ACTA ARITHMETICA
XCIV.1 (2000)

The Lucas congruence for Stirling numbers of the second kind

by

ROBERTO SÁNCHEZ-PEREGRINO (Padova)

0. Introduction. The numbers introduced by Stirling in 1730 in his *Methodus differentialis* [11], subsequently called "Stirling numbers" of the first and second kind, are of the greatest utility in the calculus of finite differences, in number theory, in the summation of series, in the theory of algorithms, in the calculation of the Bernstein polynomials [9]. In this study, we demonstrate some properties of Stirling numbers of the second kind similar to those satisfied by binomial coefficients; in particular we show that they satisfy a congruence analogous to that of Lucas, that is to: $\binom{a}{b} \equiv \prod_{i=0}^{n} \binom{a_i}{b_i} \bmod p$ with $a = \sum_{i=0}^{n} a_i p^i, b = \sum_{i=0}^{n} b_i p^i$; $0 \le a_i \le p-1$, $0 \le b_i \le p-1$. Using Proposition 4.1 we give another proof for Kaneko's recurrence formula for poly-Bernoulli numbers [10]. Some of the results are similar to those of Howard [5].

In conclusion, I wish to give my best thanks to the Geometry Group of the Dipartimento di Matematica Pura ed Applicata and Dipartimento di Metodi Matematici per le Scienze Applicate of the University of Padova, for support and help given during the preparation of this work. In particular, I wish to thank Frank Sullivan for his precious advice and suggestions.

1. Notations and definitions. In this section, we will review various definitions and notations for Stirling numbers of the second kind. Let $s, t \in \mathbb{N}$. We set

$$\left\{ {t \atop s} \right\} = \begin{cases} 1 & \text{if } t = 0,\ s = 0, \\ 0 & \text{if } t > 0,\ s = 0, \\ 0 & \text{if } t = 0,\ s > 0, \\ s\left\{ {t-1 \atop s} \right\} + \left\{ {t-1 \atop s-1} \right\} & \text{if } t > 0,\ s > 0. \end{cases}$$

2000 *Mathematics Subject Classification*: 11A07, 11B73.

Definition 1.1. The numbers represented by the symbols $\left\{ {t \atop s} \right\}$ are called *Stirling numbers of the second kind.*

As this note deals only with the Stirling numbers of the second kind, we will call them simply Stirling numbers. Stirling numbers of the second kind are also indicated in the literature with other symbols: $\mathrm{S}(t,s)$ [3]; $\mathfrak{S}_t^s$ [6]. The notation used here has been proposed by D. Knuth [4]; following his advice we may read it as "t bracket s".

With Definition 1.1 the integers t, s are assumed non-negative. Nevertheless, it is useful to simplify notation in all the necessary passages, giving a value to the number $\left\{ {t \atop s} \right\}$ even when $s<0$. In this case, we conventionally put $\left\{ {t \atop s} \right\} = 0$. For $(x)_n = x(x-1)\dots(x-n+1)$, with simple considerations one verifies that $x^n = \sum_{s=0}^n \left\{ {n \atop s} \right\}(x)_s$. Traditionally, this is the way in which Stirling numbers are introduced [6]. It should be noticed, moreover, that the Stirling number $\left\{ {n \atop k} \right\}$ is equal to the number of partitions of the set $\{1,\dots,n\}$ into k-blocks [2, 3]. Moreover, Stirling numbers have the following properties:

1. $\left\{ {t \atop s} \right\} = 0$ if $s>t$;
2. $\left\{ {t \atop s} \right\} = \dfrac{1}{s!}\displaystyle\sum_{j=0}^{s}(-1)^j\binom{s}{j}(s-j)^t$.

See [6], pages 168 and 169, for demonstrations.

2. Addition formula. As a consequence of

$$\sum_{0\le j\le i}(-1)^j\binom{i}{j}(i-j)=0$$

and of property 2, we have the following

Proposition 2.1. *For each prime number p, $p>2$, the Stirling numbers satisfy*

$$\left\{ {p \atop i} \right\} \equiv 0 \bmod p \qquad \textit{for each } i\neq 1,\ p.$$

Proposition 2.2 (addition). *Let $x,t,n\in\mathbb{N}$. Then*

$$\left\{ {x+t \atop n} \right\} = \sum_{j=0}^{t}\sum_{i=0}^{t}\binom{t}{i}n^i(-1)^{t+i+j}\left\{ {t-i \atop j} \right\}\left\{ {x \atop n-j} \right\}. \tag{1}$$

