Inequalities concerning the function $\pi(x)$: Applications

by

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Introduction. In this note we use the following standard notations: $\pi(x)$ is the number of primes not exceeding $x$, while $\theta(x) = \sum_{p \leq x} \log p$.

The best known inequalities involving the function $\pi(x)$ are the ones obtained in [6] by B. Rosser and L. Schoenfeld:

\begin{align*}
(1) & \quad \frac{x}{\log x - 1/2} < \pi(x) \quad \text{for } x \geq 67, \\
(2) & \quad \frac{x}{\log x - 3/2} > \pi(x) \quad \text{for } x > e^{3/2}.
\end{align*}

The proof of the above inequalities is not elementary and is based on the first 25,000 zeros of the Riemann function $\xi(s)$ obtained by D. H. Lehmer [4]. Then Rosser, Yohe and Schoenfeld announced that the first 3,500,000 zeros of $\xi(s)$ lie on the critical line [9]. This result was followed by two papers [7], [10]; some of the inequalities they include will be used in order to obtain inequalities (11) and (12) below.

In [6] it is proved that $\pi(x) \sim x/(\log x - 1)$. Here we will refine this expression by giving upper and lower bounds for $\pi(x)$ which both behave as $x/(\log x - 1)$ as $x \to \infty$.

New inequalities. We start by listing those inequalities in [6] and [10] that will be used further:

\begin{align*}
(3) & \quad \theta(x) < x \quad \text{for } x < 10^8, \\
(4) & \quad |\theta(x) - x| < 2.05282\sqrt{x} \quad \text{for } x < 10^8, \\
(5) & \quad |\theta(x) - x| < 0.0239922\frac{x}{\log x} \quad \text{for } x \geq 758,711, \\
(6) & \quad |\theta(x) - x| < 0.0077629\frac{x}{\log x} \quad \text{for } x \geq e^{22},
\end{align*}

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(7) \[ |\theta(x) - x| < 8.072 - \frac{x}{\log^2 x} \quad \text{for } x > 1. \]

The above inequalities are used first to prove the following lemma:

**Lemma 1.** We have

(8) \[ \theta(x) < x \left(1 + \frac{1}{3(\log x)^{1.5}}\right) \quad \text{for } x > 1, \]

(9) \[ \theta(x) > x \left(1 - \frac{2}{3(\log x)^{1.5}}\right) \quad \text{for } x \geq 6400. \]

**Proof.** For \( x \geq e^{587} \) the inequality

\[ 8.072 < \frac{1}{3(\log x)^{0.5}} \]

holds and therefore, using (7), it follows that

(10) \[ |\theta(x) - x| < \frac{x}{3(\log x)^{1.5}}. \]

For \( e^{22} \leq x < e^{587} \) we have

\[ 0.0077629 < \frac{1}{3(\log x)^{0.5}} \]

and by using (6) we obtain (10). For \( 757711 \leq x < e^{22} \) we have

\[ 0.0239922 < \frac{1}{3(\log x)^{0.5}} \]

and by using (5) we obtain again (10) for \( x \geq 757711 \). These results, together with inequality (3), obviously imply (8).

Let \( 6400 \leq x < 10^8 \). Then

\[ 2.05282 < \frac{2}{3} \cdot \frac{\sqrt{x}}{(\log x)^{1.5}} \]

which implies (9) by using (4) and (10).

Lemma 1 helps us to prove

**Theorem 1.** We have

(11) \[ \pi(x) < \frac{x}{\log x - 1 - (\log x)^{-0.5}} \quad \text{for } x \geq 6, \]

(12) \[ \pi(x) > \frac{x}{\log x - 1 + (\log x)^{-0.5}} \quad \text{for } x \geq 59. \]

