On approximation to real numbers by algebraic numbers

by

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1. Introduction. In this paper we study the problem of solvability of the inequality

\[(1.1) \quad |\xi - \alpha| < c(\xi, n)H(\alpha)^{-A}\]

in algebraic numbers \(\alpha\) of degree \(\leq n\), where \(A > 0\), \(\xi\) is a real number which is not an algebraic number of degree \(\leq n\), \(H(\alpha)\) is the height of \(\alpha\). In 1842 Dirichlet proved that for any real number \(\xi\) there exist infinitely many rational numbers \(p/q\) such that \(|\xi - p/q| < q^{-2}\). In 1961 E. Wirsing [9] proved that (1.1) has infinitely many solutions if \(A = n/2 + \gamma_n\), where \(\lim_{n \to \infty} \gamma_n = 2\). Moreover, he conjectured that the inequality (1.1) has infinitely many solutions if \(A = n + 1 - \varepsilon\), where \(\varepsilon > 0\). Further it has been conjectured [5] that the exponent \(n + 1 - \varepsilon\) can be replaced even by \(n + 1\). This problem has not been solved except in some special cases. In 1965 V. G. Sprindžuk [6] proved that the Conjecture of Wirsing holds for almost all real numbers. In 1967 H. Davenport and W. Schmidt [3] obtained new results in the theory of linear forms. These enabled them to prove the Conjecture for \(n = 2\). In 1993 [1] the following improvement of the Theorem of Wirsing was obtained: \(A = n/2 + \gamma'_n\), where \(\lim_{n \to \infty} \gamma'_n = 3\). In 1992–1997 a new method was introduced, improving the Theorem of Wirsing for \(n \leq 10\) ([7, 8]).

In this paper we prove the following

**Theorem.** For any real number \(\xi\) which is not an algebraic number of degree \(\leq n\), there exist infinitely many algebraic numbers \(\alpha\) of degree \(\leq n\) such that

\[(1.2)\quad |\xi - \alpha| \ll H(\alpha)^{-A}.
\]

Here and below \(3 \leq n \leq 7, \ll\) is the Vinogradov symbol, and \(A = A(n)\) is the positive root of the quadratic equation

\[(1.3)\quad (3n - 5)X^2 - (2n^2 + n - 9)X - n - 3 = 0.
\]

The implicit constant in \(\ll\) depends on \(\xi\) and \(n\) only.

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The following table contains the values of $A$ corresponding to Wirsing’s Theorem, the Theorem above and the Conjecture:

<table>
<thead>
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<th>$n$</th>
<th>Wirsing, 1961</th>
<th>Theorem</th>
<th>Conj.</th>
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2. Preliminaries. We can confine ourselves to the range $0 < \xi < 1/4$. We suppose that there exists a real number $0 < \xi < 1/4$ which is not an algebraic number of degree $\leq n$, such that

\begin{equation}
\forall c > 0 \exists \tilde{H}_0 > 0 \forall \alpha \in \mathbb{A}_n, \ H(\alpha) > \tilde{H}_0, \ \ |\xi - \alpha| > cH(\alpha)^{-A},
\end{equation}

where $\mathbb{A}_n$ denotes the set of algebraic numbers of degree $\leq n$. Also, we may assume that $\tilde{H}_0 > ((2n)!)^{30n}e^{60n^2}$.

By Lemma 1 of [2] we have

\begin{equation}
|\xi - \alpha| \leq n \frac{|P(\xi)|}{|P'(\xi)|},
\end{equation}

where $\alpha$ is the root of the polynomial $P(x)$ closest to $\xi$. In fact, we get

\[
\frac{|P'(\xi)|}{|P(\xi)|} = \left| \sum_{i=1}^{n} \frac{1}{\xi - \alpha_i} \right| \leq \sum_{i=1}^{n} \frac{1}{|\xi - \alpha_i|} \leq \frac{n}{|\xi - \alpha|},
\]

which gives (2.2). Put

\[
c_T = 4n^2 (n!)^{4n^3} \xi^{-2n^5}.
\]

By (2.1) and (2.2) we obtain

\begin{equation}
\exists \tilde{H}_0 > 0 \forall Q(x) \in \mathbb{Z}[x], \ \deg Q(x) \leq n, \ \frac{|Q(\xi)|}{|Q'(\xi)|} > c_T \frac{|Q|}{|Q'|}^{-A}.
\end{equation}

Throughout the paper $|L|$ denotes the height of the polynomial $L(x)$.

3. Auxiliary lemmas

**Lemma 3.1.** Let $L(x) = c_n x^n + \ldots + c_1 x + c_0$ be a polynomial with integer coefficients such that $|L(\xi)| < 1/2$. Then there is an index $j_1 \in \{1, \ldots, n\}$ such that $|c_{j_1}| = |L|$. 
Proof. Assume that \(|c_{j_1}| < |L|\) for any \(j_1 \in \{1, \ldots, n\}\). Then
\[
|L(\xi)| = \left| \sum_{\nu=0}^{n} c_{\nu} \xi^{\nu} \right| > \left| - \sum_{\nu=1}^{n} |L| \xi^{\nu} + |L| \right| - \sum_{\nu=1}^{n} \xi^{\nu} + 1 > \frac{1}{2}.
\]

Lemma 3.2. Let \(L(x)\) be a polynomial and \(j_1\) an index as in Lemma 3.1. Suppose \(|c_i|\leq \xi^{n-1} |L|\) for every \(i \in \{1, \ldots, n\}\). Then \(|L| < \xi^{-n+1} |L'(\xi)|\).

Proof. We have
\[
|L'(\xi)| = \left| \sum_{\nu=1}^{n} \nu c_{\nu} \xi^{\nu-1} \right| = \left| j_1 c_{j_1} \xi^{j_1-1} + \left( \sum_{\nu=1}^{n} \nu c_{\nu} \xi^{\nu-1} - j_1 c_{j_1} \xi^{j_1-1} \right) \right|.
\]
Since \(|j_1 c_{j_1} \xi^{j_1-1}| = j_1 |L| \xi^{j_1-1} \geq n |L| \xi^{n-1}\),
\[
\left| \sum_{\nu=1}^{n} \nu c_{\nu} \xi^{\nu-1} - j_1 c_{j_1} \xi^{j_1-1} \right| \leq \xi^{n-1} |L| \left( \sum_{\nu=1}^{n} \nu \xi^{\nu-1} - j_1 \xi^{j_1-1} \right)
\]
and \(n - \sum_{\nu=1}^{n-1} \nu \xi^{\nu-1} > 1\), we obtain
\[
|L'(\xi)| \geq |j_1 c_{j_1} \xi^{j_1-1}| - \left| \sum_{\nu=1}^{n} \nu c_{\nu} \xi^{\nu-1} - j_1 c_{j_1} \xi^{j_1-1} \right|
\]
\[
\geq n \xi^{n-1} |L| - \xi^{n-1} |L| \sum_{\nu=1}^{n-1} \nu \xi^{\nu-1} = \xi^{n-1} |L| \left( n - \sum_{\nu=1}^{n-1} \nu \xi^{\nu-1} \right)
\]
\[
> \xi^{n-1} |L|.
\]

Notations. In this section \(L^{(k)}(x)\) denotes the \(k\)th derivative of a polynomial \(L(x)\). However, in Sections 4–7 we will use \(Q^{(l)}(x)\) to denote the polynomial with indices \(l\) and \(i\).