Proof. The proof is by induction on t. When $t=1$, the right hand side of (1) reduces to

$$\binom{1}{0}n^0(-1)^1\left\{{1 \atop 0}\right\}\left\{{x \atop n}\right\}+\binom{1}{1}n^1(-1)^2\left\{{0 \atop 0}\right\}\left\{{x \atop n}\right\}$$
$$+\binom{1}{0}n^0(-1)^2\left\{{t \atop 1}\right\}\left\{{x \atop n-1}\right\}+\binom{1}{1}n(-1)^3\left\{{1 \atop 1}\right\}\left\{{x \atop n-1}\right\}$$
$$=n\left\{{x \atop n}\right\}+\left\{{x \atop n-1}\right\}.$$

However, the latter is equal to $\left\{{x+1 \atop n}\right\}$. Thus, formula (1) holds when $t=1$. Now it remains to demonstrate that if (1) is true for a natural number $t>1$, then it is also true for $t+1$:

$$\left\{{x+1+t \atop n}\right\}$$
$$=\sum_{j=0}^{t}\sum_{i=0}^{t}\binom{t}{i}n^i(-1)^{t+j+i}\left\{{t-i \atop j}\right\}\left\{{x+1 \atop n-j}\right\}$$
$$=\sum_{j=0}^{t}\sum_{i=0}^{t}\binom{t}{i}n^i(-1)^{t+j+i}\left\{{t-i \atop j}\right\}\left[(n-j)\left\{{x \atop n-j}\right\}+\left\{{x \atop n-j-1}\right\}\right]$$
$$=\sum_{j=0}^{t}\sum_{i=0}^{t}\binom{t}{i}n^{i+1}(-1)^{t+j+i}\left\{{t-i \atop j}\right\}\left\{{x \atop n-j}\right\}$$
$$+\sum_{j=0}^{t}\sum_{i=0}^{t}\binom{t}{i}n^i(-1)^{t+1+i+j}j\left\{{t-i \atop j}\right\}\left\{{x \atop n-j}\right\}$$
$$+\sum_{j=0}^{t}\sum_{i=0}^{t}\binom{t}{i}n^i(-1)^{t+i+j}\left\{{t-i \atop j}\right\}\left\{{x \atop n-j-1}\right\}$$
$$=\sum_{j=0}^{t}\sum_{i=0}^{t}\binom{t}{i}n^{i+1}(-1)^{t+j+i}\left\{{t-i \atop j}\right\}\left\{{x \atop n-j}\right\}$$
$$+\sum_{f=1}^{t}\sum_{i=0}^{t}\binom{t}{i}n^i(-1)^{t+1+f+i}f\left\{{t-i \atop f}\right\}\left\{{x \atop n-f}\right\}$$
$$+\sum_{f=1}^{t}\sum_{i=0}^{t}\binom{t}{i}n^i(-1)^{t+1+f+i}\left\{{t-i \atop f-1}\right\}\left\{{x \atop n-f}\right\}$$
$$+\sum_{i=0}^{t}\binom{t}{i}n^i(-1)^{t+t+i}\left\{{t-i \atop t}\right\}\left\{{x \atop n-t-1}\right\}$$

$$= \sum_{j=0}^{t}\sum_{i=0}^{t}\binom{t}{i}n^{i+1}(-1)^{t+j+i}\left\{{t-i \atop j}\right\}\left\{{x \atop n-j}\right\}$$

$$+ \sum_{f=1}^{t}\sum_{i=0}^{t}\binom{t}{i}n^{i}(-1)^{t+1+f+i}\left\{{t+1-i \atop f}\right\}\left\{{x \atop n-f}\right\}$$

$$+ \sum_{i=0}^{t}\binom{t}{i}n^{i}(-1)^{t+t+i}\left\{{t-i \atop t}\right\}\left\{{x \atop n-t-1}\right\}$$

$$(2) \; = \sum_{f=1}^{t}\sum_{h=1}^{t+1}\binom{t}{h-1}n^{h}(-1)^{t+h-1+f}\left\{{t+1-h \atop f}\right\}\left\{{x \atop n-f}\right\} + \left\{{x \atop n}\right\}n^{t+1}$$

$$+ \sum_{f=1}^{t}\sum_{i=0}^{t}\binom{t}{i}n^{i}(-1)^{t+1+f+i}\left\{{x \atop n-f}\right\}\left\{{t+1-i \atop f}\right\} + \left\{{x \atop n-t-1}\right\}.$$