**Proof.** We use the well-known identity

\[ \pi(x) = \frac{\theta(x)}{\log x} + \frac{x}{2} \int \frac{\theta(t)}{t \log^2 t} \, dt. \]
By (8) we obtain
\[
\pi(x) < \frac{x}{\log x} + \frac{x}{3(\log x)^{2.5}} + \frac{1}{3} \int_2^x \frac{dt}{\log^2 t} + \frac{1}{3} \int_2^x \frac{dt}{(\log t)^{3.5}}
\]
\[
= \frac{x}{\log x} \left( 1 + \frac{1}{3(\log x)^{1.5}} + \frac{1}{\log x} \right) - \frac{2}{\log^2 2} + \frac{7}{3} \int_2^x \frac{dt}{\log^3 t} + \frac{1}{3} \int_2^x \frac{dt}{(\log t)^{3.5}}.
\]
Since
\[
- \frac{2}{\log^2 2} + \frac{1}{3} \int_2^x \frac{dt}{(\log t)^{3.5}} < \frac{1}{3} \int_2^x \frac{dt}{\log^3 t}
\]
it follows that
\[
\pi(x) < \frac{x}{\log x} \left( 1 + \frac{1}{3(\log x)^{1.5}} + \frac{1}{\log x} \right) + \frac{7}{3} \int_2^x \frac{dt}{\log^3 t}.
\]
For \(x \geq e^{18.25}\) we define
\[
f(x) = \frac{2}{3} \cdot \frac{x}{(\log x)^{2.5}} - \frac{7}{3} \int_2^x \frac{dt}{\log^3 t}.
\]
Then
\[
f'(x) = \frac{2 \log x - 7(\log x)^{0.5} - 5}{3(\log x)^{3.5}} > 0,
\]
which implies that \(f\) is an increasing function. For any convex function \(g : [a, b] \to \mathbb{R}\) we have
\[
\int_a^b g(x) \, dx \leq b - a \left( g(a) + g(b) + \frac{n-1}{n} \sum_{k=1}^{n-1} g\left( a + k \frac{b-a}{n} \right) \right).
\]
For \(g(x) = 1/\log^3 x\) and \(n = 10^5\), we can apply the above inequality on each interval \([2, e], [e, e^2], \ldots, [e^{17}, e^{18}], \text{ and } [e^{18}, e^{18.25}]\) to get
\[
\int_2^{e^{18.25}} \frac{dt}{\log^3 t} < 16,870.
\]
As the referee kindly pointed out, the above inequality may also be checked using the software package Mathematica.

We have
\[
f(e^{18.25}) > \frac{1}{3}(118,507 - 118,090) > 0.
\]
Therefore \(f(x) > 0\), which implies that for \(x \geq e^{18.25}\),
\[
\pi(x) < \frac{x}{\log x} \left( 1 + \frac{1}{\log x} + \frac{1}{(\log x)^{1.5}} \right) < \frac{x}{\log x - 1 - (\log x)^{-0.5}}.
\]
Let now $x \leq e^{18.25} < 10^8$. By using (3) we obtain

$$\pi(x) = \frac{\theta(x)}{\log x} + \frac{x}{2} \frac{\theta(t)}{\log^2 t} dt < \frac{x}{\log x} + \frac{x}{2} \frac{dt}{\log^2 t}$$

$$= \frac{x}{\log x} \left( 1 + \frac{1}{\log x} \right) - \frac{2}{\log^2 2} + 2 \frac{x}{2} \frac{dt}{\log^3 t}. $$

For $4000 \leq x < 10^8$ define

$$g(x) = \frac{x}{(\log x)^{2.5}} - 2 \frac{x}{2} \frac{dt}{\log^3 t} + \frac{2}{\log^2 2}. $$

Since

$$g'(x) = \frac{\log x - 2(\log x)^{0.5} - 2.5}{(\log x)^{3.5}} > 0,$$

$g$ is an increasing function,

$$g(e^{11}) > 149 - 2 \int_2^{e^{11}} \frac{dt}{\log^3 t} > 149 - 140 > 0,$$

hence for $e^{11} \leq x < 10^8$ we have

$$\pi(x) < \frac{x}{\log x} \left( 1 + \frac{1}{\log x} + \frac{1}{(\log x)^{1.5}} \right) < \frac{x}{\log x - 1 - (\log x)^{-0.5}}. $$

