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The Waring–Goldbach problem for cubes

by

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1. Introduction. It is conjectured that all sufficiently large integers satisfying some necessary congruence conditions are sums of four cubes of primes. Such a strong result is out of reach at present. The best result in this direction is due to Hua and dates back to 1938 (see [4]):

• All sufficiently large integers are sums of nine cubes of primes;

• Almost all integers n in the set $\mathfrak{N} = \{n \ge 1 : n \not\equiv 0, \pm 2 \pmod 9\}$ can be represented as sums of five cubes of primes, i.e.

$$n = p_1^3 + p_2^3 + \ldots + p_5^3. \tag{1.1}$$

To be more precise, let $E(N)$ denote the number of integers $n \in \mathfrak{N}$ not exceeding N which cannot be written as (1.1). Then Hua's second result actually states that

$$E(N) \ll N \log^{-A} N, \tag{1.2}$$

where $A > 0$ is arbitrary.

In this paper we give the following improvement on (1.2).

Theorem 1. *For $E(N)$ defined as above, we have*

$$E(N) \ll N^{152/153}.$$

We prove the theorem by the circle method. To get a result of this strength, we have to deal with rather large major arcs, to which the Siegel–Walfisz theorem does not apply. In contrast to the previous works (see, for example, Montgomery–Vaughan [10], Gallagher [2], and Liu and Tsang [9]) which treat the enlarged major arcs by the Deuring–Heilbronn phenomenon, we prove Theorem 1 by a different approach, which has recently been used by Liu, Liu, and Zhan [8]. This approach reveals that in the situation of this paper, the possible existence of a Siegel zero does not have special influence, hence the Deuring–Heilbronn phenomenon can be avoided. The key point

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of this approach is that there are five prime variables in our problem (while there are only two in [10] and [2]), and we can take advantage of this by saving the factor $r_0^{-3/2+\varepsilon}$ in Lemma 4.3(ii) below. With this saving, our enlarged major arcs can be treated by the classical zero-density estimates (in Lemma 3.3) and the zero-free region for the Dirichlet L-functions (defined in (4.14)). Our methods not only give a better result (note that we can take, for example, $\theta = 1/20$ in (2.1)), but also lead us to a technically simpler proof.

Notation. As usual, $\varphi(n)$ and $\Lambda(n)$ stand for the Euler and von Mangoldt functions respectively, and $d(n)$ is the divisor function. We use χ mod q and χ^0 mod q to denote a Dirichlet character and the principal character modulo q, and $L(s,\chi)$ is the Dirichlet L-function. N is a large integer, $L = \log N$, and $r \sim R$ means $R < r \le 2R$. If there is no ambiguity, we write $\frac{a}{b} + \theta$ as $a/b + \theta$ or $\theta + a/b$. The same convention will be applied for quotients. The letters ε and A denote positive constants which are arbitrarily small and arbitrarily large respectively.

2. Outline of the method. In order to apply the circle method, for large $N > 0$ we set

$$P = N^{\theta}, \qquad Q = N^{1-\theta}, \tag{2.1}$$

where θ is a positive constant satisfying $\theta < 1/19.08$. Actually there is only one place (in the estimation of K_{51} right after (4.12)) where $\theta < 1/19.08$ is needed exactly. In other places, better ranges for θ suffice. By Dirichlet's lemma on rational approximations, each $\alpha \in [1/Q, 1+1/Q]$ may be written in the form

$$\alpha = a/q + \lambda, \qquad |\lambda| \le 1/(qQ), \tag{2.2}$$

for some integers a, q with $1 \le a \le q \le Q$ and $(a,q) = 1$. We denote by $\mathfrak{M}(q,a)$ the set of α satisfying (2.2), and write $\mathfrak{M}$ for the union of all $\mathfrak{M}(q,a)$ with $1 \le a \le q \le P$ and $(a,q) = 1$. The *minor arcs* $\mathfrak{m}$ are defined as the complement of $\mathfrak{M}$ in $[1/Q, 1+1/Q]$. It follows from $2P \le Q$ that the *major arcs* $\mathfrak{M}(q,a)$ are mutually disjoint. Let

$$U = (N/12)^{1/3}, \tag{2.3}$$

and define

$$S(\alpha) = \sum_{m \sim U} \Lambda(m) e(m^3 \alpha), \tag{2.4}$$

where $e(r) = \exp(i2\pi r)$ for real r. Let

$$r(n) = \sum_{\substack{n = m_1^3 + \ldots + m_5^3 \\ m_j \sim U}} \Lambda(m_1) \ldots \Lambda(m_5).$$

Then

$$r(n)=\int_0^1 S^5(\alpha)e(-n\alpha)\,d\alpha=\Big\{\int_{\mathfrak{M}}+\int_{\mathfrak{m}}\Big\}S^5(\alpha)e(-n\alpha)\,d\alpha. \tag{2.5}$$

To handle the integral on the major arcs, we need to obtain the following

Theorem 2. *Let $N/2\le n\le N$. Then uniformly for $\theta<1/19.08$, we have*

$$\int_{\mathfrak{M}} S^5(\alpha)e(-n\alpha)\,d\alpha=\mathfrak{S}(n)J(n)+O(U^2L^{-A}). \tag{2.6}$$

Here $\mathfrak{S}(n)$ is the singular series defined as in (4.3) satisfying $\mathfrak{S}(n)\gg 1$ for $n\in\mathfrak{N}$, and $J(n)=J(n;1,\dots,1)$ defined as in (4.4) satisfies

$$U^2\ll J(n)\ll U^2. \tag{2.7}$$

Sections 3 and 4 are devoted to the proof of Theorem 2.