If $y > x$, $\binom{x}{y} = 0$. Therefore, in the third addend of (2) we can extend the second sum to $i = t+1$, and (2) becomes

$$\sum_{f=1}^{t}\sum_{h=1}^{t+1}\left[\binom{t}{h-1}+\binom{t}{h}\right]n^{h}(-1)^{t+1+h+f}\left\{{x \atop h-f}\right\}\left\{{t+1-h \atop f}\right\}$$

$$+ \sum_{f=1}^{t}\binom{t}{0}(-1)^{t+1+f}\left\{{x \atop n-f}\right\}\left\{{t+1 \atop f}\right\} + \left\{{x \atop n}\right\}n^{t+1} + \left\{{x \atop n-t-1}\right\}$$

$$= \sum_{f=1}^{t}\sum_{h=1}^{t+1}\binom{t+1}{h}n^{h}(-1)^{t+1+h+f}\left\{{x \atop h-f}\right\}\left\{{t+1-h \atop f}\right\}$$

$$+ \sum_{f=1}^{t}\binom{t+1}{0}(-1)^{t+1+f}\left\{{x \atop n-f}\right\}\left\{{t+1 \atop f}\right\}$$

$$+ \left\{{x \atop n}\right\}n^{t+1} + \left\{{x \atop n-t-1}\right\}$$

$$= \sum_{f=0}^{t+1}\sum_{h=0}^{t+1}\binom{t+1}{h}n^{h}(-1)^{t+1+h+f}\left\{{t+1-h \atop f}\right\}\left\{{x \atop n-f}\right\}. \;\blacksquare$$

Corollary 2.1. *If $t = p$ then*

$$\left\{{x+p \atop n}\right\} \equiv \left\{{x+1 \atop n}\right\} + \left\{{x \atop n-p}\right\} \bmod p.$$

This formula has been demonstrated by Becker & Riordan [1], using another method.

LEMMA 2.1. *Let $t, x_0, y_0 \in \mathbb{N}$. Then*

$$\left\{ \begin{matrix} y_0 + tp \\ x_0 + tp \end{matrix} \right\} \equiv \sum_{i=0}^{t} \binom{t}{i} \left\{ \begin{matrix} y_0 + i \\ x_0 + ip \end{matrix} \right\} \bmod p. \tag{3}$$

The proof is by induction on t. If $t = 1$, then (3) is true according to Corollary 2.1. It remains to be shown that if (3) is true for a natural number $t > 1$, then it also holds for $t+1$. To do this, we can use a procedure analogous to that used to prove Newton's Binomial Formula.

REMARK 2.1. This lemma is similar to Theorem 4.2 of Howard [5] in the case $0 \le x_0, y_0 < p$.

This lemma, together with an easy induction argument on r, gives a more elementary proof of Theorem 4.4 of Howard [5].

COROLLARY 2.2. *Let $i, x, y \in \mathbb{N}$. Then*

$$\left\{ \begin{matrix} y + p^i \\ x \end{matrix} \right\} \equiv \left\{ \begin{matrix} y + 1 \\ x \end{matrix} \right\} + \sum_{j=1}^{i} \left\{ \begin{matrix} y \\ x - p^j \end{matrix} \right\} \bmod p.$$

3. First approach to the Lucas Theorem

PROPOSITION 3.1. *Let $x_0, y_0, s, t \in \mathbb{N}$ with $0 \le x_0 \le y_0 \le p-1$, $s \le t \le p-1$. Then*

$$\left\{ \begin{matrix} y_0 + tp \\ x_0 + sp \end{matrix} \right\} \equiv \left\{ \begin{matrix} y_0 + t - s \\ x_0 \end{matrix} \right\} \binom{t}{s} + \left\{ \begin{matrix} y_0 + t - s + 1 \\ x_0 + p \end{matrix} \right\} \binom{t}{s-1} \bmod p. \tag{4}$$

Proof. Let $s_0 = t - s$, from Lemma 2.1 it follows that

$$\left\{ \begin{matrix} y_0 + tp \\ x_0 + sp \end{matrix} \right\} = \left\{ \begin{matrix} y_0 + s_0 p + sp \\ x_0 + sp \end{matrix} \right\} \equiv \sum_{i=0}^{s} \binom{s}{i} \left\{ \begin{matrix} y_0 + s_0 p + i \\ x_0 + ip \end{matrix} \right\} \bmod p$$

$$\equiv \sum_{i=0}^{s} \sum_{j=0}^{s_0} \binom{s}{i} \binom{s_0}{j} \left\{ \begin{matrix} y_0 + i + j \\ x_0 + (i + j - s_0)p \end{matrix} \right\} \bmod p$$

$$\equiv \sum_{i=0}^{s} \sum_{m=i}^{s_0+i} \binom{s}{i} \binom{s_0}{m-i} \left\{ \begin{matrix} y_0 + m \\ x_0 + mp - s_0 p \end{matrix} \right\} \bmod p. \tag{5}$$

