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Comparing the distribution of $(n\alpha)$-sequences

by

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1. Introduction. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with continued fraction expansion $\alpha = [a_0, a_1, a_2, \ldots]$ (where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for all $i \geq 1$) and convergents $p_n/q_n = [a_0, a_1, \ldots, a_n]$. (Sometimes we write $a_n(\alpha)$ and $p_n(\alpha)/q_n(\alpha)$ to stress the dependence on α.) It is a classic result of P. Bohl [5], W. Sierpiński [12], [13] and H. Weyl [14], [15] that the sequence $(n\alpha)_{n\geq 1}$ is uniformly distributed modulo 1. This property is studied from a quantitative viewpoint by means of the speed of convergence in the limit relations $\lim_{N\to\infty} D_N^*(\alpha) = 0$ and $\lim_{N\to\infty} D_N(\alpha) = 0$ where the quantities

$$D_N^*(\alpha) = \sup_{0\leq x\leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} c_{[0,x)}(\{n\alpha\}) - x \right|$$

and

$$D_N(\alpha) = \sup_{0\leq x<y\leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} c_{[x,y)}(\{n\alpha\}) - (y-x) \right|$$

are called *discrepancies*. According to a theorem of W. M. Schmidt [10] the convergence is best possible if $D_N^*(\alpha) = O((\log N)/N)$ or equivalently $D_N(\alpha) = O((\log N)/N)$. It was first observed by H. Behnke [4] that this estimate is satisfied if and only if α is *of bounded density*, i.e. $\sum_{i=1}^{m} a_i = O(m)$ as $m \to \infty$. For α of bounded density the maps

$$\alpha \mapsto \nu^*(\alpha) = \limsup_{N\to\infty} ND_N^*(\alpha)/\log N$$

and

$$\alpha \mapsto \nu(\alpha) = \limsup_{N\to\infty} ND_N(\alpha)/\log N$$

are used to obtain more detailed information. The map ν^* has been studied in a number of papers. Y. Dupain and V. T. Sós [6] proved that $\inf \nu^*(B) =$

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$\nu^*(\sqrt{2})$ where B denotes the set of numbers of bounded density. J. Schoißengeier [11] expressed $\nu^*(\alpha)$ in terms of the continued fraction expansion of α. Employing these results C. Baxa [1], [2] proved the following:

(1) Let $B^{\mathrm{q}} := \{\alpha \in B \mid \alpha \text{ is a quadratic irrationality}\}$, $B^{\mathrm{t}} := \{\alpha \in B \mid \alpha \text{ is transcendental}\}$ and $B^{\mathrm{u}} := \{\alpha \in B \mid \alpha \text{ is a } U_2\text{-number}\}$. Then $\nu^*(B) = \overline{\nu^*(B^{\mathrm{q}})} = \nu^*(B^{\mathrm{t}}) = \nu^*(B^{\mathrm{u}}) = [\nu^*(\sqrt{2}), \infty)$.

(2) Let $b \geq 4$ be an even integer, $B_b := \{\alpha = [a_0, a_1, a_2, \ldots] \in B \mid a_i \geq b \text{ for all } i \geq 1\}$, $B_b^{\mathrm{q}} := \{\alpha \in B_b \mid \alpha \text{ is a quadratic irrationality}\}$, $B_b^{\mathrm{t}} := \{\alpha \in B_b \mid \alpha \text{ is transcendental}\}$ and $B_b^{\mathrm{u}} := \{\alpha \in B_b \mid \alpha \text{ is a } U_2\text{-number}\}$. Then $\nu^*(B_b) = \overline{\nu^*(B_b^{\mathrm{q}})} = \nu^*(B_b^{\mathrm{t}}) = \nu^*(B_b^{\mathrm{u}}) = [\nu^*([\overline{b}]), \infty)$ where $[\overline{b}] = [b, b, b, \ldots] = (b + \sqrt{b^2 + 4})/2$ is used as a convenient shorthand notation.

It is the purpose of the present paper to present analogous results for the map ν, more precisely we prove the following

Theorem. (1) *Let b be a positive integer. Then each of the sets $\nu(B_b)$, $\overline{\nu(B_b^{\mathrm{q}})}$, $\nu(B_b^{\mathrm{t}})$ and $\nu(B_b^{\mathrm{u}})$ contains $[\nu([\overline{b}]), \infty)$.*

(2) *If b is an even positive integer then $\nu(B_b) = \overline{\nu(B_b^{\mathrm{q}})} = \nu(B_b^{\mathrm{t}}) = \nu(B_b^{\mathrm{u}}) = [\nu([\overline{b}]), \infty)$.*

Remark. It would be desirable to prove part (2) for all b. The case $b = 1$ contains the equation $\inf \nu(B) = \nu((1 + \sqrt{5})/2)$ which is described as established in E. Hlawka's textbook [7] but it seems that there is no published proof. However, the inequality $\nu([\overline{1}]) = \nu((1 + \sqrt{5})/2) < \nu(\sqrt{2}) = \nu([1, \overline{2}])$ supports the rule of thumb that bigger partial quotients lead to bigger discrepancies, which is an interesting contrast to the fact that $\nu^*(\sqrt{2}) < \nu^*((1 + \sqrt{5})/2)$.

2. Proof of part (1). Our starting point is the following

Theorem 1. *Let α be a number of bounded density. Then*

$$\nu(\alpha) = \frac{1}{4} \limsup_{m \to \infty} \frac{1}{\log q_{m+1}} \Big(\sum_{i=0}^{m} a_{i+1} - \sum_{\substack{0 \leq i,j \leq m \\ i \equiv j \,(\mathrm{mod}\, 2)}} \varepsilon_i \varepsilon_j |s_{ij}| \Big)$$

where $s_{ij} := q_{\min(i,j)}(q_{\max(i,j)}\alpha - p_{\max(i,j)})$ and

$$\varepsilon_i := \frac{1}{2}(1 - (-1)^{a_{i+1}}) \prod_{\substack{0 \leq j \leq i \\ j \equiv i \,(\mathrm{mod}\, 2)}} (-1)^{a_{j+1}}$$

for $i, j \geq 0$.

Proof. This is a slight modification of Theorem 2.1 in [3] where we used the fact that $\lim_{m \to \infty} \log q_{m+1} / \log q_m = 1$ holds for numbers of bounded density.

REMARKS. (1) From now on we will assume without loss of generality that $\alpha \in (0,1)\backslash\mathbb{Q}$.

(2) We will repeatedly use the continuants

$$K_n(a_1,\ldots,a_n) := \begin{vmatrix} a_1 & 1 & 0 & & & \ldots & 0 \\ -1 & a_2 & 1 & 0 & & \ldots & 0 \\ 0 & -1 & a_3 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \ldots & & 0 & -1 & a_{n-1} & 1 \\ 0 & \ldots & & & 0 & -1 & a_n \end{vmatrix}.$$

(Furthermore, it is convenient to define $K_0 := 1$ and $K_{-1} := 0$.) They are connected to continued fractions by the fact that $q_m = K_m(a_1,\ldots,a_m)$ if $\alpha = [a_0,a_1,\ldots]$. Their basic property is the equation

$$\begin{aligned} K_n(a_1,\ldots,a_n) &= K_m(a_1,\ldots,a_m)K_{n-m}(a_{m+1},\ldots,a_n) \\ &\quad + K_{m-1}(a_1,\ldots,a_{m-1})K_{n-m-1}(a_{m+2},\ldots,a_n) \end{aligned}$$

for $0 \le m \le n$. This is a more convenient way to write O. Perron's "Fundamentalformeln" [8] and turns into the recursion relation for the denominators of convergents if $n = m+1$. We will drop the index and write $K(a_1,\ldots,a_n)$, which should not lead to confusion.