Now Theorem 1 is an immediate consequence of Theorem 2.

Proof of Theorem 1. We start from (2.5). The contribution of the major arcs is taken care of by Theorem 2. To treat the integral on the minor arcs, we note that each $\alpha\in\mathfrak{m}$ can be written as (2.2) for some $P<q\le Q$ and $1\le a\le q$ with $(q,a)=1$. We now apply Harman's estimate [3], which states that if $|\alpha-a/q|\le q^{-2}$, then

$$S(\alpha)\ll U^{1+\varepsilon}\bigg(\frac{1}{q}+\frac{1}{U^{1/2}}+\frac{q}{U^3}\bigg)^{1/16}.$$

Hence,

$$\sup_{\alpha\in\mathfrak{m}}|S(\alpha)|\ll U^{1+\varepsilon}P^{-1/16}. \tag{2.8}$$

Also by Hua's lemma (see [5], Theorem 4),

$$\int_0^1 |S(\alpha)|^8\,d\alpha\ll U^{5+\varepsilon}. \tag{2.9}$$

Thus, we deduce from Bessel's inequality, (2.8), and (2.9) that

$$\sum_{N/2<n\le N}\Big|\int_{\mathfrak{m}}\Big|^2\ll\max_{\alpha\in\mathfrak{m}}|S(\alpha)|^2\int_0^1|S(\alpha)|^8\,d\alpha\ll U^{7+3\varepsilon}P^{-1/8}. \tag{2.10}$$

By a standard argument, we find from (2.10) that for all $n\in\mathfrak{N}\cap(N/2,N]$ with at most $O(U^{3+5\varepsilon}P^{-1/8})$ exceptions,

$$\Big|\int_{\mathfrak{m}}\Big|\ll U^{2-\varepsilon}.$$

Consequently, by (2.5) and Theorem 2, for these unexceptional n,

$$r(n) = \mathfrak{S}(n)J(n) + O(U^2L^{-A}),$$

and therefore these n can be written as (1.1). Let $F(N)$ be the number of exceptional n above. Then

$$F(N) \ll U^{3+5\varepsilon}P^{-1/8} \ll N^{1-\theta/8+2\varepsilon} = N^{152/153}$$

on taking $\theta = 1/19.125 + 16\varepsilon$. The assertion of Theorem 1 now follows from $E(N) = \sum_{j\ge 0} F(N/2^j)$.

3. An explicit expression. The purpose of this section is to establish in Lemma 3.1 an explicit expression for the left-hand side of (2.6).

For a Dirichlet character $\chi \bmod q$, define

$$C(\chi, a) = \sum_{m=1}^{q} \overline{\chi}(m) e\left(\frac{am^3}{q}\right), \qquad C(q, a) = C(\chi^0, a), \tag{3.1}$$

where χ^0 denotes the principal character modulo q. Also, we define

$$\Phi(\lambda) = \int_U^{2U} e(\lambda u^3)\, du, \qquad \Psi(\lambda, \varrho) = \int_U^{2U} u^{\varrho-1} e(\lambda u^3)\, du. \tag{3.2}$$

The next lemma gives an asymptotic formula for $S(\alpha)$ with $\alpha \in \mathfrak{M}$.

Lemma 3.1. *Let $T = P^{9/10+2\varepsilon}$ with $\theta < 1/19.08$. Then for $\alpha = a/q + \lambda \in \mathfrak{M}$, we have*

$$S(\alpha) = S_1(\lambda) + S_2(\lambda) + S_3(\lambda)$$

with

$$S_1(\lambda) = \frac{C(q,a)}{\varphi(q)}\Phi(\lambda), \qquad S_2(\lambda) = -\frac{1}{\varphi(q)} \sum_{\chi \bmod q} C(\chi, a) \sum_{|\gamma|\le T} \Psi(\lambda, \varrho),$$

and

$$S_3(\lambda) = O\left\{ q^{1/2+\varepsilon} \frac{U}{T} (1 + |\lambda| U^3) L^2 \right\},$$

where $\varrho = \beta + i\gamma$ denotes a non-trivial zero (possibly the Siegel zero) of the Dirichlet L-function $L(s, \chi)$.