Addends of (5) which correspond to $m \ge s_0 + 2$ are null in view of property 1 of Section 1, because, in this case, $y_0 + m < x_0 + (m - s_0)p$. Addends for which $m \le s_0 - 1$ are null, since $x_0 + (m - s_0)p < 0$. Hence, we conclude that the second sum in (5) reduces to only the addends for which $m = s_0$ and $m = s_0 + 1$:

$$\sum_{i=0}^{s} \left[\binom{s}{i} \binom{s_0}{s_0 - i} \left\{ \begin{matrix} y_0 + s_0 \\ x_0 \end{matrix} \right\} + \binom{s}{i} \binom{s_0}{s_0 + 1 - i} \left\{ \begin{matrix} y_0 + s_0 + 1 \\ x_0 + p \end{matrix} \right\} \right] \tag{6}$$

$$= \left\{ \begin{matrix} y_0 + s_0 \\ x_0 \end{matrix} \right\} \sum_{i=0}^{s} \binom{s}{i} \binom{s_0}{s_0 - i} + \left\{ \begin{matrix} y_0 + s_0 + 1 \\ x_0 + p \end{matrix} \right\} \sum_{i=0}^{s} \binom{s}{i} \binom{s_0}{s_0 + 1 - i}.$$

If we use Vandermonde's equality [6] for binomial coefficients, equation (6) reduces to

$$\binom{s + s_0}{s} \left\{ \begin{matrix} y_0 + s_0 \\ x_0 \end{matrix} \right\} + \binom{s + s_0}{s_0 + 1} \left\{ \begin{matrix} y_0 + s_0 + 1 \\ x_0 + p \end{matrix} \right\}. \blacksquare$$

Observation. With the hypothesis that $y_0 + t - s + 1 < x_0 + p$, (4) reduces to

$$\left\{ \begin{matrix} y_0 + tp \\ x_0 + sp \end{matrix} \right\} \equiv \left\{ \begin{matrix} y_0 + t - s \\ x_0 \end{matrix} \right\} \binom{t}{s} \bmod p.$$

It is useful to notice that this formula is very similar to Lucas' formula for binomial coefficients.

Remark 3.1. In the case $r < p$ the congruence (4) gives the formulas (4.17) and (4.18) of Howard [5].

4. The Lucas Theorem

Proposition 4.1. *Let* $x, y, a, n \in \mathbb{N}$. *Then*

$$\left\{ \begin{matrix} y + ap^n \\ x \end{matrix} \right\} \equiv \sum_{l_0 + l_1 + \ldots + l_n = a} \binom{a}{l_0, l_1, \ldots, l_n} \left\{ \begin{matrix} y + l_0 \\ x - \sum_{k=1}^{n} l_k p^k \end{matrix} \right\} \bmod p. \tag{7}$$

Proof. The proof is by induction on a. First of all, according to Corollary 2.2 formula (7) is true when $a = 1$. It remains to be shown that if (7) is true for a natural number $a \geq 1$, then it is also true for $a + 1$:

$$\begin{aligned} &\left\{ \begin{matrix} y + p^n + ap^n \\ x \end{matrix} \right\} \\ &\qquad \equiv \sum_{l_0 + l_1 + \ldots + l_n = a} \binom{a}{l_0, l_1, \ldots, l_n} \left\{ \begin{matrix} y + p^n + l_0 \\ x - \sum_{k=1}^{n} l_k p^k \end{matrix} \right\} \bmod p \\ &\qquad \equiv \sum_{l_0 + l_1 + \ldots + l_n = a} \binom{a}{l_0, l_1, \ldots, l_n} \Bigg[\left\{ \begin{matrix} y + l_0 + 1 \\ x - \sum_{k=1}^{n} l_k p^k \end{matrix} \right\} \\ &\qquad\quad + \left\{ \begin{matrix} y + l_0 \\ x - \sum_{k=1}^{n} l_k p^k - p \end{matrix} \right\} + \left\{ \begin{matrix} y + l_0 \\ x - \sum_{k=1}^{n} l_k p^k - p^2 \end{matrix} \right\} + \ldots \\ &\qquad\quad + \left\{ \begin{matrix} y + l_0 \\ x - \sum_{k=1}^{n} l_k p^k - p^n \end{matrix} \right\} \Bigg] \bmod p \\ &\qquad = \sum_{\substack{m_0 + m_1 + \ldots + m_n = a+1 \\ m_0 \neq 0}} \binom{a}{m_0, m_1, \ldots, m_n} \left\{ \begin{matrix} y + m_0 \\ x - \sum_{k=1}^{n} m_k p^k \end{matrix} \right\} \end{aligned}$$

$$+\sum_{\substack{m_0+m_1+\ldots+m_n=a+1\\ m_1\neq 0}}\binom{a}{m_0,m_1,\ldots,m_n}\left\{\begin{matrix} y+m_0\\ x-\sum_{k=1}^n m_kp^k\end{matrix}\right\}$$

$$+\sum_{\substack{m_0+m_1+\ldots+m_n=a+1\\ m_2\neq 0}}\binom{a}{m_0,m_1,\ldots,m_n}\left\{\begin{matrix} y+m_0\\ x-\sum_{k=1}^n m_kp^k\end{matrix}\right\}+\ldots$$