LEMMA 2. *Let $M \le N$ and $t \in \{0,1\}$. Then*

$$\sum_{\substack{0\le i,j\le N \\ i\equiv j\equiv t \,(\mathrm{mod}\, 2)}} \varepsilon_i\varepsilon_j s_{ij} = \sum_{\substack{0\le i,j\le M \\ i\equiv j\equiv t \,(\mathrm{mod}\, 2)}} \varepsilon_i\varepsilon_j s_{ij} + O(N-M)$$

with an absolute implied constant.

Proof. By induction one sees that $K(a_1,\ldots,a_n) \ge 2^{(n-1)/2}$ for $n \ge 0$. Since $|s_{jj}| = (a_{j+1} + [0,a_j,\ldots,a_1] + [0,a_{j+2},\ldots])^{-1} < 1$ and $|q_j\alpha - p_j| < 1/q_{j+1}$ for $j \ge 0$ we get

$$\Big|\sum_{\substack{M<j\le N \\ j\equiv t \,(\mathrm{mod}\, 2)}} \varepsilon_j^2 s_{jj}\Big| \le \sum_{\substack{M<j\le N \\ j\equiv t \,(\mathrm{mod}\, 2)}} 1 \le N-M$$

and

$$\begin{aligned} \Big|\sum_{\substack{M<j\le N \\ 0\le i<j \\ i\equiv j\equiv t \,(\mathrm{mod}\, 2)}} \varepsilon_i\varepsilon_j s_{ij}\Big| &\le \sum_{\substack{M<j\le N \\ 0\le i<j \\ i\equiv j\equiv t \,(\mathrm{mod}\, 2)}} |s_{ij}| \le \sum_{\substack{M<j\le N \\ 0\le i<j}} \frac{q_i}{q_{j+1}} \\ &\le \sum_{\substack{M<j\le N \\ 0\le i<j}} \frac{1}{K(a_{i+1},\ldots,a_{j+1})} \le \sum_{\substack{M<j\le N \\ 0\le i<j}} 2^{-(j-i)/2} \\ &\ll N-M. \end{aligned}$$

Lemma 3. *Let $M \le N$ and $t \in \{0,1\}$. Then*

$$\sum_{\substack{0\le i,j\le N\\ i\equiv j\equiv t \ (\mathrm{mod}\ 2)}} \varepsilon_i\varepsilon_j s_{ij} = \sum_{\substack{0\le i,j\le M\\ i\equiv j\equiv t \ (\mathrm{mod}\ 2)}} \varepsilon_i\varepsilon_j s_{ij} + \sum_{\substack{M< i,j\le N\\ i\equiv j\equiv t \ (\mathrm{mod}\ 2)}} \varepsilon_i\varepsilon_j s_{ij} + O(1)$$

with an absolute implied constant.

Proof. Analogously to the above

$$\Bigg|\sum_{\substack{0\le i\le M\\ M<j\le N\\ i\equiv j\equiv t \ (\mathrm{mod}\ 2)}} \varepsilon_i\varepsilon_j s_{ij}\Bigg| \le \sum_{\substack{0\le i\le M\\ M<j\le N}} 2^{-(j-i)/2} = O(1).$$

Lemma 4. *Let $\alpha = [0,a_1,\ldots,a_M,\ldots,a_N,\ldots]$ and $\beta = [0,b_1,\ldots,b_{M'},\ldots,b_{N'},\ldots]$ with $0 \le M \le N$, $0 \le M' \le N'$, $M \equiv M' \pmod 2$ and $a_{M+j} = b_{M'+j}$ for $1 \le j \le N-M = N'-M'$. Then*

(1) $$|s_{M+j,M+j}(\alpha) - s_{M'+j,M'+j}(\beta)| \le \left(\frac{1}{2}\right)^{j-1} + \left(\frac{1}{2}\right)^{N-M-j-2} \quad \textit{for } 0 \le j < N-M.$$

(2) $$\left|\frac{q_{M+i}(\alpha)}{q_{M+j}(\alpha)} - \frac{q_{M'+i}(\beta)}{q_{M'+j}(\beta)}\right| \le 3\left(\frac{2}{3}\right)^{(j-i)/2}\left(\frac{1}{2}\right)^{i} \quad \textit{for } 0 \le i < j < N-M,\ i \equiv j \pmod 2.$$

(3) $$|s_{M+i,M+j}(\alpha) - s_{M'+i,M'+j}(\beta)| \le \left(\frac{1}{2}\right)^{(j-i-1)/2}\left(\left(\frac{1}{2}\right)^{j-1} + \left(\frac{1}{2}\right)^{N-M-j-2}\right) + 3\left(\frac{2}{3}\right)^{(j-i)/2}\left(\frac{1}{2}\right)^{i}$$
$$\textit{for } 0 \le i < j < N-M \textit{ and } i \equiv j \pmod 2.$$

(4) *Let $t \in \{0,1\}$. Then*

$$\sum_{\substack{0\le i,j< N-M\\ i\equiv j\equiv t \ (\mathrm{mod}\ 2)}} \varepsilon_{M+i}(\alpha)\varepsilon_{M+j}(\alpha)s_{M+i,M+j}(\alpha) = \sum_{\substack{0\le i,j< N'-M'\\ i\equiv j\equiv t \ (\mathrm{mod}\ 2)}} \varepsilon_{M'+i}(\beta)\varepsilon_{M'+j}(\beta)s_{M'+i,M'+j}(\beta) + O(1)$$

with an absolute implied constant.

Proof. (1) Since

$$\begin{aligned}|s_{M+j,M+j}(\alpha) - s_{M'+j,M'+j}(\beta)| &= |(a_{M+j+1} + [0,a_{M+j},\ldots,a_1] + [0,a_{M+j+2},\ldots])^{-1} \\ &\quad - (b_{M'+j+1} + [0,b_{M'+j},\ldots,b_1] + [0,b_{M'+j+2},\ldots])^{-1}|\end{aligned}$$

the assertion follows from

$$\begin{aligned}|[0,b_{M'+j},\dots,b_1]-[a_{M+j},\dots,a_1]| &= |[0,a_{M+j},\dots,a_{M+1},b_{M'},\dots,b_1]\\ &\qquad -[0,a_{M+j},\dots,a_{M+1},a_M,\dots,a_1]|\\ &\le 2^{-(j-1)}\end{aligned}$$

and

$$|[0,b_{M'+j+2},b_{M'+j+3},\dots]-[0,a_{M+j+2},a_{M+j+3},\dots]|\le 2^{-(N-M-j-2)}.$$