Proof. By introducing Dirichlet characters, the exponential sum $S(\alpha)$ can be rewritten as (see [1], §26, (2))

$$S\left(\frac{a}{q} + \lambda\right) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} C(\chi, a) \sum_{m \sim U} \Lambda(m)\chi(m) e(m^3\lambda) + O(L^2). \tag{3.3}$$

Now we apply the explicit formula (see [1], §17, (9)–(10); §19, (4)–(9))

$$\sum_{m\le x}\chi(m)\Lambda(m)=\delta_\chi x-\sum_{|\gamma|\le T}\frac{x^\varrho}{\varrho}+O\biggl(\frac{x(\log qxT)^2}{T}+(\log qx)^2\biggr),$$

where $\delta_\chi=1$ or 0 according as $\chi=\chi^0$ or not, and $\varrho=\beta+i\gamma$ is a non-trivial zero of $L(s,\chi)$. The inner sum on the right-hand side of (3.3) is equal to

$$\int_U^{2U} e(\lambda u^3)\,d\Bigl\{\sum_{m\le u}\chi(m)\Lambda(m)\Bigr\}$$
$$=\delta_\chi\Phi(\lambda)-\sum_{|\gamma|\le T}\Psi(\lambda,\varrho)+O\Bigl\{\Bigl|\int_U^{2U}e(\lambda u^3)\,dr(u)\Bigr|\Bigr\},$$

where $r(x)$ is the error term in the explicit formula. The above O-term is

$$\ll|r(2U)|+|r(U)|+U^3|\lambda|\max_{u\sim U}|r(u)|\ll\frac{U}{T}(1+|\lambda|U^3)L^2.$$

Inserting this into (3.3) and then using the Vinogradov estimate (see for example [14], Ch. VI, Problem 14b(α))

$$|C(\chi,a)|\ll q^{1/2}d^2(q), \tag{3.4}$$

we obtain the assertion of the lemma.

For $j=0,1,\ldots,5$ we define

$$I_j=\binom{5}{j}\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}e\Bigl(-\frac{an}{q}\Bigr)\int_{-\infty}^{\infty}S_1^{5-j}(\lambda)S_2^j(\lambda)e(-n\lambda)\,d\lambda. \tag{3.5}$$

Now we state the main result of this section.

LEMMA 3.2. *Let θ and n be as in Theorem 2, and $T=P^{9/10+2\varepsilon}$ as in Lemma 3.1. Then*

$$\int_{\mathfrak{M}}S^5(\alpha)e(-n\alpha)\,d\alpha=\sum_{j=0}^5 I_j+O(U^2L^{-A}).$$

To prove this result, we need the following lemma.

LEMMA 3.3. *Let $N(\sigma,T,\chi)$ denote the number of zeros of $L(s,\chi)$ in the region $\sigma\le\operatorname{Re}s\le 1$, $|\operatorname{Im}s|\le T$. Define*

$$N(\sigma,T,q)=\sum_{\chi\bmod q}N(\sigma,T,\chi),\qquad N^*(\sigma,T,x)=\sum_{q\le x}\ \sideset{}{^*}\sum_{\chi\bmod q}N(\sigma,T,\chi),$$

where $*$ means that the summation is restricted to primitive characters χ mod q. Then

$$N(\sigma,T,q)\ll(qT)^{(12/5+\varepsilon)(1-\sigma)},\qquad N^*(\sigma,T,x)\ll(x^2T)^{(12/5+\varepsilon)(1-\sigma)}.$$

It should be pointed out that the above log-free form of Lemma 3.3 is unnecessary for our purpose, and the normal form of zero-density estimates works equally well.

Proof. The lemma follows from (1.1) of Huxley [6] and Theorem 1 of Jutila [7].

Proof of Lemma 3.2. We first show that for $\alpha = a/q + \lambda \in \mathfrak{M}$,

$$S_1(\lambda), S_2(\lambda) \ll q^{-1/2+\varepsilon} \min(U, |\lambda|^{-1/3}) L^2. \tag{3.6}$$

To estimate $S_2(\lambda)$, one notes that

$$\frac{d}{du}\left(\lambda u + \frac{\gamma}{6\pi}\log u\right) = \lambda + \frac{\gamma}{6\pi u}, \qquad \frac{d^2}{du^2}\left(\lambda u + \frac{\gamma}{6\pi}\log u\right) = -\frac{\gamma}{6\pi u^2}.$$

Thus by Lemmas 4.3 and 4.5 in Titchmarsh [12], we have

$$\begin{aligned} \Psi(\lambda, \varrho) &= \frac{1}{3}\int_{U^3}^{8U^3} u^{\beta/3-1} e\left(\lambda u + \frac{\gamma}{6\pi}\log u\right) du \\ &\ll U^{\beta-3} \min\left(U^3, \frac{U^3}{\min_{U^3\le u\le 8U^3}|\gamma + 6\pi\lambda u|}, \frac{U^3}{\sqrt{|\gamma|}}\right). \end{aligned} \tag{3.7}$$