Lemma 3.3. For any polynomials \(F(x)\) and \(G(x)\) the following identity is valid:

\[
R(F, G) \equiv \begin{pmatrix}
\frac{F^{(l)}(\xi)}{l!} & \cdots & F'(\xi) & F(\xi) \\
\ldots & \cdots & \cdots & \cdots \\
\frac{F^{(m)}(\xi)}{m!} & \cdots & F'(\xi) & F(\xi) \\
\frac{G^{(m)}(\xi)}{m!} & \cdots & G'(\xi) & G(\xi)
\end{pmatrix}
\]

(3.1)
where $R(F, G)$ denotes the resultant of $F(x)$ and $G(x)$, $\xi$ is any real, complex or $p$-adic number, $\deg F(x) = l$, $\deg G(x) = m$.

**Proof.** Write

$$F(x) = \sum_{\nu=0}^{l} a_{\nu} x^{\nu} = a_{l} \prod_{\nu=1}^{l} (x - \alpha_{\nu}), \quad G(x) = \sum_{\nu=0}^{m} b_{\nu} x^{\nu} = b_{m} \prod_{\nu=1}^{m} (x - \beta_{\nu}),$$

$$\tilde{F}(x) = F(x + \xi) = \sum_{\nu=0}^{l} \tilde{a}_{\nu} x^{\nu}, \quad \tilde{G}(x) = G(x + \xi) = \sum_{\nu=0}^{m} \tilde{b}_{\nu} x^{\nu}.$$

Denote by $\Delta_{l,m}(A_{i}, B_{j})$ the determinant obtained from (3.1) by replacing $F(i)(\xi)/i!$ and $G(j)(\xi)/j!$ with $A_{i}$ and $B_{j}$, $0 \leq i \leq l$, $0 \leq j \leq m$, respectively. For example, according to the definition of resultant we have $R(F, G) = \Delta_{l,m}(a_{i}, b_{j})$. We now obtain

$$R(F, G) = a_{l} m b_{m} \prod_{i,j} (\alpha_{i} - \beta_{j}) = a_{l} m b_{m} \prod_{i,j} (\alpha_{i} - \xi - (\beta_{j} - \xi)) = \Delta_{l,m}(\tilde{a}_{i}, \tilde{b}_{j})$$

$$= \Delta_{l,m} \left( \frac{F(i)(0)}{i!}, \frac{G(j)(0)}{j!} \right) = \Delta_{l,m} \left( \frac{F(i)(\xi)}{i!}, \frac{G(j)(\xi)}{j!} \right). \quad \blacksquare$$

**Lemma 3.4.** Let $F(x), G(x) \in \mathbb{Z}[x]$ be nonzero polynomials with $\deg F(x) = l \leq n$, $\deg G(x) = m \leq n$, $lm \geq 2$. Suppose that $F(x)$ and $G(x)$ have no common root. Then at least one of the following estimates is true:

(I) $1 < c_{R} \max(|F(\xi)|, |G(\xi)|)^2 \max(\mathcal{F}, \mathcal{G})^{m+l-2}$,

(II) $1 < c_{R} \max(|F(\xi)| \cdot |F'(\xi)| \cdot |G'\xi|, |G(\xi)| \cdot |F'(\xi)|^2) \max(\mathcal{F}, \mathcal{G})^{m+l-1}$,

(III) $1 < c_{R} \max(|G(\xi)| \cdot |F'(\xi)| \cdot |G'\xi|, |F(\xi)| \cdot |G'(\xi)|^2) \max(\mathcal{F}, \mathcal{G})^{m-1} \mathcal{G}^{l-2}$,

where $0 < \xi < 1$ and $c_{R} = (2n)!((n + 1)!)^{2n-2}$.

**Proof.** Consider the identity of Lemma 3.3. Since the polynomials $F(x), G(x) \in \mathbb{Z}[x]$ have no common root, it follows that

(3.3) $|R(F, G)| \geq 1$.

We will obtain an upper bound for the absolute value of the determinant (3.1). Let us expand it with respect to the last column. Obviously, any nonzero term contains the factor $F(\xi)$ or $G(\xi)$. We distinguish two cases.

**Case A.** Suppose that some nonzero term contains $F(\xi)^2$, $G(\xi)^2$ or $F(\xi)G(\xi)$. Using the inequality

(3.4) $|L(i)(\xi)| < (n + 1)! \mathcal{I}$,

where $\deg L(x) \leq n$, we estimate other factors. Hence this term has absolute value at most

$$((n + 1)!)^{m+l-2} \max(\mathcal{F}, \mathcal{G})^{m+l-2} \max(\mathcal{F}, \mathcal{G})^{m+l-2}.$$
Case B. Suppose that some nonzero term contains \( F(\xi) \) or \( G(\xi) \) together with the other factors \( F(i)(\xi)/i! \) or \( G(i)(\xi)/j! \) where \( 1 \leq i \leq l, \ 1 \leq j \leq m \). If we expand the determinant (3.1) according to the last three columns, we see that the term considered contains one of the following expressions: \( F(\xi)F'(\xi)G'(\xi), \ G(\xi)F'(\xi)^2, \ G(\xi)F'(\xi)G'(\xi) \) or \( F(\xi)G'(\xi)^2 \). Using (3.4) we conclude that this term has absolute value at most

\[
((n+1)!)^{m+1-3} \max(|F(\xi)| \cdot |F'(\xi)| \cdot |G'(\xi)|, |G(\xi)| \cdot |F'(\xi)|^2) \cdot |F|^{m-2} |G|^{l-1}
\]

or

\[
((n+1)!)^{m+1-3} \max(|G(\xi)| \cdot |F'(\xi)| \cdot |G''(\xi)|, |F(\xi)| \cdot |G'(\xi)|^2) \cdot |F|^{m-1} |G|^{l-2}.
\]

Finally, when expanding the determinant (3.1), we obtain \((l+m)!\) terms. Combining this information with (3.3), we get (3.2)(I)–(III).

The following two lemmas are well known.

Lemma 3.5 (see [4], [5]). Let \( R(x), R_1(x), \ldots, R_v(x) \) be polynomials such that \( R(x) = R_1(x) \ldots R_v(x), \deg R(x) = l. \) Then

\[
e^{-l} \left| R_1 \ldots R_v \right| \leq |R| \leq (l+1)^{-v-1} \left| R_1 \ldots R_v \right|.
\]

Lemma 3.6. Let \( F(x) \) and \( G(x) \) be polynomials with integer coefficients of degree \( \leq l. \) Let \( F(x) \) be a polynomial irreducible over \( \mathbb{Z} \) with \( |F| > e^l |G|. \) Then \( F(x) \) and \( G(x) \) have no common root.