(2) We consider $[0,c_1,\dots,c_n]$ with $n\ge 2$. If $c_1>1$ or $c_1=c_2=1$ then $[0,c_1,\dots,c_n]<[0,1,2]=2/3$. If $c_1=1$ and $c_2>1$ then $[0,c_2,\dots,c_n]<[0,1,2]$. Using this we find

$$\begin{aligned}&\left|\frac{q_{M+i}(\alpha)}{q_{M+j}(\alpha)}-\frac{q_{M'+i}(\beta)}{q_{M'+j}(\beta)}\right|\\ &\quad=\left|\prod_{i<\kappa\le j}\frac{q_{M+\kappa-1}(\alpha)}{q_{M+\kappa}(\alpha)}-\prod_{i<\kappa\le j}\frac{q_{M'+\kappa-1}(\beta)}{q_{M'+\kappa}(\beta)}\right|\\ &\quad=\left|\prod_{i<\kappa\le j}[0,a_{M+\kappa},\dots,a_1]-\prod_{i<\kappa\le j}[0,b_{M'+\kappa},\dots,b_1]\right|\\ &\quad\le\sum_{i<\mu\le j}\Big(\prod_{i<\kappa<\mu}[0,b_{M'+\kappa},\dots,b_1]\Big)\\ &\qquad\quad\times|[0,a_{M+\mu},\dots,a_1]-[0,b_{M'+\mu},\dots,b_1]|\Big(\prod_{\mu<\kappa\le j}[0,a_{M+\kappa},\dots,a_1]\Big)\\ &\quad\le\sum_{i<\mu\le j}\left(\frac{2}{3}\right)^{(j-i-2)/2}|[0,a_{M+\mu},\dots,a_1]-[0,b_{M'+\mu},\dots,b_1]|\\ &\quad\le\left(\frac{2}{3}\right)^{(j-i-2)/2}\sum_{i\le\mu<j}\left(\frac{1}{2}\right)^{\mu}=2\left(\frac{2}{3}\right)^{(j-i-2)/2}\left(\left(\frac{1}{2}\right)^{i}-\left(\frac{1}{2}\right)^{j}\right)\\ &\quad\le 3\left(\frac{2}{3}\right)^{(j-i)/2}\left(\frac{1}{2}\right)^{i}.\end{aligned}$$

(3) Since

$$\begin{aligned}&|s_{M+i,M+j}(\alpha)-s_{M'+i,M'+j}(\beta)|\\ &\qquad\le\frac{q_{M+i}(\alpha)}{q_{M+j}(\alpha)}|s_{M+j,M+j}(\alpha)-s_{M'+j,M'+j}(\beta)|\\ &\qquad\quad+\left|\frac{q_{M+i}(\alpha)}{q_{M+j}(\alpha)}-\frac{q_{M'+i}(\beta)}{q_{M'+j}(\beta)}\right|\cdot|s_{M'+j,M'+j}(\beta)|\end{aligned}$$

the assertion follows from (1) and (2) using $q_{M+i}(\alpha)/q_{M+j}(\alpha)\le 2^{-(j-i-1)/2}$.

(4) Note that $\varepsilon_{M+i}(\alpha)\varepsilon_{M+j}(\alpha) = \varepsilon_{M'+i}(\beta)\varepsilon_{M'+j}(\beta)$ for $0 \le i,j < N-M$ and the assertion follows from (1) and (3).

LEMMA 5. *If* $\alpha = [0,\overline{a}]$ *and* $t \in \{0,1\}$ *then*

$$\sum_{\substack{0\le i,j\le m\\ i\equiv j\equiv t \ (\mathrm{mod}\ 2)}} \varepsilon_i\varepsilon_j s_{ij} = (-1)^t \frac{1-(-1)^a}{2}\cdot\frac{a}{2(a^2+4)}m + O(1)$$

with an implied constant that depends on a.

REMARK. We use $\overline{a}^{\lambda}$ as a shorthand notation for a block $a,\dots,a$ of length λ, e.g. $\overline{2}^3 = 2,2,2$.

Proof (of Lemma 5). The assertion is trivial if $2\,|\,a$. From now on let $2\nmid a$. Then

$$\begin{aligned}\sum_{\substack{0\le j\le m\\ j\equiv t \ (\mathrm{mod}\ 2)}} \varepsilon_j^2 s_{jj} &= (-1)^t \sum_{\substack{0\le j\le m\\ j\equiv t \ (\mathrm{mod}\ 2)}} q_j|q_j\alpha - p_j|\\ &= (-1)^t \sum_{\substack{0\le j\le m\\ j\equiv t \ (\mathrm{mod}\ 2)}} (a+[0,\overline{a}^j]+[0,\overline{a}])^{-1}\\ &= (-1)^t\frac{m}{2}(a+2[0,\overline{a}])^{-1}+O(1)\\ &= (-1)^t\frac{1}{2\sqrt{a^2+4}}m+O(1).\end{aligned}$$

As

$$q_n([0,\overline{a}]) = \frac{1}{\sqrt{a^2+4}}([\overline{a}]^{n+1} - (-[0,\overline{a}])^{n+1}) \quad \text{for } n\ge 0$$

we get (with $2\,|\,j$)

$$\begin{aligned}\sum_{\substack{0\le i<j\\ 2|i}} \varepsilon_i q_i &= \sum_{k=0}^{(j-2)/2} (-1)^{k+1} q_{2k}\\ &= -\frac{1}{\sqrt{a^2+4}}\sum_{k=0}^{(j-2)/2}(-1)^k([\overline{a}]^{2k+1}+[0,\overline{a}]^{2k+1})\\ &= -\frac{1}{\sqrt{a^2+4}}\left([\overline{a}]\frac{1-(-[\overline{a}]^2)^{j/2}}{1+[\overline{a}]^2} + [0,\overline{a}]\frac{1-(-[0,\overline{a}]^2)^{j/2}}{1+[0,\overline{a}]^2}\right)\\ &= -\frac{1}{\sqrt{a^2+4}}\cdot\frac{[\overline{a}]}{1+[\overline{a}]^2}(2-(-1)^{j/2}[\overline{a}]^j - (-1)^{j/2}[0,\overline{a}]^j).\end{aligned}$$

Using

$$|q_j\alpha - p_j| = \prod_{i=1}^{j+1}[0, a_i, a_{i+1}, \ldots] = [0, \overline{a}]^{j+1} \qquad \text{for } j \ge 0$$

we get

$$\begin{aligned}
\sum_{\substack{0\le i<j\le m\\ i\equiv j\equiv 0 \,(\mathrm{mod}\, 2)}} \varepsilon_i\varepsilon_j s_{ij} &= \sum_{\substack{0<j\le m\\ 2|j}} \varepsilon_j(q_j\alpha - p_j) \sum_{\substack{0\le i<j\\ 2|i}} \varepsilon_i q_i \\
&= \frac{1}{\sqrt{a^2+4}} \cdot \frac{1}{1+[\overline{a}]^2} \sum_{\substack{0<j\le m\\ 2|j}} (-1)^{j/2}[0,\overline{a}]^j \\
&\quad \times (2 - (-1)^{j/2}[\overline{a}]^j - (-1)^{j/2}[0,\overline{a}]^j) \\
&= \frac{1}{\sqrt{a^2+4}} \cdot \frac{1}{1+[\overline{a}]^2} \\
&\quad \times \sum_{\substack{0<j\le m\\ 2|j}} (-1 + 2(-1)^{j/2}[0,\overline{a}]^j - [0,\overline{a}]^{2j}) \\
&= -\frac{1}{2\sqrt{a^2+4}} \cdot \frac{1}{1+[\overline{a}]^2} m + O(1).
\end{aligned}$$

It can be proved analogously that

$$\sum_{\substack{0\le i<j\le m\\ i\equiv j\equiv 1 \,(\mathrm{mod}\, 2)}} \varepsilon_i\varepsilon_j s_{ij} = \frac{1}{2\sqrt{a^2+4}} \cdot \frac{1}{1+[\overline{a}]^2} m + O(1)$$

and therefore

$$\begin{aligned}
\sum_{\substack{0\le i,j\le m\\ i\equiv j\equiv t \,(\mathrm{mod}\, 2)}} \varepsilon_i\varepsilon_j s_{ij} &= \sum_{\substack{0\le j\le m\\ j\equiv t \,(\mathrm{mod}\, 2)}} \varepsilon_j^2 s_{jj} + 2 \sum_{\substack{0\le i<j\le m\\ i\equiv j\equiv t \,(\mathrm{mod}\, 2)}} \varepsilon_i\varepsilon_j s_{ij} \\
&= (-1)^t \frac{1}{2\sqrt{a^2+4}}\left(1 - \frac{2}{1+[\overline{a}]^2}\right) m + O(1) \\
&= (-1)^t \frac{a}{2(a^2+4)} m + O(1).
\end{aligned}$$