For $x \geq 6$ it follows immediately that $\log x - 1 - (\log x)^{-0.5} > 0$. Hence, for $6 \leq x \leq e^{11}$, the inequality to be proved is

$$h(x) = \frac{x}{\pi(x)} + 1 + (\log x)^{-0.5} - \log x > 0. $$

If $p_n$ is the $n$th prime, then $h$ is an increasing function in $[p_n, p_{n+1})$, so it suffices to prove that $h(p_n) > 0$. Since $p_n < e^{11}$, the inequality $(\log p_n)^{-0.5} > 0.3$ holds and therefore it suffices to prove that $p_n/n - \log p_n > -1.3$, which may be verified by computer for $e^{11} > p_n \geq 7$.

In order to prove inequality (12) we use (3), (9) and for $x \geq 6400$ we have

$$\pi(x) - \pi(6400) = \frac{\theta(x)}{\log x} - \frac{\theta(6400)}{\log 6400} + \frac{x}{6400} \frac{\theta(t)}{t \log^2 t} dt.$$

Since $\pi(6400) = 834$ and $\theta(6400)/\log 6400 < 6400/\log 6400 < 731$ we have

$$\pi(x) > 103 + \frac{\theta(x)}{x} + \frac{x}{6400} \frac{\theta(t)}{t \log^2 t} dt.$$
From (9) it follows that
\[
\pi(x) > 103 + \frac{x}{\log x} - \frac{2x}{3 \log^{2.5} x} + \int_{6400}^{x} \frac{dt}{\log^2 t} - \frac{2}{3} \int_{6400}^{x} \frac{dt}{\log^{3.5} t}
\]
\[
= 103 + \frac{x}{\log x} - \frac{2x}{3 \log^{2.5} x} + \int_{6400}^{x} \frac{dt}{\log^2 x} - \frac{2}{3} \int_{6400}^{x} \frac{dt}{\log^{3.5} x} + 2 \int_{6400}^{x} \frac{dt}{\log^{3.5} t} - \frac{2}{3} \int_{6400}^{x} \frac{dt}{\log^{3.5} x} \geq x \log x \left(1 + 1 - \frac{2}{3 \log^{1.5} x}\right) \geq \log x - 1 + (\log x)^{-0.5}.
\]

The last inequality is equivalent to
\[
2z^3 - 5z^2 + 3z - 1 < 0 \quad \text{where} \quad z = (\log x)^{-0.5} < 0.34.
\]

Since \( z(1 - z) < 1/4 \) it follows that \( z(1 - z)(3 - 2z) \leq (3 - z)/4 < 1 \) so that the statement is proved for \( x \geq 6400 \). For \( x < 6400 \) we have to prove that
\[
\alpha (x) = -\frac{x}{\pi(x)} + \log x - 1 + \frac{1}{\sqrt{\log x}} > 0.
\]

On \([p_n, p_{n+1})\) the function is decreasing. The checking is made for the values \( p_n - 1 \). From \( p_n - 1 \leq 6399 \) it follows that \( (\log(p_n - 1))^{-0.5} > 0.337 \) and therefore it suffices that
\[
\frac{\log(p_n - 1)}{p_n - 1} - \frac{p_n - 1}{n - 1} > 0.663,
\]
which holds for \( n \geq 36 \). Computer checking for \( n < 36 \) also gives that our inequality holds for \( x \geq 59 \). \( \blacksquare \)

**Applications.** From the large list of inequalities involving the function \( \pi(x) \) we recall
\[
(13) \quad \pi(2x) < 2\pi(x) \quad \text{for} \quad x \geq 3,
\]
suggested by E. Landau and proved by Rosser and Schoenfeld in [8].

If \( a \geq e^{1/4} \) and \( x \geq 364 \) then
\[
(14) \quad \pi(ax) < a\pi(x),
\]
as proved by C. Karanikolov in [3].

If \( 0 < \varepsilon \leq 1 \) and \( \varepsilon x \leq y \leq x \) then
\[
(15) \quad \pi(x + y) < \pi(x) + \pi(y)
\]
for \( x \) and \( y \) sufficiently large, as proved by V. Udrescu in [11].