Therefore,

$$\begin{aligned} &\sum_{|\gamma|\le T} |\Psi(\lambda, \varrho)| \\ &\ll \sum_{\substack{|\gamma|\le T \\ |\gamma|\le|\lambda|U^3}} U^{\beta-3} \min\left(U^3, \frac{U^3}{\min_{U^3\le u\le 8U^3}|\gamma + 6\pi\lambda u|}\right) \\ &\quad + \sum_{|\lambda|U^3<|\gamma|\le T} U^{\beta-3}\min\left(U^3, \frac{U^3}{\sqrt{|\gamma|}}\right) \\ &\ll \sum_{\substack{|\gamma|\le T \\ |\gamma|\le|\lambda|U^3}} U^{\beta-3}\min\left(U^3, \frac{1}{|\lambda|}\right) + \sum_{|\lambda|U^3<|\gamma|\le T} U^{\beta-3}\min\left(U^3, \frac{U^{3/2}}{|\lambda|^{1/2}}\right) \\ &\ll \min(U, |\lambda|^{-1/3}) \sum_{|\gamma|\le T} U^{\beta-1}, \end{aligned} \tag{3.8}$$

on noting that

$$\min\left(U, \frac{1}{|\lambda|U^2}\right) \ll \min\left(U, \frac{1}{|\lambda|^{1/3}}\right),$$
$$\min\left(U, \frac{1}{|\lambda|^{1/2}U^{1/2}}\right) \ll \min\left(U, \frac{1}{|\lambda|^{1/3}}\right).$$

Hence one concludes from (3.8) and Vinogradov's bound (3.4) that

$$\begin{aligned}(3.9)\qquad S_2(\lambda) &\ll q^{-1/2+\varepsilon}\sum_{\chi \bmod q}\sum_{|\gamma|\le T}|\Psi(\lambda,\varrho)|\\ &\ll q^{-1/2+\varepsilon}\min(U,|\lambda|^{-1/3})\sum_{\chi \bmod q}\sum_{|\gamma|\le T}U^{\beta-1}.\end{aligned}$$

By Lemma 3.3, we have

$$\begin{aligned}\sum_{\chi \bmod q}\sum_{|\gamma|\le T}U^{\beta-1} &= -\int_{1/2}^{1}U^{\sigma-1}\,dN(\sigma,T,q)\\ &\ll L\max_{1/2\le\sigma\le 1}N^{(\sigma-1)/3}(qT)^{(12/5+\varepsilon)(1-\sigma)}\\ &\ll L\max_{1/2\le\sigma\le 1}N^{(114\theta/25-1/3+\varepsilon)(1-\sigma)},\end{aligned}$$

where we have used $q\le P=N^{\theta}$ and $T=P^{9/10+2\varepsilon}$. The last maximum is $\ll 1$ when $\theta<1/19.08$. This in combination with (3.9) gives (3.6) for $S_2(\lambda)$.

Using the elementary estimate

$$\int_{U^3}^{8U^3}e(\lambda u)\,du \ll \min(U^3,|\lambda|^{-1})$$

and integrating by parts, one has

$$\Phi(\lambda)=\frac{1}{3}\int_{U^3}^{8U^3}u^{-2/3}e(\lambda u)\,du \ll U^{-2}\min(U^3,|\lambda|^{-1})\ll \min(U,|\lambda|^{-1/3}).$$

Thus the estimate (3.6) for $S_1(\lambda)$ now follows from this and (3.4).

Secondly we show that, on substituting $S_1(\lambda)+S_2(\lambda)$ for $S(\alpha)$ in the integral in Lemma 3.2, the resulting error is acceptable, i.e.

$$\begin{aligned}(3.10)\qquad &\int_{\mathfrak{M}}S^5(\alpha)e(-n\alpha)\,d\alpha\\ &-\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}e\Big(-\frac{an}{q}\Big)\int_{-1/(qQ)}^{1/(qQ)}\{S_1(\lambda)+S_2(\lambda)\}^5e(-n\lambda)\,d\lambda \ll U^2L^{-A}.\end{aligned}$$

By Hölder's inequality, the left-hand side of (3.10) is

$$(3.11)\qquad \ll \sum_{\substack{i+j+k=5\\k\ge1}}\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}e\Big(-\frac{an}{q}\Big)\int_{-1/(qQ)}^{1/(qQ)}|S_1|^i|S_2|^j|S_3|^k\,d\lambda$$

$$\ll \sum_{\substack{i+j+k=5\\k\ge1}} \sum_{q\le P} \varphi(q)\Big\{\int\limits_{-1/(qQ)}^{1/(qQ)} |S_1|^5\,d\lambda\Big\}^{i/5}$$

$$\times\Big\{\int\limits_{-1/(qQ)}^{1/(qQ)} |S_2|^5\,d\lambda\Big\}^{j/5}\Big\{\int\limits_{-1/(qQ)}^{1/(qQ)} |S_3|^5\,d\lambda\Big\}^{k/5}.$$