LEMMA 6. *Let $a, b \in \mathbb{N}$ and $\alpha = [0, \overline{a}^{\lambda_1}, \overline{b}^{\mu_1}, \overline{a}^{\lambda_2}, \overline{b}^{\mu_2}, \ldots]$ with $\lambda_i \equiv \mu_i \equiv 0$ (mod 2) for all $i \ge 1$. If*

$$\lambda_1 + \mu_1 + \ldots + \lambda_k + \mu_k \le m + 1 < \lambda_1 + \mu_1 + \ldots + \lambda_{k+1} + \mu_{k+1}$$

then

$$\frac{1}{\log q_{m+1}}\Big(\sum_{i=0}^{m} a_{i+1}-\sum_{\substack{0\le i,j\le m\\ i\equiv j\ (\mathrm{mod}\ 2)}}\varepsilon_i\varepsilon_j|s_{ij}|\Big)$$

$$=\frac{\Big(\sum_{i=1}^{k}\lambda_i\Big)a\Big(1-\frac{1-(-1)^a}{2}\cdot\frac{1}{a^2+4}\Big)}{\Big(\sum_{i=1}^{k}\lambda_i\Big)\log([\overline{a}])+\Big(\sum_{i=1}^{k}\mu_i\Big)\log([\overline{b}])+O(\lambda_{k+1}+\mu_{k+1}+k)}$$

$$+\frac{\Big(\sum_{i=1}^{k}\mu_i\Big)b\Big(1-\frac{1-(-1)^b}{2}\cdot\frac{1}{b^2+4}\Big)+O(\lambda_{k+1}+\mu_{k+1}+k)}{\Big(\sum_{i=1}^{k}\lambda_i\Big)\log([\overline{a}])+\Big(\sum_{i=1}^{k}\mu_i\Big)\log([\overline{b}])+O(\lambda_{k+1}+\mu_{k+1}+k)}$$

with implied constants that depend on a and b.

P r o o f. It is trivial that

$$\sum_{i=0}^{m} a_{i+1}=\Big(\sum_{i=1}^{k}\lambda_i\Big)a+\Big(\sum_{i=1}^{k}\mu_i\Big)b+O(\lambda_{k+1}+\mu_{k+1}).$$

If $n<m$ then

$$\log K(c_1,\dots,c_m)-\log K(c_1,\dots,c_n)-\log K(c_{n+1},\dots,c_m)$$
$$=\log\Big(1+\frac{K(c_1,\dots,c_{n-1})K(c_{n+2},\dots,c_m)}{K(c_1,\dots,c_n)K(c_{n+1},\dots,c_m)}\Big)<\log 2$$

and therefore

$$\log K(c_1,\dots,c_m)=\log K(c_1,\dots,c_n)+\log K(c_{n+1},\dots,c_m)+O(1).$$

As

$$\log K(\overline{c}^{\lambda})=\log([\overline{c}]^{\lambda+1}-(-[0,\overline{c}])^{\lambda+1})-\frac{1}{2}\log(c^2+4)=\lambda\log([\overline{c}])+O(1)$$

we get

$$\log q_{m+1}=\Big(\sum_{i=1}^{k}\lambda_i\Big)\log([\overline{a}])+\Big(\sum_{i=1}^{k}\mu_i\Big)\log([\overline{b}])+O(\lambda_{k+1}+\mu_{k+1}+k).$$

For our next step we introduce the shorthand notations

$$L_\varrho:=\lambda_1+\mu_1+\ldots+\lambda_{\varrho-1}+\mu_{\varrho-1}+\lambda_\varrho\quad\text{and}\quad M_\varrho:=\lambda_1+\mu_1+\ldots+\lambda_\varrho+\mu_\varrho$$

for $1\le\varrho\le k$ (and $M_0:=0$). Let $t\in\{0,1\}$. Lemmata 2 and 3 imply

$$\sum_{\substack{0\le i,j\le m\\ i\equiv j\equiv t\ (\mathrm{mod}\ 2)}}\varepsilon_i\varepsilon_j s_{ij}=\sum_{\substack{0\le i,j<M_k\\ i\equiv j\equiv t\ (\mathrm{mod}\ 2)}}\varepsilon_i\varepsilon_j s_{ij}+O(\lambda_{k+1}+\mu_{k+1})$$

$$= \sum_{\varrho=1}^{k}\Big(\sum_{\substack{M_{\varrho-1}\le i,j<L_\varrho\\ i\equiv j\equiv t\ (\mathrm{mod}\ 2)}} + \sum_{\substack{L_\varrho\le i,j<M_\varrho\\ i\equiv j\equiv t\ (\mathrm{mod}\ 2)}}\Big)\varepsilon_i\varepsilon_j s_{ij} + O(\lambda_{k+1}+\mu_{k+1}+k).$$

Employing Lemmata 4 and 5 we find that for $1\le\varrho\le k$,

$$\sum_{\substack{M_{\varrho-1}\le i,j<L_\varrho\\ i\equiv j\equiv t\ (\mathrm{mod}\ 2)}} \varepsilon_i\varepsilon_j s_{ij} = \sum_{\substack{0\le i,j<\lambda_\varrho\\ i\equiv j\equiv t\ (\mathrm{mod}\ 2)}} \varepsilon_i([0,\overline{a}])\varepsilon_j([0,\overline{a}])s_{ij}([0,\overline{a}]) + O(1)$$

$$= (-1)^t\frac{1-(-1)^a}{2}\cdot\frac{a}{2(a^2+4)}\lambda_\varrho + O(1)$$

and

$$\sum_{\substack{L_\varrho\le i,j<M_\varrho\\ i\equiv j\equiv t\ (\mathrm{mod}\ 2)}} \varepsilon_i\varepsilon_j s_{ij} = (-1)^t\frac{1-(-1)^b}{2}\cdot\frac{b}{2(b^2+4)}\mu_\varrho + O(1).$$

Altogether this gives

$$-\sum_{\substack{0\le i,j\le m\\ i\equiv j\ (\mathrm{mod}\ 2)}} \varepsilon_i\varepsilon_j|s_{ij}|$$

$$= -\Big(\sum_{i=1}^{k}\lambda_i\Big)\frac{1-(-1)^a}{2}\cdot\frac{a}{a^2+4}$$

$$-\Big(\sum_{i=1}^{k}\mu_i\Big)\frac{1-(-1)^b}{2}\cdot\frac{b}{b^2+4} + O(\lambda_{k+1}+\mu_{k+1}+k),$$

which completes the proof of Lemma 6.

LEMMA 7. *Let $1\le a<b$ be integers and $\alpha=[0,\overline{a}^{\lambda_1},\overline{b}^{\lambda_2},\overline{a}^{\lambda_3},\overline{b}^{\lambda_4},\ldots]$.*

(1) *If*

$$\limsup_{n\to\infty}\left(\lambda_{n+1} - 7\frac{\log([\overline{b}])}{\log([\overline{a}])}(\lambda_1+\ldots+\lambda_n)\right) = \infty$$

then α is transcendental.