Next, we prove two inequalities that strengthen the above results and make them more precise.
Theorem 2. If $a > 1$ and $x > e^{4(\log a)^{-2}}$ then $\pi(ax) < ap(x)$.

**Proof.** We use inequalities (11) and (12). For $ax \geq 6$,

$$\pi(ax) < \frac{ax}{\log ax - 1 - (\log ax)^{-0.5}}.$$  

For $x \geq 59$,

$$a\pi(x) > \frac{ax}{\log x - 1 + (\log x)^{-0.5}}.$$  

It remains to show that

$$\log a > (\log ax)^{-0.5} + (\log x)^{-0.5}.$$  

Since $x \geq e^{4(\log a)^{-2}}$ it follows that $\log x \geq 4(\log a)^{-2}$ and therefore

$$(\log ax)^{-0.5} + (\log x)^{-0.5} < \log a.$$  

In addition, from $x > e^{4(\log a)^{-2}}$ we obtain $ax \geq 6$ too, and the proof is complete.  

**Theorem 3.** If $a \in (0, 1]$ and $x \geq y \geq ax$, $x \geq e^{9a^{-2}}$, then

$$\pi(x + y) < \pi(x) + \pi(y).$$  

**Proof.** Since $e^{9a^{-2}} > 59$, the inequalities (11) and (12) may be applied. It suffices to prove that

$$\frac{x + y}{\log(x + y) - 1 - (\log(x + y))^{-0.5}} < \frac{x}{\log x - 1 + (\log x)^{-0.5}} + \frac{y}{\log y - 1 + (\log y)^{-0.5}},$$  

i.e.

$$(16) \quad \frac{x}{\log x - 1 + (\log x)^{-0.5}} \left( \log \left(1 + \frac{y}{x}\right) - \log(x + y)^{-0.5} - (\log x)^{-0.5} \right) \right) + \frac{y}{\log y - 1 + (\log y)^{-0.5}} \left( \log \left(1 + \frac{x}{y}\right) - (\log(x + y))^{-0.5} - (\log y)^{-0.5} \right) > 0.$$  

From $x \geq e^{9a^{-2}}$ it follows that $\log x > 9/a^2$, i.e.

$$(\log(x + y))^{-0.5} + (\log x)^{-0.5} < 2a/3.$$  

We have the inequalities

$$\log \left(1 + \frac{y}{x}\right) \geq \log(1 + a) > \frac{2a}{2a + 1} \geq \frac{2a}{3},$$  

$$\log(x + y)^{-0.5} < a/3, \quad \log y \geq \log a + \log x \geq \log a + 9a^{-2} \geq 9,$$  

i.e.

$$(\log(x + y))^{-0.5} + (\log y)^{-0.5} < \frac{a}{3} + \frac{1}{3} \leq \frac{2}{3} < \log 2 \leq \log \left(1 + \frac{x}{y}\right).$$
Therefore, the inequality (16) holds, since both expressions in parentheses are positive.

**Remark.** The inequalities (11) and (12) enable us to prove that $\pi(x+y) < \pi(x) + \pi(y)$ under less restrictive assumptions than in Theorem 3, but the amount of computation is much larger.

**Main result.** The Hardy–Littlewood inequality $\pi(x+y) \leq \pi(x) + \pi(y)$ was proved in the last section under the very particular hypothesis $ax \leq y \leq x$. The only known result in which $x$ and $y$ are not imposed to satisfy such a hypothesis, but instead they are integers with $x \geq 2$, $y \geq 2$, was obtained by H. L. Montgomery and R. C. Vaughan [5]. They prove that
\[
\pi(x+y) < \pi(x) + 2\pi(y),
\]
using the large sieve.