By (3.6),

$$\int\limits_{-1/(qQ)}^{1/(qQ)} |S_1|^5\,d\lambda \ll q^{-5/2+5\varepsilon}L^{10}\Big\{\int\limits_{0}^{U^{-3}} U^5\,d\lambda + \int\limits_{U^{-3}}^{\infty} \lambda^{-5/3}\,d\lambda\Big\}$$
$$\ll q^{-5/2+5\varepsilon}U^2L^{10},$$

and similarly,

$$\int\limits_{-1/(qQ)}^{1/(qQ)} |S_2|^5\,d\lambda \ll q^{-5/2+5\varepsilon}U^2L^{10}.$$

A similar argument also gives

$$\int\limits_{-1/(qQ)}^{1/(qQ)} |S_3|^5\,d\lambda \ll q^{5/2+5\varepsilon}\frac{U^2}{T^5}L^{10}.$$

By inserting these estimates into (3.11), the left-hand side of (3.10) is estimated as

$$\ll U^2L^{10}\sum_{1\le k\le5} T^{-k}\sum_{q\le P} q^{k-3/2+5\varepsilon} \ll U^2L^{-A},$$

on recalling that $T = P^{9/10+2\varepsilon}$. This proves (3.10).

Finally we extend the interval of integration in the second integral in (3.10) to $(-\infty,\infty)$. By (3.6), the resulting error is

$$\ll \sum_{q\le P}\sum_{a=1}^{q}\int\limits_{1/(qQ)}^{\infty} |S_1(\lambda)+S_2(\lambda)|^5\,d\lambda \ll L^{10}\sum_{q\le P} q^{-3/2+5\varepsilon}\int\limits_{1/(qQ)}^{\infty}|\lambda|^{-5/3}\,d\lambda$$
$$\ll L^{10}\sum_{q\le P} q^{-5/6+5\varepsilon}Q^{2/3} \ll U^2P^{-1/2+5\varepsilon}L^{10} \ll U^2L^{-A}.$$

Therefore (3.10) becomes

$$\int\limits_{\mathfrak{M}} S^5(\alpha)e(-n\alpha)\,d\alpha$$
$$= \sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\Big(-\frac{an}{q}\Big)\int\limits_{-\infty}^{\infty}\{S_1(\lambda)+S_2(\lambda)\}^5e(-n\lambda)\,d\lambda + O(U^2L^{-A}),$$

and Lemma 3.2 clearly follows.

4. Estimation of I_j and the proof of Theorem 2. The purpose of this section is to establish the following Lemmas 4.1 and 4.2. At the end of this section, we derive Theorem 2 from these two lemmas.

LEMMA 4.1. *Let I_j be defined as in* (3.5) *with n and θ as in Theorem* 2. *Then for all I_j with $j = 1, \ldots, 5$, we have*

$$I_j \ll U^2 L^{-A}. \tag{4.1}$$

LEMMA 4.2. *With the notation of Theorem* 2, *we have*

$$I_0 = \mathfrak{S}(n)J(n) + O(U^2 L^{-A}),$$

where $J(n) = J(n; 1, 1, \ldots, 1)$ is defined as in (4.4), *and satisfies $U^2 \ll J(n) \ll U^2$.*

We need some more notation. Let $C(\chi, a)$ and $C(q, a)$ be defined as in (3.1). If $\chi_1, \ldots, \chi_5$ are characters mod q, then we write

$$\begin{gathered} B(n, q, \chi_1, \ldots, \chi_5) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\Big(-\frac{an}{q}\Big) C(\chi_1, a) \ldots C(\chi_5, a), \\ B(n, q) = B(n, q, \chi_1^0, \ldots, \chi_5^0), \end{gathered} \tag{4.2}$$

and

$$A(n, q) = \frac{B(n, q)}{\varphi^5(q)}, \qquad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q). \tag{4.3}$$

This $\mathfrak{S}(n)$ is the singular series appearing in Theorem 2.

LEMMA 4.3. (i) *We have*

$$|A(n, q)| \ll q^{-3/2+\varepsilon}.$$

(ii) *Let χ_j mod r_j with $j = 1, \ldots, 5$ be primitive characters, $r_0 = [r_1, \ldots, r_5]$, and χ^0 the principal character mod q. Then*

$$\sum_{\substack{q \le x \\ r_0 \mid q}} \frac{1}{\varphi^5(q)} |B(n, q, \chi_1\chi^0, \ldots, \chi_5\chi^0)| \ll r_0^{-3/2+\varepsilon}.$$

Proof. We only prove (ii), since the proof for (i) is similar. Using (3.4), one has

$$B(n, q, \chi_1\chi^0, \ldots, \chi_5\chi^0) \ll \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \prod_{j=1}^{5} |C(\chi_j\chi^0, a)| \ll q^{7/2} d^{10}(q),$$

and consequently,

$$\sum_{\substack{q\le x\\ r_0|q}} \frac{1}{\varphi^5(q)}|B(n,q,\chi_1\chi^0,\dots,\chi_5\chi^0)| \ll \sum_{\substack{q\le x\\ r_0|q}} \frac{q^{7/2}d^{10}(q)}{\varphi^5(q)} \ll \sum_{\substack{q\le x\\ r_0|q}} q^{-3/2+\varepsilon} \ll r_0^{-3/2+\varepsilon},$$

as required.