(2) *If even $\limsup_{n\to\infty}\lambda_{n+1}/(\lambda_1+\ldots+\lambda_n)=\infty$ then α is a U_2-number.*

Proof. This can be proved in the same way as Corollary 6 in [2]. The modification made in part (1) is necessary to make possible the inclusion of $a=1$.

LEMMA 8. *Let $\alpha=[0,a_1,a_2,\ldots]$, $\alpha'=[0,a_1',a_2',\ldots]$ and*

$$L(m) = \sum_{i=1}^{m+1}(a_i'-a_i)$$

where $a'_i \geq a_i$ *and* $a'_i \equiv a_i \pmod 2$ *for all* $i \geq 1$. *If* $L(m) = o(m)$ *as* $m \to \infty$ *then* $\nu(\alpha') = \nu(\alpha)$.

Proof. This follows from Lemmata 5.4 and 5.9 in [1].

Completion of the proof of part (1). If we use Lemma 6 as a starting point, the inclusions $[\nu([\overline{b}]), \infty) \subseteq \nu(B_b)$ and $[\nu([\overline{b}]), \infty) \subseteq \overline{\nu(B_b^{\mathrm{q}})}$ can be proved just as the analogous assertions in [1]. Lemmata 7 and 8 lead to the inclusions $[\nu([\overline{b}]), \infty) \subseteq \nu(B_b^{\mathrm{t}})$ and $[\nu([\overline{b}]), \infty) \subseteq \nu(B_b^{\mathrm{u}})$ in the same way as for the analogous results in [2].

3. Proof of part (2). We mimic the proof of Y. Dupain and V. T. Sós [6] of $\inf \nu^*(B) = \nu^*(\sqrt{2})$. The idea is to partition the sequence $(a_n)_{n\geq 1}$ of partial quotients into blocks $a_{k_j+1}, \ldots, a_{k_{j+1}}$ and to prove

$$\sum_{m=k_j}^{k_{j+1}-1} \Big(a_{m+1} - \varepsilon_m^2 |s_{mm}| - 2 \sum_{\substack{0\leq i<m \\ i\equiv m \pmod 2}} \varepsilon_i \varepsilon_m |s_{im}| \Big) \geq 4\nu([\overline{b}]) \log \frac{q_{k_{j+1}}}{q_{k_j}}$$

for each kind of block. The assertion follows from Theorem 1 by addition of these inequalities. We will use the following types of blocks:

Type 1: $a_{m+1} \geq b+2$ where $2 \mid a_{m+1}$;
Type 2: $a_{m+1} \geq b+3$ where $2 \nmid a_{m+1}$;
Type 3: $a_{m+1} = \ldots = a_{m+k} = b$ where $a_m \geq b+1$ and $k \geq 1$;
Type 4: $a_{m+1} = \ldots = a_{m+k} = b+1$ where $a_m \geq b+2$ and $k \geq 1$;
Type 5: $a_{m+1} = \ldots = a_{m+k} = b$, $a_{m+k+1} = \ldots = a_{m+k+l} = b+1$ where $a_m \geq b+2$ or $(a_{m-1}, a_m) \in \{(b, b+1), (b+1, b+1)\}$ and $k, l \geq 1$;
Type 6: $a_{m+1} = b+1$, $a_{m+2} = \ldots = a_{m+k+1} = b$, $a_{m+k+2} = \ldots = a_{m+k+l+1} = b+1$ where $a_m \geq b+2$ and $k, l \geq 1$.

REMARKS. (1) The labelling of blocks is chosen to stress the analogy with the proof of Y. Dupain and V. T. Sós. In fact first the blocks of type 5 and 6 should be determined and then the blocks of type 3 and 4 among the remaining partial quotients.

(2) For types 4–6 the case $b = 2$ deserves more careful estimations than the case $b \geq 4$. We will normally present the calculations for $b \geq 4$ and briefly indicate the necessary changes for $b = 2$.

(3) Let

$$S_m := a_{m+1} - \varepsilon_m^2 |s_{mm}| - 2 \sum_{\substack{0\leq i<m \\ i\equiv m \pmod 2}} \varepsilon_i \varepsilon_m |s_{im}|.$$

We want to show that

$$\sum_{m=k_j}^{k_{j+1}-1} S_m \geq \frac{b}{\log([\bar{b}])} \log \frac{q_{k_{j+1}}}{q_{k_j}}.$$

To this end we estimate S_m from below and $q_{k_{j+1}}/q_{k_j}$ from above using Lemmata 9 and 10 below.

(4) Since $\nu(\alpha) = \nu(\alpha')$ if α and α' are equivalent [9], [3] we may assume $m \geq 2$ for blocks of type 3–6.

Lemma 9. *Let $|\varepsilon_m| = 1$. If $[0, a_m, \ldots, a_1] \geq c$ and $[0, a_{m+2}, a_{m+3}, \ldots] \geq d$ then $S_m \geq a_{m+1} - (a_{m+1} + c + d)^{-1}$.*

Proof. Since

$$\sum_{\substack{0 \leq i < m \\ i \equiv m \,(\mathrm{mod}\, 2)}} \varepsilon_i \varepsilon_m q_i \leq 0$$

this follows from

$$\begin{aligned} \varepsilon_m^2 |s_{mm}| + 2 &\sum_{\substack{0 \leq i < m \\ i \equiv m \,(\mathrm{mod}\, 2)}} \varepsilon_i \varepsilon_m |s_{im}| \\ &\leq q_m |q_m \alpha - p_m| = (a_{m+1} + [0, a_m, \ldots, a_1] + [0, a_{m+2}, \ldots])^{-1} \\ &\leq (a_{m+1} + c + d)^{-1}. \end{aligned}$$

Lemma 10. *In this lemma b denotes an arbitrary positive integer.*

(1) *If $a_m \geq b + 1$ then $K(a_1, \ldots, a_m, \overline{b}^k) \leq [\bar{b}]^k K(a_1, \ldots, a_m)$ for all $k \geq 0$.*

(2) *If $a_m \geq b + 2$ then*

$$K(a_1, \ldots, a_m, \overline{b}^k, \overline{b+1}^l) \leq [\bar{b}]^k [\overline{b+1}]^l K(a_1, \ldots, a_m)$$

for all $k, l \geq 0$. This also holds for $k \geq 1$ and $l \geq 0$ if $(a_{m-1}, a_m) \in \{(b, b+1), (b+1, b+1)\}$.

(3) *If $a_m \geq b + 2$ then*

$$K(a_1, \ldots, a_m, b+1, \overline{b}^k, \overline{b+1}^l) \leq [\bar{b}]^k [\overline{b+1}]^{l+1} K(a_1, \ldots, a_m)$$

for all $k, l \geq 0$.

