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# Inequalities concerning the function $\pi(x)$: Applications

by

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**Introduction.** In this note we use the following standard notations: $\pi(x)$ is the number of primes not exceeding $x$, while $\theta(x) = \sum_{p \le x} \log p$.

The best known inequalities involving the function $\pi(x)$ are the ones obtained in [6] by B. Rosser and L. Schoenfeld:

$$\frac{x}{\log x - 1/2} < \pi(x) \quad \text{for } x \ge 67, \tag{1}$$

$$\frac{x}{\log x - 3/2} > \pi(x) \quad \text{for } x > e^{3/2}. \tag{2}$$

The proof of the above inequalities is not elementary and is based on the first 25 000 zeros of the Riemann function $\xi(s)$ obtained by D. H. Lehmer [4]. Then Rosser, Yohe and Schoenfeld announced that the first 3 500 000 zeros of $\xi(s)$ lie on the critical line [9]. This result was followed by two papers [7], [10]; some of the inequalities they include will be used in order to obtain inequalities (11) and (12) below.

In [6] it is proved that $\pi(x) \sim x/(\log x - 1)$. Here we will refine this expression by giving upper and lower bounds for $\pi(x)$ which both behave as $x/(\log x - 1)$ as $x \to \infty$.

**New inequalities.** We start by listing those inequalities in [6] and [10] that will be used further:

$$\theta(x) < x \quad \text{for } x < 10^8, \tag{3}$$

$$|\theta(x) - x| < 2.05282\sqrt{x} \quad \text{for } x < 10^8, \tag{4}$$

$$|\theta(x) - x| < 0.0239922\frac{x}{\log x} \quad \text{for } x \ge 758\,711, \tag{5}$$

$$|\theta(x) - x| < 0.0077629\frac{x}{\log x} \quad \text{for } x \ge e^{22}, \tag{6}$$

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$$|\theta(x) - x| < 8.072\frac{x}{\log^2 x} \qquad \text{for } x > 1. \tag{7}$$

The above inequalities are used first to prove the following lemma:

Lemma 1. *We have*

$$\theta(x) < x\left(1 + \frac{1}{3(\log x)^{1.5}}\right) \qquad \textit{for } x > 1, \tag{8}$$

$$\theta(x) > x\left(1 - \frac{2}{3(\log x)^{1.5}}\right) \qquad \textit{for } x \geq 6\,400. \tag{9}$$

Proof. For $x \geq e^{587}$ the inequality

$$8.072 < \frac{1}{3}(\log x)^{0.5}$$

holds and therefore, using (7), it follows that

$$|\theta(x) - x| < \frac{x}{3(\log x)^{1.5}}. \tag{10}$$

For $e^{22} \leq x < e^{587}$ we have

$$0.0077629 < \frac{1}{3(\log x)^{0.5}}$$

and by using (6) we obtain (10). For $757\,711 \leq x < e^{22}$ we have

$$0.0239922 < \frac{1}{3(\log x)^{0.5}}$$

and by using (5) we obtain again (10) for $x \geq 757\,711$. These results, together with inequality (3), obviously imply (8).

Let $6\,400 \leq x < 10^8$. Then

$$2.05282 < \frac{2}{3} \cdot \frac{\sqrt{x}}{(\log x)^{1.5}}$$

which implies (9) by using (4) and (10). ∎

Lemma 1 helps us to prove

Theorem 1. *We have*

$$\pi(x) < \frac{x}{\log x - 1 - (\log x)^{-0.5}} \qquad \textit{for } x \geq 6, \tag{11}$$

$$\pi(x) > \frac{x}{\log x - 1 + (\log x)^{-0.5}} \qquad \textit{for } x \geq 59. \tag{12}$$