$$+\sum_{\substack{m_0+m_1+\ldots+m_n=a+1\\ m_n\neq 0}}\binom{a}{m_0,m_1,\ldots,m_n}\left\{\begin{matrix} y+m_0\\ x-\sum_{k=1}^n m_kp^k\end{matrix}\right\}$$

$$=\sum_{m_0+m_1+\ldots+m_n=a+1}\binom{a}{m_0,m_1,\ldots,m_n}m_0\left\{\begin{matrix} y+m_0\\ x-\sum_{k=1}^n m_kp^k\end{matrix}\right\}$$

$$+\sum_{m_0+m_1+\ldots+m_n=a+1}\binom{a}{m_0,m_1,\ldots,m_n}m_1\left\{\begin{matrix} y+m_0\\ x-\sum_{k=1}^n m_kp^k\end{matrix}\right\}$$

$$+\sum_{m_0+m_1+\ldots+m_n=a+1}\binom{a}{m_0,m_1,\ldots,m_n}m_2\left\{\begin{matrix} y+m_0\\ x-\sum_{k=1}^n m_kp^k\end{matrix}\right\}+\ldots$$

$$+\sum_{m_0+m_1+\ldots+m_n=a+1}\binom{a}{m_0,m_1,\ldots,m_n}m_n\left\{\begin{matrix} y+m_0\\ x-\sum_{k=1}^n m_kp^k\end{matrix}\right\}$$

$$=\sum_{m_0+m_1+\ldots+m_n=a+1}\binom{a+1}{m_0,m_1,\ldots,m_n}\left\{\begin{matrix} y+m_0\\ x-\sum_{k=1}^n m_kp^k\end{matrix}\right\}. \blacksquare$$

Applying Proposition 4.1 now gives the following theorem.

THEOREM 4.1 (Lucas). *Let $x,y,x_i,y_i\in\mathbb{N}$, $i=1,\ldots,m$, $0\le x_i,y_i\le p-1, y_n=y_{n0}+y_{n1}+\ldots+y_{nn}$, $y_{ni}\in\mathbb{N}$, $n=1,\ldots,m$. Then*

$$\left\{\begin{matrix} y+y_1p+y_2p^2+\ldots+y_mp^m\\ x+x_1p+x_2p^2+\ldots+x_mp^m\end{matrix}\right\}$$
$$\equiv \sum_{y_{10}+y_{11}=y_1}\ \sum_{y_{20}+y_{21}+y_{22}=y_2}\cdots\sum_{y_{m0}+y_{m1}+\ldots+y_{mm}=y_m}$$
$$\times\binom{y_1}{y_{10},y_{11}}\binom{y_2}{y_{20},y_{21},y_{22}}\cdots\binom{y_m}{y_{m0},y_{m1},\ldots,y_{mm}}$$
$$\times\left\{\begin{matrix} y+y_{10}+y_{20}+\ldots+y_{m0}\\ x+(x_1-y_{m1}-\ldots-y_{21}-y_{11})p\\ +(x_2-y_{m2}-\ldots-y_{22})p^2+(x_m-y_{mm})p^m\end{matrix}\right\}\bmod p.$$

5. An application to Clausen–Von Staudt's congruence for the poly-Bernoulli numbers. For every integer k, we define a sequence of rational numbers B_n^k $(n = 0, 1, \ldots)$, which we refer to as *poly-Bernoulli numbers*, by

$$\frac{1}{z}\,\mathrm{Li}_k(z)|_{z=1-e^{-x}} = \sum_{n=0}^{\infty} \mathrm{B}_n^k \frac{x^n}{n!}. \tag{8}$$

Here, for any integer k, $\mathrm{Li}_k(z)$ denotes the formal power series $\sum_{m=1}^{\infty} z^m/m^k$, the kth polylogarithm if $k \geq 1$ and a rational function if $k \leq 0$. When $k = 1$, B_n^1 is the usual Bernoulli number (with $\mathrm{B}_1^1 = 1/2$) [8].

Throughout this section, ν_p is the standard p-adic valuation on $\mathbb{Q}$. The rational p-adic integers $\mathbb{Z}_p \cap \mathbb{Q}$ are the rational numbers r such that $\nu_p(r) \geq 0$. We have the following expansions of the numbers B_n^k in terms of the Stirling numbers of second kind.

Theorem 5.1.

$$\mathrm{B}_n^k = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m!}{(m+1)^k} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}.$$

(See [7] for demonstration.)