We record the following lemma, which is a modification of Lemma 4.7 of Liu and Tsang [9].

LEMMA 4.4. *Let ϱ_j be any complex numbers with $0 < \operatorname{Re}\varrho_j \le 1$, $j = 1,\dots,5$. Then*

$$\int_{-\infty}^{\infty} e(-n\lambda)\prod_{j=1}^{5}\Psi(\lambda,\varrho_j)\,d\lambda = \frac{1}{3^5}\int_{\mathfrak{D}} u_1^{\varrho_1/3-1}\dots u_5^{\varrho_5/3-1}du_1\dots du_4 =: J(n;\varrho_1,\dots,\varrho_5), \tag{4.4}$$

where

$$\mathfrak{D} = \{(u_1,\dots,u_4) : U^3 \le u_1,\dots,u_5 \le 8U^3\} \tag{4.5}$$

with $u_5 = n - u_1 - \dots - u_4$.

Now we establish the main results of this section.

Proof of Lemma 4.1. We treat the case $j = 5$ in (4.1) in detail. In other cases, the proof for (4.1) is similar and better ranges of θ (in (2.1)) suffice, so we will only give a sketch.

By (3.5), Lemma 4.4 and (4.2), we have

$$I_5 = \sum_{q\le P}\frac{1}{\varphi^5(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\Big(-\frac{an}{q}\Big) \times \int_{-\infty}^{\infty}\Big\{\sum_{\chi \bmod q} C(\chi,a)\sum_{|\gamma|\le T}\Psi(\lambda,\varrho)\Big\}^5 e(-n\lambda)\,d\lambda \tag{4.6}$$

$$= \sum_{q\le P}\sum_{\chi_1 \bmod q}\dots\sum_{\chi_5 \bmod q}\frac{B(n,q,\chi_1,\dots,\chi_5)}{\varphi^5(q)}\sum_{|\gamma_1|\le T}\dots\sum_{|\gamma_5|\le T} J(n;\varrho_1,\dots,\varrho_5),$$

where $J(n;\varrho_1,\dots,\varrho_5)$ denotes the quantity on the right hand side of (4.4) with $\mathfrak{D}$ defined by (4.5). Now we recall that if a primitive character $\chi \bmod r$ induces a character $\psi \bmod k$, then $r\mid k$ and $\psi = \chi\chi^0$, where χ^0 is the principal character modulo k. Collecting all contributions made by an individual primitive character, and then appealing to Lemma 4.3 with $r_0 = [r_1, \ldots, r_5]$, one has

$$(4.7)\quad I_5 = \sum_{r_1 \le P} \cdots \sum_{r_5 \le P} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \cdots \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \sum_{|\gamma_1| \le T} \cdots \sum_{|\gamma_5| \le T} J(n; \varrho_1, \ldots, \varrho_5) \times \sum_{\substack{q \le P \\ r_0 | q}} \frac{B(n, q, \chi_1\chi^0, \ldots, \chi_5\chi^0)}{\varphi^5(q)}$$

$$\ll \sum_{r_1 \le P} \cdots \sum_{r_5 \le P} r_0^{-3/2+\varepsilon} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \cdots \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \sum_{|\gamma_1| \le T} \cdots \sum_{|\gamma_5| \le T} |J(n; \varrho_1, \ldots, \varrho_5)|.$$

Now we come to an upper bound estimate for $J(n; \varrho_1, \ldots, \varrho_5)$. By definition,

$$J(n; \varrho_1, \ldots, \varrho_5) = \int_{U^3}^{8U^3} u_1^{\varrho_1/3-1} du_1 \int_{U^3}^{8U^3} u_2^{\varrho_2/3-1} du_2 \ldots \int_{\max(U^3, x-8U^3)}^{\min(8U^3, x-U^3)} u_4^{\varrho_4/3-1} (x-u_4)^{\varrho_5/3-1} du_4,$$

where we have written

$$x = n - u_1 - \ldots - u_3.$$

Clearly, if $x \le 2U^3$ then $x - U^3 \le U^3$, and hence the innermost integral is 0. For $x > 2U^3$, we make the substitution $u_4 = xu$, so that

$$\int_{\max(U^3, x-8U^3)}^{\min(8U^3, x-U^3)} u_4^{\varrho_4/3-1} (x-u_4)^{\varrho_5/3-1} du_4 \ll x^{\beta_4/3+\beta_5/3-1} \int_0^{1-U^3/x} u^{\beta_4/3-1} (1-u)^{\beta_5/3-1}\, du$$

$$\ll U^{\beta_4+\beta_5-3} \int_0^1 u^{-5/6} (1-u)^{-5/6}\, du \ll U^{\beta_4+\beta_5-3}.$$

Estimating the other integrals in $J(n; \varrho_1, \ldots, \varrho_5)$ trivially, we obtain the following bound:

$$(4.8)\qquad J(n; \varrho_1, \ldots, \varrho_5) \ll U^2 U^{\beta_1+\ldots+\beta_5-5}.$$