Proof. Assume that \( F(x) \) and \( G(x) \) have a common root. Then there exists a polynomial \( \tilde{F}(x) \in \mathbb{Z}[x], \tilde{F}(x) \neq 1, \) dividing both \( F(x) \) and \( G(x). \) Since \( F(x) \) is irreducible, we have \( \tilde{F}(x) \equiv F(x). \) Therefore \( G(x) = F(x) \tilde{G}(x), \) where \( \tilde{G}(x) \in \mathbb{Z}[x]. \) By (3.5) we have \( |G| \geq e^{-l} |F| |\tilde{G}| \geq e^{-l} |F|. \)

Lemma 3.7. Consider the following system of inequalities:

\[
\begin{cases}
|a_{11}x_1 + \ldots + a_{1n}x_n| \leq A_1, \\
|a_{21}x_1 + \ldots + a_{2n}x_n| \leq A_2, \\
\vdots \\
|a_{n1}x_1 + \ldots + a_{nn}x_n| \leq A_n,
\end{cases}
\]

where \( a_{ij} \in \mathbb{R}, \ A_i \in \mathbb{R}^+, \ 1 \leq i, j \leq n. \) Suppose that

1. \((I)\) for any \( 1 \leq j \leq n, \max_{2 \leq i \leq n} (|a_{ij}|) \leq B_j, \min_{1 \leq j \leq n-1} (B_j) \geq B_n > 0; \)
2. \((II)\) \( \max_{1 \leq j \leq n-1} (|a_{1j}|) \leq |a_{1n}|, \ a_{1n} \neq 0; \)
3. \((III)\) \( \max_{2 \leq j \leq n} (A_j) \leq A_n; \)
4. \((IV)\) \( |\Delta| > c_d |a_{1n}| B_1 \ldots B_{n-1}, \) where \( \Delta \) is the determinant of the system (3.6), and \( c_d \) is some positive constant.

Then for any solution \( (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{R}^n \) of the system (3.6) the following estimates hold:

\[
|\tilde{x}_l| < \frac{n!}{c_d} B_{l-1} \max \left( \frac{A_1 B_n}{|a_{1n}|}, A_n \right) \quad (1 \leq l \leq n).
\]
Proof. Using the Theorem of Cramer, we have

\[(3.8) \quad |\tilde{\lambda}_l| = \frac{|\Delta_l|}{|\Delta|} \quad (1 \leq l \leq n),\]

where $\Delta_l$ is the determinant obtained from $\Delta$ by replacing $l$th column with $[\theta_1 A_1, \ldots, \theta_n A_n]^T$. $|\theta_{\nu}| \leq 1, 1 \leq \nu \leq n$.

When expanding $\Delta_l$ with respect to the $l$th column, we get

\[(3.9) \quad |\Delta_l| \leq n \max(A_1|M_1|, \ldots, A_n|M_n|),\]

where $M_\nu$ are the minors corresponding to $\theta_\nu A_\nu$ for $1 \leq \nu \leq n$.

By (I) we have

\[(3.10) \quad |M_1| \leq (n - 1)! B_1 \ldots B_n B_l^{-1}.

Let us show that

\[(3.11) \quad |M_\nu| \leq (n - 1)! |a_{1n}| B_1 \ldots B_{n-1} B_l^{-1} \quad (2 \leq \nu \leq n).\]

In fact, by (II) the absolute values of $a_{1j}$ from the first line of the minors $M_\nu$, $2 \leq \nu \leq n$, are less than or equal to $|a_{1n}|$. On the other hand, by (I) the absolute values of any minors $m_{\nu j}$ of $M_\nu$ which correspond to the elements $a_{1j}$ are less than or equal to $(n - 2)! B_1 \ldots B_{n-1} B_l^{-1}$. This gives (3.11).

Using (III) and (3.9)–(3.11), we get

\[(3.12) \quad |\Delta_l| \leq n! B_1 \ldots B_{n-1} B_l^{-1} \max(A_1 B_n, A_n|a_{1n}|).\]

By substituting the estimate (IV) and (3.12) into (3.8), we obtain (3.7). ■

4. Construction of $\tilde{Q}_1^{(0)}(x), \ldots, \tilde{Q}_1^{(n-1)}(x)$. Fix some $h \in \mathbb{N}$, $h > \tilde{H}_0$. We consider the finite set of polynomials $P(x) \in \mathbb{Z}[x]$ with $\deg P(x) \leq n$, $|P| \leq h$. Their values at $\xi$ are distinct. Hence we can choose a unique (up to sign) polynomial $\tilde{P}_0(x) \in \mathbb{Z}[x]$, $\tilde{P}_0(x) \not\equiv 0$, with minimal absolute value at $\xi$.

Put

\[c_p = n! \xi^{-n^2}.

We now increase $h$ until a polynomial $\tilde{P}_1(x) \in \mathbb{Z}[x]$, $\tilde{P}_1(x) \not\equiv 0$, of degree $\leq n$ with $|\tilde{P}_1| \leq h$, $|\tilde{P}_1(\xi)| < c_p^{-1}|\tilde{P}_0(\xi)|$ appears. If there are several polynomials of this kind, pick one with minimal absolute value at $\xi$. It is clear that $\tilde{H}_0 < |\tilde{P}_1|$. We increase $h$ again until a polynomial $\tilde{P}_2(x) \in \mathbb{Z}[x]$ of degree $\leq n$ with $\tilde{H}_0 < |\tilde{P}_1| < |\tilde{P}_2| \leq h$, $|\tilde{P}_2(\xi)| < c_p^{-1}|\tilde{P}_1(\xi)|$ appears. By repeating this process, we obtain a sequence of polynomials $\tilde{P}_l(x) \in \mathbb{Z}[x]$,
\[ \deg \tilde{P}_i(x) \leq n, \text{ such that} \]

\[
\begin{align*}
(i) & \quad 1/2 > |\tilde{P}_1(\xi)| > c_p|\tilde{P}_2(\xi)| > \ldots > c_p^{k-1}|\tilde{P}_k(\xi)| > \ldots, \\
(ii) & \quad \tilde{H}_0 < |\tilde{P}_1| < |\tilde{P}_2| < \ldots < |\tilde{P}_k| < \ldots, \\
(iii) & \quad \forall P(x) \in \mathbb{Z}[x], \ P(x) \neq 0, \ \deg P(x) \leq n, \ |P| < |\tilde{P}_{k+1}|, \ \\
& \quad |P(\xi)| \geq c_p^{-1}|\tilde{P}_k(\xi)|.
\end{align*}
\]

For any natural \( i \) we set

\[ \tilde{Q}_i^{(0)}(x) = \tilde{P}_i(x). \]

Write \( \tilde{Q}_i^{(0)}(x) = a_0^{(0)} x^n + \ldots + a_1^{(0)} x + a_0^{(0)} \). By Lemma 3.1 there is an index \( j_1 \in \{1, \ldots, n\} \) such that \( |a_{j_1}^{(0)}| = \frac{\lvert \tilde{Q}_i^{(0)} \rvert}{\lvert \tilde{P}_i \rvert} \).