We set $m + 1 = q_r p^r + a_{r+1} p^{r+1} + \ldots + a_l p^l$ with $q_r \in [1, p-1]$ and $a_i \in [0, p-1]$ for $i = r+1, \ldots, l$. Then

$$\begin{aligned} \nu_p\left(\frac{m!}{(m+1)^k}\right) & \\ &= -r(k+1) + \frac{p^r - 1}{p-1} + \frac{(q_r - 1)(p^r - 1) + \sum_{i=r+1}^{l} a_i (p^i - 1)}{p-1}. \end{aligned} \tag{9}$$

Remark 5.1. We denote the right hand side of (9) by $\square$.

Let $(p-1)k_0 = n$, and let $n = b_0 + b_1 p + \ldots + b_l p^l$ be the p-adic expansion of n. Put $s(n) = \sum_{i=0}^{l} b_i$. Then

Remark 5.2. We have $(p-1) \mid n \Leftrightarrow (p-1) \mid s(n)$.

We establish the following lemma which will be constantly used below.

Lemma 5.1.

$$\left\{ \begin{matrix} n \\ p-1 \end{matrix} \right\} \equiv \begin{cases} 1 \bmod p & \text{if } (p-1) \mid n, \\ 0 \bmod p & \text{if } (p-1) \nmid n. \end{cases}$$

Proof. With the notation introduced after Remark 5.1, it follows from Proposition 4.1 and Remark 5.2 that

$$\left\{ \begin{matrix} n \\ p-1 \end{matrix} \right\} \equiv \left\{ \begin{matrix} s(n) \\ p-1 \end{matrix} \right\} = \left\{ \begin{matrix} (p-1)k_1 \\ p-1 \end{matrix} \right\} \bmod p, \quad \text{where} \quad (p-1)k_1 = s(n).$$

We can iterate this procedure and at the end we obtain

$$\left\{ {n \atop p-1} \right\} \equiv \left\{ {r(p-1) \atop p-1} \right\} \bmod p \quad \text{with } r \le p. \tag{10}$$

Since $r \le p$ and by Proposition 3.1 congruence (10) is equivalent to $\left\{ {p-1 \atop p-1} \right\} \bmod p$, by Proposition 2.1 we obtain the first equality. For the second case the proof is the same.

THEOREM 5.2. *If* $k \ge 2$, $(p-1) \mid n$, *and* $k+2 \le p \le n+1$, *then*

$$p^k \mathrm{B}_n^k + 1 \in p\mathbb{Z}_p, \quad \textit{i.e.} \quad p^k \mathrm{B}_n^k \equiv -1 \bmod p.$$

P r o o f. Let p be a positive prime. By Theorem 5.1 we obtain

$$p^k \mathrm{B}_n^k = p^k \sum_{m=0}^{n} \frac{(-1)^m m!}{(m+1)^k} \left\{ {n \atop m} \right\}. \tag{11}$$

We set $m+1 = q_r p^r + a_{r+1} p^{r+1} + \ldots + a_l p^l$ with $q_r \in [1, p-1]$ and $a_i \in [0, p-1]$ for $i = r+1, \ldots, l$. Then by (9) equation (11) is equivalent to

$$p^k \mathrm{B}_n^k = \sum_{r=0} p^k \frac{(-1)^m m!}{(m+1)^k} \left\{ {n \atop m} \right\} + \sum_{r=1,\, \square>0} p^k \frac{(-1)^m m!}{(m+1)^k} \left\{ {n \atop m} \right\} \tag{12}$$

$$+ \sum_{r \ge 2} p^k \frac{(-1)^m m!}{(m+1)^k} \left\{ {n \atop m} \right\} + (-1)^{p-1} (p-1)! \left\{ {n \atop p-1} \right\}. \tag{13}$$

Because $\left\{ {n \atop m} \right\}$ is an integer the right hand side of (12) is an element of $p\mathbb{Z}_p$. Since p is a prime and $\left\{ {n \atop m} \right\}$ is an integer, we can prove that the sum in (13) is an element of $p\mathbb{Z}_p$. By Lemma 5.1 and by Wilson's Theorem the second sum in (13) is equivalent to $-1 \pmod p$. Thus we finally obtain the assertion of Theorem 5.2.

THEOREM 5.3. *If* $k \ge 2, (p-1) \nmid n$ *and* $k+2 \le p \le n+1$ *then* $p^{k-1} \mathrm{B}_n^k \in \mathbb{Z}_p$.

The proof of this theorem is similar to that of Theorem 5.2.

REMARK 5.3. The case $k = 2$ of Theorem 5.2 is given by Kaneko [7].

For $p < k+2$ the behaviour of the $\nu_p(\mathrm{B}_n^k)$ is chaotic. We show this in the case $p = 2, 3,\ k = 3$.

PROPOSITION 5.1. *If* n *is even, then* $\nu_2(\mathrm{B}_n^3) = -4$.