Proof. We use the well-known identity

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t\log^2 t}\,dt.$$

By (8) we obtain

$$\begin{aligned}\pi(x) &< \frac{x}{\log x} + \frac{x}{3(\log x)^{2.5}} + \int_2^x \frac{dt}{\log^2 t} + \frac{1}{3}\int_2^x \frac{dt}{(\log t)^{3.5}} \\ &= \frac{x}{\log x}\left(1 + \frac{1}{3(\log x)^{1.5}} + \frac{1}{\log x}\right) - \frac{2}{\log^2 2} + 2\int_2^x \frac{dt}{\log^3 t} + \frac{1}{3}\int_2^x \frac{dt}{(\log t)^{3.5}}.\end{aligned}$$

Since

$$-\frac{2}{\log^2 2} + \frac{1}{3}\int_2^x \frac{dt}{(\log t)^{3.5}} < \frac{1}{3}\int_2^x \frac{dt}{\log^3 t}$$

it follows that

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{3(\log x)^{1.5}} + \frac{1}{\log x}\right) + \frac{7}{3}\int_2^x \frac{dt}{\log^3 t}.$$

For $x \ge e^{18.25}$ we define

$$f(x) = \frac{2}{3}\cdot\frac{x}{(\log x)^{2.5}} - \frac{7}{3}\int_2^x \frac{dt}{\log^3 t}.$$

Then

$$f'(x) = \frac{2\log x - 7(\log x)^{0.5} - 5}{3(\log x)^{3.5}} > 0,$$

which implies that $f$ is an increasing function. For any convex function $g : [a,b] \to \mathbb{R}$ we have

$$\int_a^b g(x)\,dx \le \frac{b-a}{n}\left(g(a) + g(b) + \sum_{k=1}^{n-1} g\left(a + k\frac{b-a}{n}\right)\right).$$

For $g(x) = 1/\log^3 x$ and $n = 10^5$, we can apply the above inequality on each interval $[2,e]$, $[e,e^2]$, …, $[e^{17},e^{18}]$, and $[e^{18},e^{18.25}]$ to get

$$\int_2^{e^{18.25}} \frac{dt}{\log^3 t} < 16\,870.$$

As the referee kindly pointed out, the above inequality may also be checked using the software package *Mathematica*.

We have

$$f(e^{18.25}) > \frac{1}{3}(118\,507 - 118\,090) > 0.$$

Therefore $f(x) > 0$, which implies that for $x \ge e^{18.25}$,

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1}{(\log x)^{1.5}}\right) < \frac{x}{\log x - 1 - (\log x)^{-0.5}}.$$

Let now $x \le e^{18.25} < 10^8$. By using (3) we obtain

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t\log^2 t}\,dt < \frac{x}{\log x} + \int_2^x \frac{dt}{\log^2 t}$$
$$= \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right) - \frac{2}{\log^2 2} + 2\int_2^x \frac{dt}{\log^3 t}.$$

For $4\,000 \le x < 10^8$ define

$$g(x) = \frac{x}{(\log x)^{2.5}} - 2\int_2^x \frac{dt}{\log^3 t} + \frac{2}{\log^2 2}.$$

Since

$$g'(x) = \frac{\log x - 2(\log x)^{0.5} - 2.5}{(\log x)^{3.5}} > 0,$$

$g$ is an increasing function,

$$g(e^{11}) > 149 - 2\int_2^{e^{11}} \frac{dt}{\log^3 t} > 149 - 140 > 0,$$

hence for $e^{11} \le x < 10^8$ we have

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1}{(\log x)^{1.5}}\right) < \frac{x}{\log x - 1 - (\log x)^{-0.5}}.$$

For $x \ge 6$ it follows immediately that $\log x - 1 - (\log x)^{-0.5} > 0$. Hence, for $6 \le x \le e^{11}$, the inequality to be proved is

$$h(x) = \frac{x}{\pi(x)} + 1 + (\log x)^{-0.5} - \log x > 0.$$

If $p_n$ is the $n$th prime, then $h$ is an increasing function in $[p_n, p_{n+1})$, so it suffices to prove that $h(p_n) > 0$. Since $p_n < e^{11}$, the inequality $(\log p_n)^{-0.5} > 0.3$ holds and therefore it suffices to prove that $p_n/n - \log p_n > -1.3$, which may be verified by computer for $e^{11} > p_n \ge 7$.