We successively construct nonzero polynomials \( \tilde{Q}_i^{(0)}(x), \ldots, \tilde{Q}_i^{(n-1)}(x) \) in \( \mathbb{Z}[x] \) of degree \( \leq n \) and distinct integers \( j_1, \ldots, j_n \) from \( \{1, \ldots, n\} \). We write \( \tilde{Q}_i^{(l)}(x) = a_n^{(l)} x^n + \ldots + a_1^{(l)} x + a_0^{(l)} \), \( 0 \leq l \leq n-1 \). The polynomials \( \tilde{Q}_i^{(l)}(x) \) and the numbers \( j_{l+1} \) (which we call the indices of the \( \tilde{Q}_i \)-system) will have the following properties:

\[
\begin{align*}
(1) & \quad |\tilde{Q}_i^{(l)}(\xi)| < c_p^{-1}|\tilde{P}_{l-1}(\xi)|, \\
(2) & \quad |a_j^{(l)}| \leq c_p^{-1}|\tilde{Q}_i^{(\mu-1)}| \quad (\mu = 1, \ldots, l), \\
(3) & \quad |a_{j_{l+1}}^{(l)}| > \xi^{n-1}|\tilde{Q}_i^{(l)}|
\end{align*}
\]

(if \( l = 0 \), we have \((1), (3)\) only). Moreover, if for some \( 0 \leq l \leq n-1 \) any nonzero polynomial \( Q(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x] \) satisfies

\[
\begin{align*}
|Q(\xi)| & < c_p^{-1}|\tilde{P}_{l-1}(\xi)|, \\
|a_{j_1}| & \leq c_p^{-1}|\tilde{Q}_i^{(\mu-1)}| \quad (\mu = 1, \ldots, l)
\end{align*}
\]

(if \( l = 0 \), we have \( |Q(\xi)| < c_p^{-1}|\tilde{P}_{l-1}(\xi)| \) only), then \( |Q| \geq |\tilde{Q}_i^{(l)}| \). In other words, \( \tilde{Q}_i^{(l)}(x) \) has minimum height among nonzero polynomials in \( \mathbb{Z}[x] \) with \((1), (2)\). We call this the minimality property of \( \tilde{Q}_i^{(l)}(x) \), \( 0 \leq l \leq n-1 \).

The pair \( (\tilde{Q}_i^{(0)}(x), j_1) \) has the desired properties. Suppose \( 0 \leq t < n-1 \), and \( (\tilde{Q}_i^{(0)}(x), j_1), \ldots, (\tilde{Q}_i^{(l)}(x), j_{l+1}) \) have been constructed so that \((1), (2), (3)\) with \( l = 0, \ldots, t \) and the minimality property hold, and \( j_1, \ldots, j_{t+1} \) are distinct integers in \( \{1, \ldots, n\} \). By Minkowski’s Theorem on linear forms there is a nonzero polynomial \( Q(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x] \)
having

\[ |Q(\xi)| < c_p^{-1} |\tilde{P}_{t-1}(\xi)|, \]

\[ |a_{j_\mu}| \leq c_p^{-1} |\bar{Q}_t^{(\mu-1)}| \quad (\mu = 1, \ldots, t + 1), \]

\[ |a_{k_\eta}| \leq \left( c_p^{-t-2} |\tilde{P}_{t-1}(\xi)| \prod_{\nu=0}^{t-2} |\tilde{Q}_t^{(\nu)}| \right)^{-1/(n-t-1)} \quad (\eta = 1, \ldots, n - t - 1), \]

where \( \{k_1, \ldots, k_{n-t-1}\} = \{1, \ldots, n\} \setminus \{j_1, \ldots, j_{t+1}\}. \)

If there are several polynomials of this kind, pick one whose height is minimal. We denote it by \( \tilde{Q}_t^{(l+1)}(x) \). By Lemma 3.1, there is an index \( j \) in \( \{1, \ldots, n\} \) such that \( |a_j^{(l+1)}| = |\tilde{Q}_t^{(l+1)}| \). On the other hand, by the minimality property of \( \tilde{Q}_t^{(l)}(x) \) we have \( \tilde{Q}_t^{(l)} \leq \tilde{Q}_t^{(l+1)} \) for any \( 0 \leq l \leq t \). Hence \( |a_j^{(l+1)}| < |\tilde{Q}_t^{(\mu-1)}| \leq |\tilde{Q}_t^{(l+1)}| \) for \( \mu = 1, \ldots, t + 1 \). Therefore \( j \) is distinct from \( j_1, \ldots, j_{t+1} \). We set \( j_{t+2} = j \). Then \((1_{t+1}), (2_{t+1}), (3_{t+1})\), and the minimality property hold for \( \tilde{Q}_t^{(l+1)}(x) \). In Section 5 we will slightly modify the construction of the polynomials \( Q_t^{(0)}(x), \ldots, Q_t^{(n-1)}(x) \) (see (5.7) and Remark 5.8). Therefore we use the inequality \( |a_j^{(l)}| > \xi^{n-1} |\tilde{Q}_t^{(l)}| \) instead of \( |a_{j_{t+1}}^{(l)}| = |\tilde{Q}_t^{(l)}| \), \( 0 \leq l \leq n - 1 \).

In this way \( (\tilde{Q}_t^{(0)}(x), j_1), \ldots, (\tilde{Q}_t^{(n-1)}(x), j_n) \) can be constructed. Clearly

\[ (4.3) \quad |\tilde{Q}_t^{(0)}| \leq |\tilde{Q}_t^{(1)}| \leq \ldots \leq |\tilde{Q}_t^{(n-1)}|. \]

5. Properties of \( \tilde{Q}_t^{(0)}(x), \ldots, \tilde{Q}_t^{(n-1)}(x) \). Using Lemma 3.1, the last two inequalities from (4.2), and (4.3), we deduce

\[ (5.1) \quad |\tilde{Q}_t^{(l)}| \leq c_p^{l/(l+1)/(n-l)} \left( |\tilde{P}_{t-1}(\xi)| \prod_{\nu=0}^{l-1} |\tilde{Q}_t^{(\nu)}| \right)^{-1/(n-l)} \quad (1 \leq l \leq n - 1). \]