In [1] and [2], the authors take into account the possibility that the general Hardy–Littlewood inequality might be false, and propose an alternative (evidently weaker) conjecture
\[
\pi(x+y) \leq 2\pi(x/2) + \pi(y).
\]

Below, using inequalities (11) and (12), we prove the following

**Theorem 4.** If $x$ and $y$ are positive integers with $x \geq y \geq 2$ and $x \geq 6$, then
\[
(17) \quad \pi(x+y) \leq 2\pi(x/2) + \pi(y).
\]

Before giving the proof, we note that the method we use cannot be adapted to prove $\pi(x+y) < \pi(x) + \pi(y)$.

**Lemma 2.** If $x \geq y$ and $x \geq 7500$, $y \geq 2000$ then (17) holds.

**Proof.** Taking into account inequalities (11) and (12) it follows that
\[
2\pi(x/2) + \pi(y) - \pi(x+y)
\]

\[
> \frac{x \left(\log \left(1 + \frac{y}{x}\right) + \log 2 - \frac{1}{\sqrt{\log(x/2)}} - \frac{1}{\sqrt{\log(x+y)}}\right)}{\left(\log (x/2) - 1 + \frac{1}{\sqrt{\log(x/2)}}\right)\left(\log(x+y) - 1 - \frac{1}{\sqrt{\log(x+y)}}\right)}
\]

\[
+ \frac{y \left(\log \left(1 + \frac{x}{y}\right) - \frac{1}{\sqrt{\log y}} - \frac{1}{\sqrt{\log(x+y)}}\right)}{\left(\log y - 1 + \frac{1}{\sqrt{\log y}}\right)\left(\log(x+y) - 1 - \frac{1}{\sqrt{\log(x+y)}}\right)}.
\]
The lemma follows using the inequalities
\[
\frac{1}{\sqrt{\log y}} + \frac{1}{\sqrt{\log (x + y)}} \leq \frac{1}{\sqrt{\log 2000}} + \frac{1}{\sqrt{\log 9500}} < \log 2 \leq \log \left(1 + \frac{x}{y}\right),
\]
\[
\frac{1}{\sqrt{\log(x/2)}} + \frac{1}{\sqrt{\log (x + y)}} \leq \frac{1}{\sqrt{\log 3750}} + \frac{1}{\sqrt{\log 9500}} < \log 2.
\]

**Lemma 3.** If \(x \geq 25000\), then
\[
(18) \quad \pi(x + 2000) < 2\pi(x/2).
\]

**Proof.** Using again inequalities (11) and (12) we have
\[
2\pi(x/2) - \pi(x + 2000) > \frac{f(x)g(x) - 2000}{\log(x + 2000) - 1 - \frac{1}{\sqrt{\log(x + 2000)}}}
\]
where
\[
f(x) = \frac{x}{\log(x/2) - 1 + \frac{1}{\sqrt{\log(x/2)}}}
\]
and
\[
g(x) = \log \left(2 + \frac{4000}{x}\right) - \frac{1}{\sqrt{\log(x/2)}} - \frac{1}{\sqrt{\log(x + 2000)}}
\]
For \(x \geq 195000\),
\[
g(x) > \log 2 - \frac{1}{\sqrt{\log 97500}} - \frac{1}{\sqrt{\log 197000}} > 0.1116
\]
and
\[
f(x) > f(195000) > 18084.6;
\]
then \(f(x)g(x) > 2000\), therefore \(\pi(x + 2000) < 2\pi(x/2)\).

Computer check for prime \(x + 2000\) and \(x < 195000\) shows that the inequality (18) holds for \(x \geq 25000\).

**Proof of Theorem 4.** By Lemma 3 it follows that the inequality (17) holds for \(x \geq 25000\) and \(y < 2000\). By Lemma 3 it also holds for positive integers \(x\) and \(y\) satisfying \(x \geq 25000\).

Computer check for the cases \(y \leq x < 25000\) completes the proof of the theorem.

**Remark.** Because \(\pi(y) \leq 2\pi(y/2)\) for \(y \geq 6\), after some easy computations using the former theorem we obtain the statement:

If \(x\) and \(y\) are positive integers with \(x, y \geq 4\) then
\[
\pi(x + y) \leq 2(\pi(x/2) + \pi(y/2)).
\]

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References

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