Since $r_0 = [r_1, \ldots, r_5] \ge r_j$ for $j = 1, \ldots, 5$, one has

$$(4.9)\qquad r_0^{-3/2+\varepsilon} \le r_1^{-3/10+\varepsilon} \ldots r_5^{-3/10+\varepsilon}.$$

Inserting (4.8) and (4.9) into (4.7), one gets

$$
\begin{aligned}
(4.10)\qquad I_5 &\ll U^2 \sum_{r_1\le P} r_1^{-3/10+\varepsilon} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \sum_{|\gamma_1|\le T} U^{\beta_1-1} \dots \\
&\quad\times \sum_{r_5\le P} r_5^{-3/10+\varepsilon} \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \sum_{|\gamma_5|\le T} U^{\beta_5-1} \\
&\ll U^2 \Big\{ \sum_{r\le P} r^{-3/10+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \sum_{|\gamma|\le T} U^{\beta-1} \Big\}^5 \\
&=: U^2\{J_5+K_5\}^5,
\end{aligned}
$$

where J_5 and K_5 denote the contribution from $r \le L^A$ and $L^A < r \le P$ respectively. By Lemma 3.3, K_5 can be easily estimated as

$$
\begin{aligned}
(4.11)\qquad K_5 &= \sum_{L^A<r\le P} r^{-3/10+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \sum_{|\gamma|\le T} U^{\beta-1} \\
&\ll L \max_{L^A<R\le P} R^{-3/10+\varepsilon} \sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \sum_{|\gamma|\le T} U^{\beta-1} \\
&\ll L \max_{L^A<R\le P} \max_{1/2\le\sigma\le 1} R^{-3/10+\varepsilon} (R^2T)^{(12/5+\varepsilon)(1-\sigma)} U^{\sigma-1}.
\end{aligned}
$$

The total exponent of R in the last line is

$$
f(\sigma) := \left(\frac{24}{5}+2\varepsilon\right)(1-\sigma) - \frac{3}{10} + \varepsilon,
$$

say. Obviously $f(\sigma) \ge 0$ for $1/2 \le \sigma \le \sigma_0$, and $f(\sigma) < 0$ for $\sigma_0 < \sigma \le 1$, where $\sigma_0 = (45+30\varepsilon)/(48+20\varepsilon)$. Thus, (4.11) becomes

$$
\begin{aligned}
(4.12)\qquad K_5 &\ll L \max_{1/2\le\sigma\le\sigma_0} P^{f(\sigma)} T^{(12/5+\varepsilon)(1-\sigma)} U^{\sigma-1} \\
&\quad + L \max_{\sigma_0<\sigma\le 1} L^{Af(\sigma)} T^{(12/5+\varepsilon)(1-\sigma)} U^{\sigma-1} \\
&=: K_{51} + K_{52},
\end{aligned}
$$

say. Clearly,

$$
K_{51} \ll L \max_{1/2\le\sigma\le\sigma_0} N^{\{(174/25+8\varepsilon)(1-\sigma)-3/10+\varepsilon\}\theta-(1-\sigma)/3} \ll L^{-A}
$$

if $\theta < 1/19.08$. We remark that it is this estimate for K_{51} that requires $\theta < 1/19.08$ exactly. To estimate K_{52}, we note that

$$
K_{52} \ll L \max_{\sigma_0<\sigma\le 99/100} N^{\{(54/25+7\varepsilon)\theta-1/3\}(1-\sigma)} + L \max_{99/100\le\sigma\le 1} L^{Af(\sigma)}.
$$

For $\sigma_0 < \sigma \le 99/100$, the exponent of N above is $\le -1/460$; while for $99/100 \le \sigma \le 1$, we have $f(\sigma) \le -1/4$. Hence, we have $K_{52} \ll L^{-A/5}$, and consequently (4.12) becomes

$$K_5 \ll L^{-A/5}. \tag{4.13}$$

Now we turn to J_5. By Satz VIII.6.2 of Prachar [11], there exists a positive constant c_1 such that $\prod_{\chi \bmod q} L(s,\chi)$ is zero-free in the region

$$\sigma \ge 1 - c_1/\max\{\log q, \log^{4/5} N\}, \qquad |t| \le N, \tag{4.14}$$

except for the possible Siegel zero. But by Siegel's theorem (see [1], §21), the Siegel zero does not exist in this situation, since $q \le L^A$. Let $\eta(N) = c_1 \log^{-4/5} N$. Then Lemma 3.3 gives

$$\begin{aligned} J_5 &\ll \sum_{r \le L^A} \sideset{}{^*}\sum_{\chi \bmod r} \sum_{|\gamma| \le T} U^{\beta-1} \\ &\ll L \max_{1/2 \le \sigma \le 1-\eta(N)} (L^{2A}T)^{(12/5+\varepsilon)(1-\sigma)} U^{\sigma-1} \\ &\ll L^{4A} \max_{1/2 \le \sigma \le 1-\eta(N)} N^{(54\theta/25-1/3+8\varepsilon)(1-\sigma)} \ll L^{4A} N^{-c_2\eta(N)} \\ &\ll L^{4A} \exp\{-c_3 L^{1/5}\} \end{aligned} \tag{4.15}$$

provided $\theta < 1/19.08$. Inserting (4.13) and (4.15) into (4.10), we get (4.1) for $j = 5$. This completes the proof of Lemma 4.1 for $j = 5$.