Applying (4.3) to (5.1) with \( l = n - 1 \), we get

\[ (5.2) \quad |\tilde{Q}_t^{(n-1)}| \leq c_p^n |\tilde{P}_{t-1}(\xi)|^{-1} \left( \prod_{\nu=0}^{n-2} |\tilde{Q}_t^{(\nu)}| \right)^{-1} \leq c_p^n |\tilde{P}_{t-1}(\xi)|^{-1} |\tilde{P}_t|^{n+1}. \]

Similarly, (4.3) and (5.1) imply that

\[ (5.3) \quad |\tilde{Q}_t^{(l)}| \leq |\tilde{Q}_t^{(n-2)}| \leq c_p^{(n-1)/2} |\tilde{P}_{t-1}(\xi)|^{-1/2} \left( \prod_{\nu=0}^{n-3} |\tilde{Q}_t^{(\nu)}| \right)^{-1/2} \leq c_p^{(n-1)/2} |\tilde{P}_{t-1}(\xi)|^{-1/2} |\tilde{P}_t|^{1-n/2} \quad (0 \leq l \leq n - 2). \]
Lemma 5.1. Let \( i \) be any natural number \( > 1 \). Suppose that for some \( 0 \leq l \leq n - 1 \) the polynomial \( \tilde{Q}_i^{(l)}(x) \) satisfies the conditions of Lemma 3.2. Then

\[
(5.4) \quad \left| \tilde{Q}_i^{(l)} \right|^{-1} < (c_T c_p \xi^{n-1})^{-1/(A-1)} |\tilde{P}_{i-1}(\xi)|^{1/(A-1)}.
\]

Proof. By Lemma 3.2 we obtain \( |\tilde{Q}_i^{(l)}| < \xi^{-n+1} |\tilde{Q}_i^{(l)'}(\xi)| \). On the other hand, \( |\tilde{Q}_i^{(l)}| > \tilde{H}_0 \). Therefore by (2.3) and (1) we get

\[
c_T \left| \frac{\tilde{Q}_i^{(l)}}{\tilde{Q}_i^{(l)'}}(\xi) \right| \quad \xi^{-n+1} |\tilde{Q}_i^{(l)}(\xi)| < c_p^{-1} |\tilde{P}_{i-1}(\xi)||\tilde{Q}_i^{(l)}|^{-1},
\]

hence

\[
|\tilde{Q}_i^{(l)}|^{-1} < c_T^{-1} c_p^{-1} \xi^{-n+1} |\tilde{P}_{i-1}(\xi)|,
\]

and the result follows. \( \blacksquare \)

Define

\[
(5.5) \quad c_M = \min_{P(x) \in \mathbb{Z}[x], P(x) \neq 0, \deg P(x) \leq n} (|\tilde{P}(\xi)|),
\]

\[
(5.6) \quad H_0 = c_M^{-30n} c_p^{15n} e^{60n^2 |\tilde{P}|}.
\]

By (4.1)(ii) there exists an index \( k_0 \in \mathbb{N} \) such that \( |\tilde{P}_{k_0}| \leq H_0 < |\tilde{P}_{k_0+1}| \).

From now on

\[
(5.7) \quad Q_i^{(l)}(x) = \tilde{Q}_{k_0+l}^{(l)}(x) \quad \text{for any } i \in \mathbb{N} \text{ and } l = 0, \ldots, n - 1.
\]

In particular,

\[
(5.8) \quad P_i(x) = \tilde{P}_{k_0+l}(x) \quad \text{for any } i \in \mathbb{N}.
\]

Lemma 5.2. For any natural \( i > 1 \) we have

(I) \( |\tilde{P}_{i-1}(\xi)| < |\tilde{P}|^{-(n-1)(A-1)/(A-2)} \),

(II) \( \prod_{\nu=0}^{n-2} |Q^{(\nu)}| < c_p^{-n} |\tilde{P}_{i-1}(\xi)|^{-(A-2)/(A-1)} \).
and so, by the definitions of $c_T$ and $c_p$, we obtain

$$|\tilde{P}_{i-1}(\xi)|^{(A-2)/(A-1)} < |\tilde{P}_i|^{n+1},$$

which gives (5.8)(I).

Similarly, substituting (5.1) with $l = n - 1$ into (5.4) and keeping (5.7) in mind, we deduce

$$\left(c_p^m|P_{i-1}(\xi)|^{-1} \left(\prod_{\nu=0}^{n-2} Q_i^{(\nu)}\right)^{-1}\right)^{-1} < (c_T c_p \xi^{n-1})^{-1/(A-1)}|P_{i-1}(\xi)|^{1/(A-1)},$$

hence

$$\prod_{\nu=0}^{n-2} Q_i^{(\nu)} < c_p^n (c_T c_p \xi^{n-1})^{-1/(A-1)}|P_{i-1}(\xi)|^{-(A-2)/(A-1)}.$$

Using the definitions of $c_T$ and $c_p$, we get (5.8)(II).

**Corollary 5.3.** For any natural $i > 1$ we have

(5.9)

\begin{align*}
&\text{(I)} \quad |P_{i-1}(\xi)| < |\tilde{P}_i|^{-(n-1)(A-1)/(A-2)}, \\
&\text{(II)} \quad |P_{i-1}(\xi)| < |\tilde{P}_i|^{-n}.
\end{align*}

**Proof.** The inequality (5.9)(I) immediately follows from (5.7) and (5.8)(I). To obtain (5.9)(II) we must use (5.9)(I) and the inequality $A < n + 1$:

$$|P_{i-1}(\xi)| < |\tilde{P}_i|^{-(n-1)(A-1)/(A-2)} < |\tilde{P}_i|^{-(n-1)(n+1-1)/(n+1-2)} = |\tilde{P}_i|^{-n}.$$

**Lemma 5.4.** For any $i \in \mathbb{N}$ the polynomials $P_i(x)$ are irreducible over $\mathbb{Z}$ and have degree $n$.

**Proof.** Assume that $P_i(x) = P_{i_1}(x) \ldots P_{i_\lambda}(x)$, $1 \leq \gamma \leq n$, where $P_{i_1}(x), \ldots, P_{i_\lambda}(x)$ are irreducible over $\mathbb{Z}$, have degree $< n$ and integer coefficients. Let the heights of $P_{i_1}(x), \ldots, P_{i_\lambda}(x)$ be greater than $e^n |\tilde{P}_1|$ and the heights of the others be at most $e^n |\tilde{P}_1|$. It is obvious that $\lambda \leq n$. We now show that $\lambda \geq 1$. In fact, assume that the heights of $P_{i_1}(x), \ldots, P_{i_\gamma}(x)$ do not exceed $e^n |\tilde{P}_1|$. Then by (3.5) we get

$$|\tilde{P}_i| \leq (n+1)^{\gamma-1} |P_{i_1}| \ldots |P_{i_\gamma}| \leq (n+1)^{\gamma-1} \left(e^n |\tilde{P}_1|\right)^\gamma,$$

hence $|\tilde{P}_i| \leq (n+1)^{n-1} e^{n^2 |\tilde{P}_1|^n}$. On the other hand, (5.6) and (5.7) yield (5.10)

$$|\tilde{P}_i| > c_M^{-30n} c_R^{15n} e^{60n^2 |\tilde{P}_1|^n}$$

for any $i \in \mathbb{N}$. This gives a contradiction.
We now prove that there exists an index $1 \leq j_0 \leq \lambda$ such that
\begin{equation}
|P_{j_0}(\xi)| < c_R^{-1/2} |P_{j_0}|^{-(n-1)(A-1)/(A-2) + 1/30}. \tag{5.11}
\end{equation}