In order to prove inequality (12) we use (3), (9) and for $x \ge 6\,400$ we have

$$\pi(x) - \pi(6\,400) = \frac{\theta(x)}{\log x} - \frac{\theta(6\,400)}{\log 6\,400} + \int_{6\,400}^x \frac{\theta(t)}{t\log^2 t}\,dt.$$

Since $\pi(6\,400) = 834$ and $\theta(6\,400)/\log 6\,400 < 6\,400/\log 6\,400 < 731$ we have

$$\pi(x) > 103 + \frac{\theta(x)}{x} + \int_{6\,400}^x \frac{\theta(t)}{t\log^2 t}\,dt.$$

From (9) it follows that

$$\begin{aligned}\pi(x) &> 103 + \frac{x}{\log x} - \frac{2x}{3\log^{2.5} x} + \int_{6\,400}^{x} \frac{dt}{\log^2 t} - \frac{2}{3}\int_{6\,400}^{x} \frac{dt}{\log^{3.5} t} \\ &= 103 + \frac{x}{\log x} - \frac{2x}{3\log^{2.5} x} + \frac{x}{\log^2 x} - \frac{6\,400}{\log^2 6\,400} \\ &\quad + 2\int_{6\,400}^{x} \frac{dt}{\log^3 t} - \frac{2}{3}\int_{6\,400}^{x} \frac{dt}{\log^{3.5} t} \\ &> \frac{x}{\log x}\left(1 + \frac{1}{\log x} - \frac{2}{3\log^{1.5} x}\right) > \frac{x}{\log x - 1 + (\log x)^{-0.5}}.\end{aligned}$$

The last inequality is equivalent to

$$2z^3 - 5z^2 + 3z - 1 < 0 \quad \text{where} \quad z = (\log x)^{-0.5} < 0.34.$$

Since $z(1-z) < 1/4$ it follows that $z(1-z)(3-2z) \le (3-z)/4 < 1$ so that the statement is proved for $x \ge 6\,400$. For $x < 6\,400$ we have to prove that

$$\alpha(x) = -\frac{x}{\pi(x)} + \log x - 1 + \frac{1}{\sqrt{\log x}} > 0.$$

On $[p_n, p_{n+1})$ the function is decreasing. The checking is made for the values $p_n - 1$. From $p_n - 1 \le 6\,399$ it follows that $(\log(p_n - 1))^{-0.5} > 0.337$ and therefore it suffices that

$$\frac{\log(p_n - 1)}{p_n - 1} - \frac{p_n - 1}{n - 1} > 0.663,$$

which holds for $n \ge 36$. Computer checking for $n < 36$ also gives that our inequality holds for $x \ge 59$. ∎

**Applications.** From the large list of inequalities involving the function $\pi(x)$ we recall

(13) $$\pi(2x) < 2\pi(x) \quad \text{for } x \ge 3,$$

suggested by E. Landau and proved by Rosser and Schoenfeld in [8].

If $a \ge e^{1/4}$ and $x \ge 364$ then

(14) $$\pi(ax) < a\pi(x),$$

as proved by C. Karanikolov in [3].

If $0 < \varepsilon \le 1$ and $\varepsilon x \le y \le x$ then

(15) $$\pi(x + y) < \pi(x) + \pi(y)$$

for $x$ and $y$ sufficiently large, as proved by V. Udrescu in [11].

Next, we prove two inequalities that strengthen the above results and make them more precise.