To conclude the proof, we need to sketch how to estimate $I_1, \ldots, I_4$. As an example, we only consider I_4. By definition, and an argument similar to that leading to (4.6) and (4.7), we have

$$\begin{aligned} I_4 &= 5 \sum_{q \le P} \sum_{\substack{a=1 \\ (a,q)=1}}^{5} e\left(-\frac{an}{q}\right) \int_{-\infty}^{\infty} S_1(\lambda) S_2^4(\lambda) e(-n\lambda)\, d\lambda \\ &= 5 \sum_{q \le P} \sum_{\chi_1 \bmod q} \cdots \sum_{\chi_4 \bmod q} \frac{B(n,q,\chi_1,\ldots,\chi_4,\chi^0)}{\varphi^5(q)} \\ &\quad \times \sum_{|\gamma_1| \le T} \cdots \sum_{|\gamma_4| \le T} J(n;\varrho_1,\ldots,\varrho_4,1) \\ &\ll \sum_{r_1 \le P} \cdots \sum_{r_4 \le P} r_0^{-3/2+\varepsilon} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \cdots \sideset{}{^*}\sum_{\chi_4 \bmod r_4} \sum_{|\gamma_1| \le T} \cdots \sum_{|\gamma_4| \le T} |J(n;\varrho_1,\ldots,\varrho_4,1)|, \end{aligned}$$

where $r_0 = [r_1, \ldots, r_4, 1]$. Now instead of (4.8) and (4.9), we have respectively

$$J(n;\varrho_1,\ldots,\varrho_4,1) \ll U^2 U^{\beta_1+\ldots+\beta_4-4},$$

and

$$r_0^{-3/2+\varepsilon} \le r_1^{-3/8+\varepsilon} \ldots r_4^{-3/8+\varepsilon}.$$

So corresponding to (4.10), I_4 can now be estimated as

$$I_4 \ll U^2\Big\{\sum_{r\le P} r^{-3/8+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \sum_{|\gamma|\le T} U^{\beta-1}\Big\}^4.$$

Since $r^{-3/8+\varepsilon} \le r^{-3/10+\varepsilon}$, we have $I_4 \ll L^{-A}$ for $\theta < 1/19.08$ by the same method as for J_5 and K_5 above. Actually a better range for θ suffices for the purpose.

This completes the proof of Lemma 4.1.

Proof of Lemma 4.2. By (3.5),

$$I_0 = \sum_{q\le P} \frac{1}{\varphi^5(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} C^5(q,a)e\left(-\frac{an}{q}\right) \int_{-\infty}^{\infty} \Phi^5(\lambda)e(-n\lambda)\,d\lambda.$$

Using Lemma 4.4 again, one gets

$$I_0 = \sum_{q\le P} \frac{B(n,q)}{\varphi^5(q)} J(n;1,\ldots,1) = J(n)\sum_{q\le P} A(n,q), \tag{4.16}$$

where we recall that $J(n) = J(n;1,\ldots,1)$. By Lemma 4.3(i), we have

$$\sum_{q\le P} A(n,q) = \mathfrak{S}(n) + O(P^{-1/2+\varepsilon})$$

and consequently (4.16) becomes

$$I_0 = \mathfrak{S}(n)J(n) + O(U^2L^{-A}).$$

Here in the O-term we have used the upper bound of the estimate

$$U^2 \ll J(n) \ll U^2, \tag{4.17}$$

which will be established in the next paragraph.

We first note that the second inequality in (4.17) is a consequence of (4.8). To bound $J(n)$ from below, we define the set

$$\mathfrak{D}^* = \{(u_1,\ldots,u_4) : U^3 \le u_1,\ldots,u_4 \le 5U^3/4\}.$$

For $(u_1,\ldots,u_4)\in\mathfrak{D}^*$, one deduces from $N/2 < n \le N$ that

$$U^3 \le u_5 = n - u_1 - \ldots - u_4 \le 8U^3.$$

Thus $\mathfrak{D}^*$ is a subset of the $\mathfrak{D}$ in (4.5), and consequently,

$$J(n) \ge \frac{1}{3^5}\int_{\mathfrak{D}^*} u_1^{-2/3}\ldots u_4^{-2/3}u_5^{-2/3}du_1\ldots du_4 \gg U^2.$$

This proves (4.17) hence Lemma 4.2.

Proof of Theorem 2. The absolute convergence and positivity of $\mathfrak{S}(n)$ have been proved in Lemmas 8.10 and 8.12 of Hua [5] respectively. Other assertions of Theorem 2 follow from Lemmas 3.2, 4.1, and 4.2.

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