Assume the contrary. Then by (5.9)(I), the definition of $c_M$, (3.5), and (5.10) we have
\[
|P_{j_0}(\xi)| = \prod_{\nu=1}^{\gamma} |P_{j_0}(\xi)| \geq c_M^{-\lambda} \prod_{\nu=1}^{\lambda} |P_{j_0}(\xi)|
\]
\[
\geq c_M^{-\lambda} \prod_{\nu=1}^{\lambda} |P_{j_0}(\xi)|^{-(n-1)(A-1)/(A-2) + 1/30}
\]
\[
|P_{j_0}(\xi)| \geq c_M^{-n/2} (e^n |P_1|)^{-(n-1)(A-1)/(A-2) + 1/30}
\]
\[
= c_M^{-n/2} e^{-n(n-1)(A-1)/(A-2) + n/30} |P_1|^{1/30} |P_1|^{-(n-1)(A-1)/(A-2)}
\]
\[
> |P_{j_0}|^{-(n-1)(A-1)/(A-2)},
\]
which is impossible.

Since $1 \leq j_0 \leq \lambda$, we have $|P_{j_0}| > e^n |P_1|$. Therefore there exists an index $k \in \mathbb{N}$ such that
\begin{equation}
(5.12) \quad e^n |\widetilde{P}_k| < |P_{j_0}| \leq e^n |\widetilde{P}_{k+1}|.
\end{equation}

Combining (5.8)(I) with (5.12), then using the inequality $|P_{j_0}| > \tilde{H}_0 > c_R^{15} e^{60n^2}$, we obtain
\begin{equation}
(5.13) \quad |\widetilde{P}_k(\xi)| < |\widetilde{P}_{k+1}|^{-(n-1)(A-1)/(A-2)} \leq (e^{-n} |P_{j_0}|)^{-(n-1)(A-1)/(A-2)}
\]
\[
= e^{n(n-1)(A-1)/(A-2)} |P_{j_0}|^{-1/30} |P_{j_0}|^{-(n-1)(A-1)/(A-2) + 1/30}
\]
\[
< c_R^{-1/2} |P_{j_0}|^{-(n-1)(A-1)/(A-2) + 1/30}.
\]

Since $|P_{j_0}| > e^n |\widetilde{P}_k|$ and $P_{j_0}(x)$ is irreducible over $\mathbb{Z}$, by Lemma 3.6 the polynomials $\widetilde{P}_k(x)$ and $P_{j_0}(x)$ have no common root. Moreover, $\deg \widetilde{P}_k(x) \geq 2$ and $\deg P_{j_0}(x) \geq 2$, since otherwise we get
\[
\frac{|\widetilde{P}_k(\xi)|}{|P_k(\xi)|} = \frac{|\widetilde{P}_k(\xi)|}{|P_k(\xi)|} < \frac{|P_{j_0}|^{-(n-1)(A-1)/(A-2) + 1/30 - 1}}{\widetilde{P}_k},
\]
and a simple calculation shows that
\[
-(n-1) \frac{A-1}{A-2} - \frac{29}{30} < -A,
\]
hence
\[
\frac{|\tilde{P}_k(\xi)|}{|P_k^\prime(\xi)|} < |\tilde{P}_k|^{-A},
\]
which contradicts (2.3). The same holds for \(P_{i_0}(x)\). Thus, we can apply (3.2) to \(\tilde{P}_k(x)\) and \(P_{i_0}(x)\).

(a) Substituting (5.11) and (5.13) into (3.2)(I), then using (5.12), we deduce
\[
1 < c_R \max(|\tilde{P}_k(\xi)|, |P_{i_0}(\xi)|)^2 \max\left(\tilde{P}_k, \frac{1}{P_{i_0}}\right)^{2n-3} \\
< c_R c_R^{-1} \left|\tilde{P}_{i_0}\right|^{-2(n-1)(A-1)/(A-2)+1/15} \left|P_{i_0}\right|^{2n-3} \\
= \left|\tilde{P}_{i_0}\right|^{-2(n-1)(A-1)/(A-2)+2n-44/15}.
\]
Here we have used the inequalities \(\deg \tilde{P}_k(x) \leq n\), \(\deg P_{i_0}(x) \leq n-1\). It is easy to verify that
\[-2(n-1)\frac{A-1}{A-2} + 2n - \frac{44}{15} < 0 \quad \text{for} \quad n = 3, \ldots, 7,
\]
and we obtain a contradiction.

Since \(\min(\tilde{P}_k, P_{i_0}) > \tilde{H}_0\), we can apply (2.3) to the polynomials \(\tilde{P}_k(x)\) and \(P_{i_0}(x)\).

(b) Applying (2.3) to (3.2)(II)–(III), then using (5.11)–(5.13) and the definitions of \(c_T\) and \(c_R\), we have
\[
1 < c_R c_T^{-2} \max(|\tilde{P}_k(\xi)|, |P_{i_0}(\xi)|)^3 \max\left(\tilde{P}_k, \frac{1}{P_{i_0}}\right)^{2n-4} \\
< c_R c_T^{-3/2} c_T^{-2} \left|\tilde{P}_{i_0}\right|^{-3(n-1)(A-1)/(A-2)+1/10} \left|P_{i_0}\right|^{2A+2n-4} \\
< \left|\tilde{P}_{i_0}\right|^{-3(n-1)(A-1)/(A-2)+2A+2n-39/10}.
\]
Since
\[-3(n-1)\frac{A-1}{A-2} + 2A + 2n - \frac{39}{10} < 0 \quad \text{for} \quad n = 3, \ldots, 7,
\]
we come to a contradiction again. This completes the proof. 

**Lemma 5.5.** For any natural \(i > 1\) we have
\[
|P_{i-1}(\xi)|^{-1} < \left|\tilde{P}_i\right|^{2A+n-2/3} \left|\tilde{P}_{i-1}\right|^{(n-1)/3}.
\]

**Proof.** By Lemma 5.4 the polynomials \(P_{i-1}(x)\) and \(P_i(x)\) are irreducible over \(\mathbb{Z}\) and have degree \(n\). Therefore they have no common root. Moreover, \(\deg P_{i-1}(x) \geq 2\) and \(\deg P_i(x) \geq 2\), since otherwise by (5.9)(II) we get
\[
\frac{|P_i(\xi)|}{|P_i^\prime(\xi)|} = \frac{|P_i(\xi)|}{|P_i^\prime(\xi)|} < \left|\tilde{P}_i\right|^{-n-1},
\]
which contradicts (2.3). The same holds for \(P_{i-1}(x)\). Thus, we can apply (3.2) to \(P_{i-1}(x)\) and \(P_i(x)\).