THEOREM 2. *If $a > 1$ and $x > e^{4(\log a)^{-2}}$ then $\pi(ax) < a\pi(x)$.*

Proof. We use inequalities (11) and (12). For $ax \ge 6$,

$$\pi(ax) < \frac{ax}{\log ax - 1 - (\log ax)^{-0.5}}.$$

For $x \ge 59$,

$$a\pi(x) > \frac{ax}{\log x - 1 + (\log x)^{-0.5}}.$$

It remains to show that

$$\log a > (\log ax)^{-0.5} + (\log x)^{-0.5}.$$

Since $x \ge e^{4(\log a)^{-2}}$ it follows that $\log x \ge 4(\log a)^{-2}$ and therefore

$$(\log ax)^{-0.5} + (\log x)^{-0.5} < \log a.$$

In addition, from $x > e^{4(\log a)^{-2}}$ we obtain $ax \ge 6$ too, and the proof is complete. ■

THEOREM 3. *If $a \in (0,1]$ and $x \ge y \ge ax$, $x \ge e^{9a^{-2}}$, then*

$$\pi(x+y) < \pi(x) + \pi(y).$$

Proof. Since $e^{9a^{-2}} > 59$, the inequalities (11) and (12) may be applied. It suffices to prove that

$$\frac{x+y}{\log(x+y) - 1 - (\log(x+y))^{-0.5}} < \frac{x}{\log x - 1 + (\log x)^{-0.5}} + \frac{y}{\log y - 1 + (\log y)^{-0.5}},$$

i.e.

$$\frac{x}{\log x - 1 + (\log x)^{-0.5}}\left(\log\left(1+\frac{y}{x}\right) - \log(x+y)^{-0.5} - (\log x)^{-0.5}\right) + \frac{y}{\log y - 1 + (\log y)^{-0.5}}\left(\log\left(1+\frac{x}{y}\right) - (\log(x+y))^{-0.5} - (\log y)^{-0.5}\right) > 0. \tag{16}$$

From $x \ge e^{9a^{-2}}$ it follows that $\log x > 9/a^2$, i.e.

$$(\log(x+y))^{-0.5} + (\log x)^{-0.5} < 2a/3.$$

We have the inequalities

$$\log\left(1+\frac{y}{x}\right) \ge \log(1+a) > \frac{2a}{2a+1} \ge \frac{2a}{3},$$

$$(\log(x+y))^{-0.5} < a/3, \qquad \log y \ge \log a + \log x \ge \log a + 9a^{-2} \ge 9,$$

i.e.

$$(\log(x+y))^{-0.5} + (\log y)^{-0.5} < \frac{a}{3} + \frac{1}{3} \le \frac{2}{3} < \log 2 \le \log\left(1+\frac{x}{y}\right).$$

Therefore, the inequality (16) holds, since both expressions in parentheses are positive. ■

REMARK. The inequalities (11) and (12) enable us to prove that $\pi(x+y) < \pi(x)+\pi(y)$ under less restrictive assumptions than in Theorem 3, but the amount of computation is much larger.

**Main result.** The Hardy–Littlewood inequality $\pi(x+y) \le \pi(x)+\pi(y)$ was proved in the last section under the very particular hypothesis $ax \le y \le x$. The only known result in which $x$ and $y$ are not imposed to satisfy such a hypothesis, but instead they are integers with $x \ge 2$, $y \ge 2$, was obtained by H. L. Montgomery and R. C. Vaughan [5]. They prove that

$$\pi(x+y) < \pi(x) + 2\pi(y),$$

using the large sieve.

In [1] and [2], the authors take into account the possibility that the general Hardy–Littlewood inequality might be false, and propose an alternative (evidently weaker) conjecture

$$\pi(x+y) \le 2\pi(x/2) + \pi(y).$$

Below, using inequalities (11) and (12), we prove the following

THEOREM 4. *If $x$ and $y$ are positive integers with $x \ge y \ge 2$ and $x \ge 6$, then*

$$\pi(x+y) \le 2\pi(x/2) + \pi(y). \tag{17}$$

Before giving the proof, we note that the method we use cannot be adapted to prove $\pi(x+y) < \pi(x)+\pi(y)$.