P r o o f. Using (9) we see that the only summands in

$$\mathrm{B}_n^3 = \sum_{m=0}^{n} \frac{(-1)^m m!}{(m+1)^3} \left\{ {n \atop m} \right\}$$

which have valuation less than -3 are $\frac{-3!}{4^3} \left\{ {n \atop 3} \right\}$ and $\frac{-7!}{8^3} \left\{ {n \atop 7} \right\}$, but $\left\{ {n \atop 3} \right\} \equiv 2 \bmod 4$ and $\left\{ {n \atop 7} \right\} \equiv 0 \bmod 4$.

Proposition 5.2. *If* $n \equiv 2 \bmod 3$, *then* $\nu_3(\mathrm{B}_n^3) = -4$.

Proof. Using (9) we see that the only summand in B_n^3 which gives the valuation of B_n^3 is $\frac{8!}{9^3}\left\{ {n \atop 8} \right\}$, but $8!\left\{ {n \atop 8} \right\} \equiv 9 \bmod 3^3$.

Proposition 5.3. *If* $n = 16 + 18\alpha_2$ *or* $22 + 18\alpha_3$, *then* $\nu_3(\mathrm{B}_n^3) = -3$.

Proof. By (9) the only summands in B_n^3 which give a contribution to the valuation are $\frac{2!}{3^3}\left\{ {n \atop 2} \right\}$, $\frac{-5!}{6}\left\{ {n \atop 5} \right\}$, and $\frac{8!}{3^6}\left\{ {n \atop 8} \right\}$. We find by Lemma 5.1 and by induction that $\left\{ {n \atop 2} \right\} = 1 + 3d$, $\left\{ {n \atop 5} \right\} \equiv 0 \bmod 3$, $8!\left\{ {16+18\alpha_2 \atop 8} \right\} = (2 + 3\alpha_2)3^3 + e3^5$ and $8!\left\{ {22+18\alpha_3 \atop 8} \right\} = (1 + \alpha_3)3^4 + h3^5$, with $d, e, h \in \mathbb{N}$. In the first case we have

$$\nu_3\left(\left\{ {16 + 18\alpha_2 \atop 3} \right\}\right) = \nu_3\left(\frac{4}{3^3} + \frac{d + \alpha_2}{3^2} + \frac{e}{3}\right);$$

in the second case we have

$$\nu_3\left(\left\{ {22 + 18\alpha_3 \atop 3} \right\}\right) = \nu_3\left(\frac{2}{3^3} + \frac{d + \alpha_3}{3^2} + \frac{h}{3}\right).$$

Proposition 5.4. *Let* $n = 10 + 18k$. *If* $n = 10 + 54\alpha_1$, *then*

$$\nu_3(\mathrm{B}_n^3) = \begin{cases} -1 & \text{if } \alpha_1 \equiv 0 \bmod 3, \\ -1 & \text{if } \alpha_1 \equiv 1 \bmod 3, \\ \geq 0 & \text{if } \alpha_1 \equiv 2 \bmod 3. \end{cases}$$

In the remaining cases $n = 28 + 54\alpha_2$ *and* $n = 46 + 54\alpha_3$ *we have* $\nu_3(\mathrm{B}_n^3) = -2$.

Proof. By direct calculation and by induction we obtain $\left\{ {n \atop 2} \right\} = 25 + 3^3 d$, $5!\left\{ {n \atop 5} \right\} = 3^3 e$, and $8!\left\{ {10+54\alpha_1 \atop 8} \right\} = (13 + 9\alpha_1)3^3 + 3^6 f$ with $d, e, f \in \mathbb{N}$. In the cases $\alpha_1 \equiv 0 \bmod 3$ and $\alpha_1 \equiv 1 \bmod 3$ we obtain

$$\nu_3(\mathrm{B}_{10+54\alpha_1}^3) = \nu_3\left(\frac{7 + \alpha_1}{3} + d + f\right);$$

in the case $\alpha_1 = 2 + 3a$ we obtain

$$\nu_3(\mathrm{B}_{10+54\alpha_1}^3) = \min\left\{\nu_3(a + 2 + d + f), \nu_3\left(\frac{e}{8}\right)\right\} \geq 0.$$

If $n = 28 + 54\alpha_2$ we obtain $8!\left\{ {28+54\alpha_2 \atop 8} \right\} = (9\alpha_2 + 7)3^3 + g3^6$ with $g \in \mathbb{N}$ and so

$$\nu_3(\mathrm{B}_{28+54\alpha_2}^3) = \nu_3\left(\frac{19 + 3\alpha_2}{3^2} + d + g\right).$$

If $n = 46 + 54\alpha_3$ we obtain $8!\left\{ {46+54\alpha_3 \atop 8} \right\} = (9\alpha_3 + 1)3^3 + 3^6 h$ with $h \in \mathbb{N}$ and so

$$\nu_3(\mathrm{B}_{46+54\alpha_3}^3) = \nu_3\left(\frac{17 + 3\alpha_3}{3^2} + d + h\right).$$

Remark 5.4. In Propositions 5.3 and 5.4, $n \equiv 1 \bmod 3$.