(a) Substituting (5.9)(II) into (3.2)(I) and using (4.1)(ii), we obtain
\[
1 < c_R \max(|P_{i-1}(\xi)|, |P_i(\xi)|)^2 \max(|P_{i-1}|, |P_i|)^{2n-2}
\]
\[
< c_R |P_{i-1}|^{-2n} |P_i|^{2n-2} = c_R |P_i|^{-2},
\]

hence \(|P_i|^2 < c_R\). This gives a contradiction with (5.10).

Since \(\min(\{P_{i-1}, |P_i|\}) \geq \tilde{H}_0\), we can apply (2.3) to the polynomials \(P_{i-1}(x)\) and \(P_i(x)\).

(b) Applying (2.3) to (3.2)(II), then using (4.1)(i), (4.1)(ii), and the definitions of \(c_T\) and \(c_R\), we deduce
\[
1 < c_R \max(|P_{i-1}(\xi)\cdot |P_{i-1}'(\xi)|\cdot |P_i'(\xi)|, |P_i(\xi)\cdot |P_{i-1}(\xi)|^2) |P_{i-1}|^{-n-2} |P_i|^{-n-1}
\]
\[
< c Rc_T^{-2} |P_{i-1}(\xi)|^3 |P_{i-1}A| |P_iA| |P_{i-1}A|^{-2} |P_i|^{-n-1}
\]
\[
= c Rc_T^{-2} |P_{i-1}(\xi)|^3 |P_iA| |P_{i-1}A|^{-2} |P_i|^{-n-1}
\]
\[
< |P_{i-1}(\xi)|^3 |P_i|^{-2A+n-2} |P_{i-1}|^{-n-1}.
\]

(c) Similarly, by (2.3), (3.2)(III), (4.1)(i), (4.1)(ii), and the definitions of \(c_T\) and \(c_R\), we have
\[
1 < c_R \max(|P_i(\xi)\cdot |P_{i-1}'(\xi)|\cdot |P_i'(\xi)|, |P_{i-1}(\xi)\cdot |P_i'(\xi)|^2) |P_{i-1}|^{-n-1} |P_i|^{-n-2}
\]
\[
< c Rc_T^{-2} |P_{i-1}(\xi)|^3 |P_iA| |P_{i-1}A|^{-1} |P_i|^{-n-2}
\]
\[
= c Rc_T^{-2} |P_{i-1}(\xi)|^3 |P_iA| |P_{i-1}A|^{-2} |P_i|^{-n-1}
\]
\[
< |P_{i-1}(\xi)|^3 |P_i|^{-2A+n-2} |P_{i-1}|^{-n-1}.
\]

It is easy to see that either one of the above two inequalities gives (5.14). ■

**Lemma 5.6.** For any natural \(i > 1\) we have

\[
\prod_{\nu=0}^{n-2} Q_i^{(\nu)} < c_p^{-n} |P_{i-1}(\xi)|^{-1/2} |P_i|^{-1+n/2}.
\]

**Proof.** From (5.8)(II) we deduce

\[
\prod_{\nu=0}^{n-2} Q_i^{(\nu)}
\]
\[
< c_p^{-n} |P_{i-1}(\xi)|^{-(A-2)/(A-1)}
\]
\[
= c_p^{-n} |P_{i-1}(\xi)|^{-(A-3)/(2(A-1))} |P_{i-1}(\xi)|^{-1+n/2} |P_{i-1}(\xi)|^{-1+n/2}.
\]

We now prove that

\[
|P_{i-1}(\xi)|^{-(A-3)/(2(A-1))} |P_i|^{-1-n/2} < 1.
\]
If the result were false, we should have

\[ |P_{l-1}(\xi)| \leq \left| \frac{P}{P} \right|^{-(n-2)(A-1)/(A-3)}. \]

Substituting this into (5.14), we get

\[
1 < |P_{l-1}(\xi)| \left| \frac{P}{P} \right|^{(2A+n-2)/3} \left| \frac{P}{P} \right|^{(n-1)/3} \\
\leq \left| \frac{P}{P} \right|^{-(n-2)(A-1)/(A-3)+(2A+n-2)/3} \left| \frac{P}{P} \right|^{(n-1)/3} \\
< \left| \frac{P}{P} \right|^{-(n-2)(A-1)/(A-3)+(2A+n-2)/3+(n-1)/3}.
\]

A simple calculation shows that

\[
-\frac{(n-2)(A-1)}{A-3} + \frac{2A + 2n - 3}{3} < 0 \quad \text{for } n = 3, \ldots, 7,
\]

and we obtain a contradiction. This gives (5.17). Finally, (5.16) and (5.17) imply (5.15).

\textbf{Lemma 5.7.} Let \( i \) be any natural number > 1. Then for any \( 0 \leq l \leq n - 2 \) there exist at least two indices \( \{k_1, k_2\} \subset \{1, \ldots, n\} \) such that

\[ |a^{(l)}_{k_i}| > \xi^{-1}Q^{(l)}_i \quad (\nu = 1, 2). \]

\textbf{Proof.} By Lemma 3.1 for any \( 0 \leq l \leq n - 1 \) there exists an index \( k_1 \in \{1, \ldots, n\} \) such that \( |a^{(l)}_{k_1}| = Q^{(l)}_i \).

Fix some \( 0 \leq l \leq n - 2 \) and suppose that \( |a^{(l)}_k| \leq \xi^{-1}Q^{(l)}_i \) for all \( k \in \{1, \ldots, n\} \setminus \{k_1\} \). Then the polynomial \( Q^{(l)}_i(x) \) satisfies the conditions of Lemma 3.2. Therefore we can apply Lemma 5.1 to \( Q^{(l)}_i(x) \). Substituting (5.3) into (5.4) and keeping (5.7) in mind, we obtain

\[
(c_p^{(n-1)/2}P_{l-1}(\xi))^{-1/2} P_i^{-1-n/2} < (c_T c_p^{(n-1)/2})^{1/(A-1)} |P_{l-1}(\xi)|^{1/(A-1)}.
\]

This inequality can be written as

\[
|P_{l-1}(\xi)|^{(A-3)/(2(A-1))} \left| \frac{P}{P} \right|^{-1+n/2} < (c_T c_p^{(n-1)/2})^{1/(A-1)} c_p^{(n-1)/2},
\]

and so, by the definitions of \( c_T \) and \( c_p \), we get

\[
|P_{l-1}(\xi)|^{(A-3)/(2(A-1))} \left| \frac{P}{P} \right|^{-1+n/2} < 1,
\]

which contradicts (5.17). \( \blacksquare \)

