = 1$.

Proof. Let $\mathrm{Ad}_{\mathrm{GL}_n(q)}$ be the adjoint representation of $\mathrm{GL}_n(q)$ and $\mathrm{Ad}_{\mathrm{SL}_n(q)}$ be the adjoint representation of $\mathrm{SL}_n(q)$. Note that $\mathfrak{gl}_n(q) = \mathfrak{sl}_n(q) \oplus \mathbb{F}_q \cdot e_{nn}$, where $e_{nn} = \mathrm{diag}(0,\ldots,0,1)$. Thus for any $g \in \mathrm{SL}_n(q)$, we get

$$\mathrm{Ad}_{\mathrm{GL}_n(q)}|_{\mathrm{SL}_n(q)}(x) = \begin{pmatrix} \mathrm{Ad}_{\mathrm{SL}_n(q)}(x) & * \\ 0 & * \end{pmatrix}.$$

However, since

$$\mathrm{tr}(\mathrm{Ad}_{\mathrm{GL}_n(q)}|_{\mathrm{SL}_n(q)}(g).e_{nn}) = \mathrm{tr}(ge_{nn}g^{-1}) = 1,$$

we have $ge_{nn}g^{-1} - e_{nn} \in \mathfrak{sl}_n(q)$. Therefore

$$\mathrm{Ad}_{\mathrm{GL}_n(q)}|_{\mathrm{SL}_n(q)}(g) = \begin{pmatrix} \mathrm{Ad}_{\mathrm{SL}_n(q)}(g) & * \\ 0 & 1 \end{pmatrix}.$$

This proves our lemma by Lemma 3.1. ∎

By Lemma 5.1, if we want to get the Gauss sum of the adjoint representation of $\mathrm{SL}_n(q)$, it is enough to calculate

$$\sum_{x\in \mathrm{SL}_n(q)} \lambda(\mathrm{tr}(x)\,\mathrm{tr}(x^{-1}) - 1).$$

Now we decompose $\mathrm{GL}_n(q)$ into the disjoint union of left cosets of $\mathrm{SL}_n(q)$.

Lemma 5.2. *Let n and $q-1$ be relatively prime. Then*

$$\mathrm{GL}_n(q) = \coprod_{t\in \mathbb{F}_q^\times} t\mathrm{SL}_n(q).$$

Proof. For any $g \in \mathrm{GL}_n(q)$, let $\det(g) = \alpha$. Since n and $q-1$ are relatively prime, there is a unique $t \in \mathbb{F}_q^\times$ such that $t^n = \alpha$. Thus $t^{-1}g \in \mathrm{SL}_n(q)$ and $g \in t\mathrm{SL}_n(q)$. ∎

From Lemma 5.2, we have

$$\begin{aligned}\sum_{x\in \mathrm{GL}_n(q)} \lambda(\mathrm{tr}(x)\,\mathrm{tr}(x^{-1})) &= \sum_{t\in\mathbb{F}_q^\times}\sum_{y\in \mathrm{SL}_n(q)} \lambda(\mathrm{tr}(ty)\,\mathrm{tr}((ty)^{-1})) \\ &= \sum_{t\in\mathbb{F}_q^\times}\sum_{y\in \mathrm{SL}_n(q)} \lambda(\mathrm{tr}(y)\,\mathrm{tr}(y^{-1})) \\ &= (q-1)\sum_{y\in \mathrm{SL}_n(q)} \lambda(\mathrm{tr}(y)\,\mathrm{tr}(y^{-1})).\end{aligned}$$

Therefore we get the following lemma.