LEMMA 2. *If $x \ge y$ and $x \ge 7\,500$, $y \ge 2\,000$ then* (17) *holds.*

Proof. Taking into account inequalities (11) and (12) it follows that

$$\begin{aligned}
&2\pi(x/2) + \pi(y) - \pi(x+y)\\
&\quad > \frac{x\left(\log\left(1+\frac{y}{x}\right) + \log 2 - \frac{1}{\sqrt{\log(x/2)}} - \frac{1}{\sqrt{\log(x+y)}}\right)}{\left(\log(x/2) - 1 + \frac{1}{\sqrt{\log(x/2)}}\right)\left(\log(x+y) - 1 - \frac{1}{\sqrt{\log(x+y)}}\right)}\\
&\qquad + \frac{y\left(\log\left(1+\frac{x}{y}\right) - \frac{1}{\sqrt{\log y}} - \frac{1}{\sqrt{\log(x+y)}}\right)}{\left(\log y - 1 + \frac{1}{\sqrt{\log y}}\right)\left(\log(x+y) - 1 - \frac{1}{\sqrt{\log(x+y)}}\right)}.
\end{aligned}$$

The lemma follows using the inequalities

$$\frac{1}{\sqrt{\log y}}+\frac{1}{\sqrt{\log (x+y)}} \le \frac{1}{\sqrt{\log 2\,000}}+\frac{1}{\sqrt{\log 9\,500}} < \log 2 \le \log\left(1+\frac{x}{y}\right),$$

$$\frac{1}{\sqrt{\log(x/2)}}+\frac{1}{\sqrt{\log(x+y)}} \le \frac{1}{\sqrt{\log 3\,750}}+\frac{1}{\sqrt{\log 9\,500}} < \log 2. \ \blacksquare$$

LEMMA 3. *If* $x \ge 25\,000$, *then*

$$\pi(x+2\,000) < 2\pi(x/2). \tag{18}$$

Proof. Using again inequalities (11) and (12) we have

$$2\pi(x/2)-\pi(x+2\,000) > \frac{f(x)g(x)-2\,000}{\log(x+2\,000)-1-\dfrac{1}{\sqrt{\log(x+2\,000)}}}$$

where

$$f(x)=\frac{x}{\log(x/2)-1+\dfrac{1}{\sqrt{\log (x/2)}}}$$

and

$$g(x)=\log\left(2+\frac{4\,000}{x}\right)-\frac{1}{\sqrt{\log(x/2)}}-\frac{1}{\sqrt{\log(x+2\,000)}}.$$

For $x \ge 195\,000$,

$$g(x) > \log 2-\frac{1}{\sqrt{\log 97\,500}}-\frac{1}{\sqrt{\log 197\,000}} > 0.1116$$

and

$$f(x) > f(195\,000) > 18084.6;$$

then $f(x)g(x) > 2\,000$, therefore $\pi(x+2\,000) < 2\pi(x/2)$.

Computer check for prime $x+2\,000$ and $x < 195\,000$ shows that the inequality (18) holds for $x \ge 25\,000$. ■

*Proof of Theorem 4.* By Lemma 3 it follows that the inequality (17) holds for $x \ge 25\,000$ and $y < 2\,000$. By Lemma 3 it also holds for positive integers $x$ and $y$ satisfying $x \ge 25\,000$.

Computer check for the cases $y \le x < 25\,000$ completes the proof of the theorem. ■

REMARK. Because $\pi(y) \le 2\pi(y/2)$ for $y \ge 6$, after some easy computations using the former theorem we obtain the statement:

If $x$ and $y$ are positive integers with $x, y \ge 4$ then

$$\pi(x+y) \le 2(\pi(x/2)+\pi(y/2)).$$

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