Proof. (1) This is trivial if $k = 0$. If $k = 1$ then

$$K(a_1, \ldots, a_m, b)/K(a_1, \ldots, a_m) = [b, a_m, \ldots, a_1] < [\bar{b}]$$

and for $k \geq 2$ we use the induction step

$$\begin{aligned} K(a_1, \ldots, a_m, \overline{b}^k) &= bK(a_1, \ldots, a_m, \overline{b}^{k-1}) + K(a_1, \ldots, a_m, \overline{b}^{k-2}) \\ &\leq (b[\bar{b}]^{k-1} + [\bar{b}]^{k-2}) K(a_1, \ldots, a_m) = [\bar{b}]^k K(a_1, \ldots, a_m). \end{aligned}$$

(2) First let $a_m \geq b+2$. If $l=0$ or $(k,l)=(0,1)$ this follows from (1). We now check the assertion for $k=l=1$:

$$\begin{aligned}
&K(a_1,\ldots,a_m,b,b+1)\\
&\qquad = K(b,b+1)K(a_1,\ldots,a_m) + (b+1)K(a_1,\ldots,a_{m-1})\\
&\qquad = (b^2+b+1+(b+1)[0,a_m,\ldots,a_1])K(a_1,\ldots,a_m)\\
&\qquad \leq \left(b^2+b+1+\frac{b+1}{b+2}\right)K(a_1,\ldots,a_m)\\
&\qquad = \left(b^2+b+2-\frac{1}{b+2}\right)K(a_1,\ldots,a_m)\\
&\qquad < [\overline{b}][\overline{b+1}]K(a_1,\ldots,a_m).
\end{aligned}$$

The last inequality is true if $b=1$. If $b\geq 2$ it can be checked as follows:

$$\begin{aligned}
[\overline{b}][\overline{b+1}] &> [b,b,b][b+1,b+1,b+1]\\
&= \left(b+\frac{b}{b^2+1}\right)\left(b+1+\frac{b+1}{b^2+2b+2}\right)\\
&= b^2+b+2-\frac{4}{(b^2+1)(b^2+2b+2)} > b^2+b+2-\frac{1}{b+2}.
\end{aligned}$$

If $l=1$ and $k\geq 2$ then

$$\begin{aligned}
&K(a_1,\ldots,a_m,\overline{b}^k,b+1)\\
&\quad = K(a_1,\ldots,a_m)K(\overline{b}^k,b+1) + K(a_1,\ldots,a_{m-1})K(\overline{b}^{k-1},b+1)\\
&\quad = K(a_1,\ldots,a_m)(bK(\overline{b}^{k-1},b+1)+K(\overline{b}^{k-2},b+1))\\
&\qquad + K(a_1,\ldots,a_{m-1})(bK(\overline{b}^{k-2},b+1)+K(\overline{b}^{k-3},b+1))\\
&\quad = b(K(a_1,\ldots,a_m)K(\overline{b}^{k-1},b+1)+K(a_1,\ldots,a_{m-1})K(\overline{b}^{k-2},b+1))\\
&\qquad + K(a_1,\ldots,a_m)K(\overline{b}^{k-2},b+1)+K(a_1,\ldots,a_{m-1})K(\overline{b}^{k-3},b+1)\\
&\quad = bK(a_1,\ldots,a_m,\overline{b}^{k-1},b+1)+K(a_1,\ldots,a_m,\overline{b}^{k-2},b+1)\\
&\quad \leq b[\overline{b}]^{k-1}[\overline{b+1}]K(a_1,\ldots,a_m)+[\overline{b}]^{k-2}[\overline{b+1}]K(a_1,\ldots,a_m)\\
&\quad = [\overline{b}]^k[\overline{b+1}]K(a_1,\ldots,a_m).
\end{aligned}$$

Note that this calculation remains valid if $k=2$ since in this case $K(\overline{b}^{k-3}, b+1)=1$. The assertion is now proved for $k\geq 0$ and $l\in\{0,1\}$. For $l\geq 2$ the proof is finished by the induction step

$$\begin{aligned}
&K(a_1,\dots,a_m,\overline{b}^k,\overline{b+1}^l)\\
&\quad = (b+1)K(a_1,\dots,a_m,\overline{b}^k,\overline{b+1}^{l-1}) + K(a_1,\dots,a_m,\overline{b}^k,\overline{b+1}^{l-2})\\
&\quad \le (b+1)[\overline{b}]^k[\overline{b+1}]^{l-1}K(a_1,\dots,a_m) + [\overline{b}]^k[\overline{b+1}]^{l-2}K(a_1,\dots,a_m)\\
&\quad = [\overline{b}]^k[\overline{b+1}]^l K(a_1,\dots,a_m).
\end{aligned}$$

Now let $(a_{m-1},a_m)\in\{(b,b+1),(b+1,b+1)\}$. If $l=0$ the assertion follows from (1). If $k=l=1$ then

$$\begin{aligned}
&K(a_1,\dots,a_m,b,b+1)\\
&\quad = (b^2+b+1+[0,a_m,\dots,a_1](b+1))K(a_1,\dots,a_m)\\
&\quad \le (b^2+b+1+[0,b+1,b+2](b+1))K(a_1,\dots,a_m)\\
&\quad = \left(b^2+b+2-\frac{1}{b^2+3b+3}\right)K(a_1,\dots,a_m) < [\overline{b}][\overline{b+1}]K(a_1,\dots,a_m)
\end{aligned}$$

where the last inequality is checked as in the case $a_m\ge b+2$. Consider the case $(k,l)=(2,1)$. Then

$$\begin{aligned}
&K(a_1,\dots,a_m,b,b,b+1)\\
&\qquad = K(a_1,\dots,a_m)K(b,b,b+1) + K(a_1,\dots,a_{m-1})K(b,b+1)\\
&\qquad = (b^3+b^2+2b+1+(b^2+b+1)[0,a_m,\dots,a_1])K(a_1,\dots,a_m)\\
&\qquad \le (b^3+b^2+2b+1+(b^2+b+1)[0,b+1,b+2])K(a_1,\dots,a_m)\\
&\qquad = \left(b^3+b^2+3b+1+\frac{2}{b^2+3b+3}\right)K(a_1,\dots,a_m)\\
&\qquad < [\overline{b}]^2[\overline{b+1}]K(a_1,\dots,a_m).
\end{aligned}$$

The proof is then finished as in the case $a_m\ge b+2$.

(3) If $l=0$ or $(k,l)=(0,1)$ the assertion follows from (1). Let $k=l=1$. Then

$$\begin{aligned}
&K(a_1,\dots,a_m,b+1,b,b+1)\\
&\quad = K(a_1,\dots,a_m)K(b+1,b,b+1) + K(a_1,\dots,a_{m-1})K(b,b+1)\\
&\quad = (b^3+2b^2+3b+2+(b^2+b+1)[0,a_m,\dots,a_1])K(a_1,\dots,a_m)\\
&\quad \le \left(b^3+2b^2+3b+2+\frac{b^2+b+1}{b+2}\right)K(a_1,\dots,a_m)\\
&\quad = \left(b^3+2b^2+4b+1+\frac{3}{b+2}\right)K(a_1,\dots,a_m) < [\overline{b}][\overline{b+1}]^2K(a_1,\dots,a_m)
\end{aligned}$$

and the proof is completed as in (2).

Proof for blocks of type 1. Since $S_m = a_{m+1}$ and $q_{m+1}/q_m = [a_{m+1}, \ldots, a_1] \le a_{m+1} + 1/b$ we have to prove that

$$a_{m+1}\log([\bar{b}]) - b\log(a_{m+1} + 1/b) > 0 \quad \text{for } a_{m+1} \ge b+2.$$

We check this by considering $f_b : [b,\infty) \to \mathbb{R}$, $x \mapsto x\log([\bar{b}]) - b\log(x+1/b)$:

$$\begin{aligned} f_b'(x) &= \log([\bar{b}]) - \frac{b}{x+1/b} \ge \log([\bar{b}]) - \frac{b}{b+1/b} \\ &= \left(b+\frac{1}{b}\right)^{-1}\left(\left(b+\frac{1}{b}\right)\log([\bar{b}]) - b\right). \end{aligned}$$

This is positive if $b = 2$. If $b \ge 4$ then

$$\left(b+\frac{1}{b}\right)\log([\bar{b}]) - b > \left(b+\frac{1}{b}\right)\log b - b = b(\log b - 1) + \frac{1}{b}\log b > 0.$$

Therefore, f_b is strictly increasing and it suffices to check that $f_b(b+2) > 0$. This is true if $b = 2$. For $b \ge 4$ we estimate

$$\begin{aligned} (b+2)\log([\bar{b}]) - b\log(b+2+1/b) &> (b+2)\log b - b\log(b+5/2) \\ &= 2\log b - b\log(1+5/(2b)). \end{aligned}$$

The right hand side is positive if $b = 4$ and

$$\begin{aligned} \frac{d}{dx}\left(2\log x - x\log\left(1+\frac{5}{2x}\right)\right) &= \frac{2}{x} - \log\left(1+\frac{5}{2x}\right) + \frac{5}{2x+5} \\ &> \frac{2}{x} - \frac{5}{2x} + \frac{5}{2x+5} = \frac{8x-5}{2x(2x+5)} > 0 \end{aligned}$$

for all real $x \ge 4$.