Proposition 5.5. *If* $n = 12 + 18\alpha_0$ *or* $24 + 18\alpha_2$, *then* $\nu_3(\mathrm{B}_n^3) = -3$.

P r o o f. By (9) the only summands in B_n^3 which may give a contribution to valuation are $\frac{2!}{3^3}\left\{ {n \atop 2} \right\}$, $\frac{-5!}{6^3}\left\{ {n \atop 5} \right\}$ and $\frac{8!}{9^3}\left\{ {n \atop 8} \right\}$. Using Lemma 5.1 we obtain $\left\{ {n \atop 2} \right\} = 1 + 3a$ and by induction we get $\left\{ {n \atop 5} \right\} \equiv 0 \bmod 3$.

In the first case we obtain $8!\left\{ {12+18\alpha_0 \atop 8} \right\} = \frac{2+6b}{3^3} + 3^5 c$ so

$$\nu_3(\mathrm{B}^3_{12+18\alpha_0}) = \nu_3\left(\frac{4}{3^3} + \frac{2a+b}{3^2} + \frac{c}{3}\right).$$

In the second case we obtain $8!\left\{ {24+18\alpha_2 \atop 8} \right\} = 3^4 d$ and thus

$$\nu_3(\mathrm{B}^3_{24+18\alpha_2}) = \nu_3\left(\frac{2+3a}{3^3}\right).$$

PROPOSITION 5.6. *Let $n = 18\alpha$. We have*

$$\nu_3(\mathrm{B}^3_{18\alpha}) = \begin{cases} -1 & \textit{if } \alpha \equiv 3 \bmod 9, \\ \geq 0 & \textit{if } \alpha \equiv 6 \bmod 9, \\ -2 & \textit{otherwise}. \end{cases}$$

P r o o f. By induction and by direct calculation we obtain $\left\{ {n \atop 2} \right\} = 13 + 3^3 a$, $5!\left\{ {n \atop 5} \right\} = 18 + 3^3 b$, and $8!\left\{ {n \atop 8} \right\} = (1+6k)3^3 + 3^6 d$. Then

$$\nu_3(\mathrm{B}^3_n) = \nu_3\left(\frac{2!}{3^3}\left\{ {n \atop 2} \right\} - \frac{5!}{6^3}\left\{ {n \atop 5} \right\} + \frac{8!}{9^3}\left\{ {n \atop 8} \right\}\right) = \nu_3\left(\frac{8k-3}{43^2}\right).$$

REMARK 5.5. In Propositions 5.5 and 5.6, $n \equiv 0 \bmod 3$.

Acknowledgments. We wish to thank M. Kaneko for his advice on this subject, and we wish to express our sincere thanks to the referee for his helpful comments and suggestions that led to a considerable improvement of this paper.

References

[1] H. W. Becker and J. Riordan, *The arithmetic of Bell and Stirling numbers*, Amer. J. Math. 70 (1948), 385–394.

[2] L. Carlitz, *Weighted Stirling numbers of the first and second kind*, *I*, Fibonacci Quart. 18 (1970), 147–162.

[3] L. Comtet, *Analyse Combinatoire*, Tomes i, ii, Presses Univ. de France, Paris, 1970.

[4] A. E. Fekete, *Apropos two notes on notation*, Amer. Math. Monthly 99 (1992), 403–422.

[5] F. T. Howard, *Congruences for the Stirling numbers and associated Stirling numbers*, Acta Arith. 55 (1990), 29–41.

[6] C. Jordan, *Calculus of Finite Differences*, Chelsea, New York, 1960.

[7] M. Kaneko, *Poly–Bernoulli numbers*, J. Théor. Nombres Bordeaux 9 (1997), 221–228.

[8] N. Koblitz, *p-adic Numbers*, *p-adic Analysis*, *and Zeta-Functions*, Grad. Texts in Math. 58, Springer, Berlin, 1977.

[9] R. Sánchez-Peregrino, *Identité de Bernstein pour une fonction homogène à singularité isolée*, Rend. Sem. Mat. Univ. Padova 81 (1989), 221–227.
[10] —, *Another proof of Kaneko's recurrence formula for the Bernoulli numbers*, preprint.
[11] J. Stirling, *Methodus differentialis, sive tractatus de summatione et interpolazione serierum infinitarum*, Londini, 1730.

Dipartimento de Matematica Pura ed Applicata
Università di Padova
Via Belzoni, 7
35131 Padova, Italy
E-mail: sanchez@math.unipd.it

Received on 8.12.1998
and in revised form on 11.10.1999 (3526)