Lemma 5.3. *Let n and $q-1$ be relatively prime. Then*

$$\sum_{x\in \mathrm{SL}_n(q)} \lambda(\mathrm{tr}(x)\,\mathrm{tr}(x^{-1})) = \frac{1}{q-1}\sum_{x\in \mathrm{GL}_n(q)} \lambda(\mathrm{tr}(x)\,\mathrm{tr}(x^{-1})).$$

For $\mathrm{SL}_n(q)$, we take H_n to be the standard maximal $\mathbb{F}_q$-split torus in $\mathrm{SL}_n(q)$, that is,

$$H_n = C_{\mathrm{SL}_n(q)}(H_n) = \Big\{\mathrm{diag}(x_1,\dots,x_n) \;\Big|\; x_i \in \mathbb{F}_q^\times,\ 1\le i\le n,\ \prod_{i=1}^n x_i = 1\Big\}.$$

Hence, we have

$$\begin{aligned}\mathcal{H}(\mathrm{SL}_n(q),\mathrm{Ad}) &= \sum_{x_1,\dots,x_{n-1}\in\mathbb{F}_q^\times} \lambda\bigg(\bigg(x_1+\ldots+x_{n-1}+\frac{1}{x_1\ldots x_{n-1}}\bigg) \\ &\qquad\times\bigg(\frac{1}{x_1}+\ldots+\frac{1}{x_{n-1}}+x_1\ldots x_{n-1}\bigg)\bigg) \\ &= \sum_{\substack{x_1,\dots,x_n\in\mathbb{F}_q^\times\\ x_1\ldots x_n=1}} \lambda\bigg((x_1+\ldots+x_n)\bigg(\frac{1}{x_1}+\ldots+\frac{1}{x_n}\bigg)\bigg).\end{aligned}$$

Then we have:

LEMMA 5.4. *Let n and $q-1$ be relatively prime. Then*

$$\mathcal{H}(\mathrm{SL}_n(q),\mathrm{Ad}_{\mathrm{SL}_n(q)}) = \frac{1}{q-1}\mathcal{H}(\mathrm{GL}_n(q),\mathrm{Ad}_{\mathrm{GL}_n(q)}).$$

Proof. For $\alpha,t\in\mathbb{F}_q^\times$, we denote $X_\alpha=\{(x_1,\dots,x_n)\in(\mathbb{F}_q^\times)^n \mid x_1\ldots x_n=\alpha\}$ and $t(x_1,\dots,x_n)=(tx_1,\dots,tx_n)$. Since n and $q-1$ are relatively prime, for any $\alpha\in\mathbb{F}_q^\times$, there is $t\in\mathbb{F}_q^\times$ such that $t^n=\alpha$. Hence $X_\alpha=tX_1$ and $(\mathbb{F}_q^\times)^n=\prod_{t\in\mathbb{F}_q^\times} tX_1$. Therefore

$$\begin{aligned}\mathcal{H}(\mathrm{GL}_n(q),\mathrm{Ad}) &= \sum_{x_1,\dots,x_n\in\mathbb{F}_q^\times} \lambda\bigg((x_1+\ldots+x_n)\bigg(\frac{1}{x_1}+\ldots+\frac{1}{x_n}\bigg)\bigg) \\ &= \sum_{t\in\mathbb{F}_q^\times}\sum_{tX_1}\lambda\bigg((tx_1+\ldots+tx_n)\bigg(\frac{1}{tx_1}+\ldots+\frac{1}{tx_n}\bigg)\bigg) \\ &= \sum_{t\in\mathbb{F}_q^\times}\sum_{X_1}\lambda\bigg((x_1+\ldots+x_n)\bigg(\frac{1}{x_1}+\ldots+\frac{1}{x_n}\bigg)\bigg) \\ &= (q-1)\mathcal{H}(\mathrm{SL}_n(q),\mathrm{Ad}). \ \blacksquare\end{aligned}$$

Summarizing the above results, we have the following proposition.

PROPOSITION 5.5. *Let n and $q-1$ be relatively prime. Then*

$$\mathcal{G}(\mathrm{SL}_n(q),\mathrm{Ad},\chi,\lambda) = \lambda(-1)\bigg\{\frac{1}{q-1}L_{n,0} + q^{\binom{n}{2}}\sum_{k=0}^{[n/2]} c_k\mathcal{H}(\mathrm{SL}_{n-2k}(q),\mathrm{Ad})\bigg\}.$$

We note that $\frac{1}{q-1}L_{n,0}$ are polynomials in q.

We also note that the finite projective special linear group $\mathrm{PSL}_n(q)$ is isomorphic to $\mathrm{SL}_n(q)$, since we are assuming n and $q-1$ are relatively prime.

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