Blocks of type 2. Estimating $S_m \ge a_{m+1} - 1/a_{m+1}$ we get

$$S_m\log([\bar{b}]) - b\log(q_{m+1}/q_m) \ge (a_{m+1} - 1/a_{m+1})\log([\bar{b}]) - b\log(a_{m+1} + 1/b).$$

Consider $f_b : [b+3,\infty) \to \mathbb{R}$, $x \mapsto (x-1/x)\log([\bar{b}]) - b\log(x+1/b)$. Since

$$\begin{aligned} (x+1/b)f_b'(x) &= (1+1/x^2)(x+1/b)\log([\bar{b}]) - b \\ &> (b+3+1/b)\log b - b \\ &= b(\log b - 1) + (3+1/b)\log b > 0 \end{aligned}$$

for all even $b > 0$ it suffices to check that $f_b(b+3) > 0$. This is true if $b = 2$ and

$$f_b(b+3) \ge \left(b+\frac{14}{5}\right)\log b - b\log\left(b+\frac{7}{2}\right) = \frac{14}{5}\log b - b\log\left(1+\frac{7}{2b}\right).$$

The right hand side is positive if $b = 4$ and it can be checked as above that

$$\frac{d}{dx}\left(\frac{14}{5}\log x - x\log\left(1+\frac{7}{2x}\right)\right) > 0 \quad \text{for all real } x \ge 4.$$

Blocks of type 3. Using Lemma 10(1) we get

$$(S_m + \ldots + S_{m+k-1})\log([\bar{b}]) - b\log\frac{q_{m+k}}{q_m} \ge kb\log([\bar{b}]) - b\log([\bar{b}]^k) = 0.$$

Blocks of type 4. First let $k \ge 2$ and $b \ge 4$. Lemma 9 yields

$$S_m \ge b + 1 - (b + 1 + [0, b+2])^{-1},$$
$$S_{m+j} \ge b + 1 - (b + 1 + [0, b+1, b+1] + [0, b+1, b])^{-1} \quad \text{for } 1 \le j \le k-2,$$
$$S_{m+k-1} \ge b + 1 - (b + 1 + [0, b+1, b+1])^{-1}.$$

As $\frac{d}{dx}\big(x+1+\frac{1}{x+2}\big) > 0$ for real $x > 0$ the sequence $(1/(b+1+[0,b+2]))_{b\ge1}$ is decreasing. The same argument shows that $(1/(b+1+[0,b+1,b+1]+[0,b+1,b]))_{b\ge1}$ and $(1/(b+1+[0,b+1,b+1]))_{b\ge1}$ are decreasing. Using Lemma 10(1) we get

$$(S_m + \ldots + S_{m+k-1})\log([\bar{b}]) - b\log(q_{m+k}/q_m) \ge f_b(k)$$

where $f_b : [1,\infty) \to \mathbb{R}$ with

$$\begin{aligned} f_b(x) = {} & (x(b+1-(b+1+[0,b+1,b+1]+[0,b+1,b])^{-1}) \\ & + 2(b+1+[0,b+1,b+1]+[0,b+1,b])^{-1} \\ & - (b+1+[0,b+2])^{-1} - (b+1+[0,b+1,b+1])^{-1})\log([\bar{b}]) \\ & - xb\log([\overline{b+1}]). \end{aligned}$$

We estimate

$$\begin{aligned} f_b'(x) &> (b+1-(5+[0,5,5]+[0,5,4])^{-1})\log b - b\log([b+1,b+1]) \\ &= \left(b + \frac{2393}{2939}\right)\log b - b\log([b+1,b+1]). \end{aligned}$$

This is positive if $b = 4$. If $b \ge 6$ we use

$$\log([b+1,b+1]) = \log b + \log(1+b^{-1}+b^{-1}(b+1)^{-1}) < \log b + b^{-1} + b^{-1}(b+1)^{-1}$$

to get

$$f_b'(x) > \frac{2393}{2939}\log b - 1 - \frac{1}{b+1}.$$

As the right hand side is increasing and positive if $b = 6$ it suffices to check $f_b(2) \ge 0$. This is true if $b = 4$. If $b \ge 6$ we estimate

$$\begin{aligned} f_b(2) &> (2b+2-(5+[0,6])^{-1} - (5+[0,5,5])^{-1})\log b \\ &\quad - 2b(\log b + b^{-1} + b^{-1}(b+1)^{-1}) \\ &= \frac{6754}{4185}\log b - 2 - \frac{2}{b+1}. \end{aligned}$$

As the right hand side is positive for $b = 6$ and increasing we have proved the assertion for $k \ge 2$ and $b \ge 4$. If $b = 2$ the above estimates are too weak for small k. This case can be dealt with by replacing Lemma 10(1) by the better estimate

$$K(a_1,\dots,a_m,\overline{3}^k) \le \frac{1971}{2000}[\overline{3}]^k K(a_1,\dots,a_m) \qquad (\text{for } a_m \ge 4 \text{ and } k \ge 1),$$

which can be proved in the same way as Lemma 10(1). It then remains to check that the resulting function $\widetilde{f}_2$ has the properties $\widetilde{f}_2' > 0$ and $\widetilde{f}_2(2) > 0$.

Now let $k = 1$. Then

$$S_m \log([\overline{b}]) - b\log(q_{m+1}/q_m) \ge f_b(a_m)$$

where

$$f_b(x) = \left(b+1-\frac{1}{b+1+1/(x+1)}\right)\log([\overline{b}]) - b\log(b+1+1/x).$$

Since

$$b(b+1) - \log([\overline{b}]) > b(b+1) - \log([b,b]) > b^2 + b - \log b - 1/b^2 > 0,$$
$$2b(b+1)(b+2) - \log([\overline{b}]) > 2b^3 - \log b - 1/b^2 > 0,$$

we get

$$\begin{aligned}((b+1)x^2 + x)((b+1)x+b+2)^2 f_b'(x) &= (b+1)(b(b+1) - \log([\overline{b}]))x^2 \\ &\quad + (2b(b+1)(b+2) - \log([\overline{b}]))x + b(b+2)^2 > 0\end{aligned}$$

for all real $x > 0$ and it suffices to check $f_b(b+2) \ge 0$. This is true if $b \in \{2,4\}$. If $b \ge 6$ then

$$\begin{aligned}f_b(b+2) &> \left(b+1-\frac{b+3}{(b+2)^2}\right)\log b - b\log b - 1 - \frac{1}{b+2} \\ &= \left(1-\frac{b+3}{(b+2)^2}\right)\log b - 1 - \frac{1}{b+2}.\end{aligned}$$

This is increasing and positive if $b = 6$, which completes the proof for blocks of type 4.

Blocks of type 5. First let $l \ge 2$ and $b \ge 4$. Due to Lemma 9,

$$S_m = \ldots = S_{m+k-1} = b,$$
$$S_{m+k} \ge b+1 - (b+1+[0,b,b]+[0,b+2])^{-1},$$
$$S_{m+k+j} \ge b+1 - (b+1+[0,b+1,b]+[0,b+2])^{-1} \qquad \text{for } 1 \le j \le l-2,$$
$$S_{m+k+l-1} \ge b+1 - (b+1+[0,b+1,b])^{-1},$$

where the rational functions on the right hand side are monotone. Therefore, Lemma 10(2) yields

$$(S_m + \ldots + S_{m+k+l-1})\log([\overline{b}]) - b\log(q_{m+k+l}/q_m) \ge f_b(l)$$

where

$$\begin{aligned} f_b(x) = {}& (x(b+1-(b+1+[0,b+1,b]+[0,b+2])^{-1}) \\ &+ 2(b+1+[0,b+1,b]+[0,b+2])^{-1} \\ &- (b+1+[0,b,b]+[0,b+2])^{-1} - (b+1+[0,b+1,b])^{-1}) \log([\overline{b}]) \\ &- xb\log([\overline{b+1}]). \end{aligned}$$

Now $f_4' > 0$ and if $b \geq 6$ then

$$\begin{aligned} f_b'(x) &> (b+1-(5+[0,5,4]+[0,6])^{-1}) \log b - b\log([b+1,b+1]) \\ &\geq \left(b+\frac{61}{75}\right) \log b - b(\log b + b^{-1} + b^{-1}(b+1)^{-1}) \\ &= \frac{61}{75}\log b - 1 - \frac{1}{b+1} > 0. \end{aligned}$$

Checking $f_4(2) > 0$ and estimating

$$\begin{aligned} f_b(2) &> (2b+2-(5+[0,4,4]+[0,6])^{-1} \\ &\quad - (5+[0,5,4])^{-1}) \log b - 2b(\log b + b^{-1} + b^{-1}(b+1)^{-1}) \\ &= \frac{97429}{60059}\log b - 2 - \frac{2}{b+1} \end{aligned}$$

(which is positive for $b \geq 6$) completes the proof. If $b = 2$ the above proof works for $l \geq 3$ (i.e. $f_2' > 0$ and $f_2(3) > 0$) but if $l = 2$ we use the estimate

$$K(a_1,\ldots,a_m,\overline{2}^k,\overline{3}^2) \leq \frac{10823}{1000}[\overline{2}]^k K(a_1,\ldots,a_m)$$

(where $k \geq 1$, $a_m \geq 4$ or $(a_{m-1},a_m) \in \{(2,3),(3,3)\}$) instead of Lemma 10(2).

Now let $l = 1$ and $b \geq 4$. Then due to Lemma 10(2),

$$\begin{aligned} &(S_m + \ldots + S_{m+k}) \log([\overline{b}]) - b\log(q_{m+k+1}/q_m) \\ &\quad \geq (b+1-(b+1+[0,b,b])^{-1}) \log([\overline{b}]) - b\log([\overline{b+1}]) \\ &\qquad\qquad\qquad\qquad \text{which is positive if } b = 4 \\ &\quad > (b+1-(5+[0,4,4])^{-1}) \log b - b(\log b + b^{-1} + b^{-1}(b+1)^{-1}) \\ &\quad = \frac{72}{89}\log b - 1 - \frac{1}{b+1} \qquad \text{which is positive if } b \geq 6. \end{aligned}$$

For $b = 2$ we use the estimate

$$K(a_1,\ldots,a_m,\overline{2}^k,3) \leq \frac{3287}{1000}[\overline{2}]^k K(a_1,\ldots,a_m)$$

(where $k \geq 1$, $a_m \geq 4$ or $(a_{m-1},a_m) \in \{(2,3),(3,3)\}$) instead of Lemma 10(2).

Blocks of type 6. First let $l \geq 2$ and $b \geq 4$. Due to Lemma 9

$$S_m \geq b+1-(b+1+[0,b,b])^{-1},$$

$$S_{m+1} = \ldots = S_{m+k} = b,$$
$$S_{m+k+1} \geq b+1-(b+1+[0,b,b]+[0,b+1,b])^{-1},$$
$$S_{m+k+j} \geq b+1-(b+1+2[0,b+1,b])^{-1} \quad \text{for } 2 \leq j \leq l-1,$$
$$S_{m+k+l} \geq b+1-(b+1+[0,b+1,b])^{-1}$$

where the rational functions on the right hand side are monotone. Due to Lemma 10(3)

$$(S_m + \ldots + S_{m+k+l})\log([\overline{b}]) - b\log(q_{m+k+l+1}/q_m) \geq f_b(l)$$

where

$$\begin{aligned} f_b(x) = ((x+1)&(b+1-(b+1+2[0,b+1,b])^{-1}) \\ &+ 3(b+1+2[0,b+1,b])^{-1} \\ &- (b+1+[0,b,b])^{-1} - (b+1+[0,b,b]+[0,b+1,b])^{-1} \\ &- (b+1+[0,b+1,b])^{-1})\log([\overline{b}]) - (x+1)b\log([\overline{b+1}]). \end{aligned}$$

Now $f_4' > 0$ and if $b \geq 6$ then

$$\begin{aligned} f_b'(x) &> (b+1-(5+2[0,5,4])^{-1})\log b - b(\log b + b^{-1} + b^{-1}(b+1)^{-1}) \\ &= \frac{92}{113}\log b - 1 - \frac{1}{b+1} > 0. \end{aligned}$$

Checking $f_4(2) > 0$ and estimating

$$\begin{aligned} f_b(2) > (3b+3&-(5+[0,4,4])^{-1} - (5+[0,4,4]+[0,5,4])^{-1} \\ &- (5+[0,5,4])^{-1})\log b - 3b(\log b + b^{-1} + b^{-1}(b+1)^{-1}) \\ &= \frac{45699740}{18790837}\log b - 3 - \frac{3}{b+1} \end{aligned}$$

(which is positive for $b \geq 6$) completes the proof. If $b = 2$ the above proof works for $l \geq 3$ (i.e. $f_2' > 0$ and $f_2(3) > 0$) but if $l = 2$ we use the estimate

$$K(a_1, \ldots, a_m, 3, \overline{2}^k, \overline{3}^2) \leq \frac{1759}{50}[\overline{2}]^k K(a_1, \ldots, a_m)$$

(where $k \geq 1$, $a_m \geq 4$) instead of Lemma 10(3).

Now let $l = 1$. Then due to Lemmata 9 and 10(3),

$$\begin{aligned} &(S_m + \ldots + S_{m+k+1})\log([\overline{b}]) - b\log(q_{m+k+2}/q_m) \\ &\quad \geq 2(b+1-(b+1+[0,b,b])^{-1})\log([\overline{b}]) - 2b\log([\overline{b+1}]) \\ &\qquad\qquad\qquad\qquad \text{which is positive if } b = 4 \\ &\quad > 2(b+1-(5+[0,4,4])^{-1})\log b - 2b(\log b + b^{-1} + b^{-1}(b+1)^{-1}) \\ &\quad = \frac{144}{89}\log b - 2 - \frac{2}{b+1} \qquad \text{which is positive if } b \geq 6. \end{aligned}$$

For $b = 2$ we use the estimate

$$K(a_1, \ldots, a_m, 3, \overline{2}^k, 3) \le \frac{107}{10}[\overline{2}]^k K(a_1, \ldots, a_m)$$

(for $k \ge 1$ and $a_m \ge 4$) instead of Lemma 10(3).

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