\textbf{Remark 5.8.} We now can slightly modify the construction of the polynomials \( Q^{(0)}_i(x), \ldots, Q^{(n-1)}_i(x) \). By Lemma 5.7 there are at least two indices \( \{k_1, k_2\} \subset \{1, \ldots, n\} \) such that

\[ |a^{(0)}_{k_\nu}| > \xi^{-1}Q^{(0)}_i \quad (\nu = 1, 2). \]

We may suppose that \( k_1 \in \{1, \ldots, n-1\} \) and set \( j_1 = k_1 \). We now construct \((Q^{(1)}_i(x), j_2), \ldots, (Q^{(n-1)}_i(x), j_n)\) with this (possibly new) value of \( j_1 \). Again
there are at least two indices \( \{k_1, k_2\} \subset \{1, \ldots, n\} \) with
\[
|a_{k_\nu}^{(1)}| > \xi^{n-1}Q_i^{(1)} \quad (\nu = 1, 2).
\]
Since \( |a_{j_1}^{(1)}| \leq c_p^{-1}Q_i^{(0)} < \xi^{n-1}Q_i^{(1)} \), these indices are distinct from \( j_1 \).
So, we can pick \( j_2 \in \{1, \ldots, n-1\} \setminus \{j_1\} \), etc. In this way we can arrange \( j_1, \ldots, j_{n-1} \) so that \( \{j_1, \ldots, j_{n-1}\} = \{1, \ldots, n-1\} \). Below, we assume this is true.

6. Three statements. The following results are of great importance for this paper.

**Statement 6.1.** Let \( i \) be any natural number \( > 1 \). Write
\[
P_{i-1}(x) = b_nx^n + \ldots + b_1x + b_0.
\]
Then the polynomials \( P_{i-1}(x), Q_i^{(0)}(x), \ldots, Q_i^{(n-2)}(x) \) are linearly independent and also
\[
|\Delta| = \begin{vmatrix}
q_{j_1}^{(n-2)} & \ldots & q_{j_{n-1}}^{(n-2)} & Q_i^{(n-2)}(\xi) \\
\vdots & \ddots & \vdots & \vdots \\
q_{j_1}^{(0)} & \ldots & q_{j_{n-1}}^{(0)} & Q_i^{(0)}(\xi) \\
b_{j_1} & \ldots & b_{j_{n-1}} & P_{i-1}(\xi)
\end{vmatrix}
> \xi^{n^2}|P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} Q_i^{(\nu)},
\]
where \( j_1, \ldots, j_{n-1} \) are the indices of the \( Q_i \)-system.

**Proof.** From this moment on, we will take into account the notation (5.7) when using the formulas from Section 4. By (2) with \( 1 \leq l \leq n - 2 \) and (4.3) we have
\[
|a_{j_\nu}^{(l)}| \leq c_p^{-1}Q_i^{(\mu-1)} \leq c_p^{-1}Q_i^{(l)} \quad (1 \leq \mu \leq l),
\]
hence
\[
|a_{j_\nu}^{(n-2)}| \leq c_p^{-1}Q_i^{(\mu-1)} \leq c_p^{-1}Q_i^{(l)} \quad (1 \leq \mu \leq l),
\]
and
\[
|a_{j_\nu}^{(n-2)}| \leq c_p^{-1}Q_i^{(\mu-1)} \leq c_p^{-1}Q_i^{(l)} \quad (1 \leq \mu \leq l),
\]
Applying (3) with \( l = 0, \ldots, n - 2 \) to \( \prod_{\nu=0}^{n-2} |a_{j_{\nu+1}}^{(\nu)}| \), we obtain
\[
\text{(*)} = \xi^{n^2} \prod_{\nu=0}^{n-2} Q_i^{(\nu)} - \frac{(n-1)!}{c_p} \prod_{\nu=0}^{n-2} Q_i^{(\nu)}
\]
On the other hand, by (4.1)(ii) and (4.3) the absolute values of other minors from the first \( n - 1 \) columns of the determinant \( \Delta \) are less than or
equal to \((n-1)! \prod_{\nu=0}^{n-2} Q_i^{(\nu)}\). Hence by \((1_i)\) with \(l = 0, \ldots, n-2\), (6.2) and the definition of \(c_p\), we get

\[
\begin{align*}
|\Delta| &> \left( \xi^{(n-1)^2} - \frac{(n-1)!}{c_p} |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} Q_i^{(\nu)} \right) \\
&\quad - (n-1)! \left( \sum_{\nu=0}^{n-2} |Q_i^{(\nu)}(\xi)| \prod_{\nu=0}^{n-2} Q_i^{(\nu)} \right) \\
&\quad > \left( \xi^{(n-1)^2} - \frac{(n-1)!}{c_p} |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} Q_i^{(\nu)} \right) \\
&\quad - \frac{(n-1)! (n-1)}{c_p} |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} Q_i^{(\nu)} \\
&\quad > \xi^{n^2} |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} Q_i^{(\nu)}.
\end{align*}
\]

This gives (6.1). Finally, since \(|\Delta| > 0\), the polynomials \(P_{i-1}(x), Q_i^{(0)}(x), \ldots, Q_i^{(n-2)}(x)\) are linearly independent. ■

**Statement 6.2.** Let \(i\) and \(\tau\) be natural numbers such that

\(\frac{P_{i-1}}{P_{\tau}} \leq c_h \frac{P_{\tau}}{P_{\tau}}, \quad 1 \leq \tau \leq i-1, \quad i > 1,\)

where

\(c_h = 4 (n!)^2 c_p^{2n}.\)

Let also \(L(x)\) be a nonzero polynomial satisfying

\[
\begin{align*}
|L(\xi)| &< |P_{i-1}(\xi)|^{1/2} |P_{\tau}|^{-1+n/2} |P_{\tau}|^{-n+1}, \\
|L'(\xi)| &< |P_{i-1}(\xi)|^{1-A/2} |P_{\tau}|^{(n-2) (1-A/2)} |P_{\tau}|^{-n+2}, \\
|L| &< \xi^{-n+1} |L'(\xi)|.
\end{align*}
\]

Then

\[
\begin{align*}
\frac{|L(\xi)|}{|L'(\xi)|} &< (c_h \xi^{-1})^{(n-1)A} |L|^{-A}.
\end{align*}
\]

**Proof.** By (6.4), (6.3), (5.9)(II), and (5.14) we get

\[
\begin{align*}
|L(\xi)| &< |P_{i-1}(\xi)|^{1/2} |P_{\tau}|^{-1+n/2} |P_{\tau}|^{-n+1} \\
&\leq c_h^{-1} |P_{i-1}(\xi)|^{1/2} |P_{\tau}|^{-1+n/2} |P_{i-1}|^{-n+1} \\
&= c_h^{-1} |P_{i-1}(\xi)|^{1/2+\alpha_1-\alpha_2} |P_{i-1}(\xi)|^{-\alpha_1} |P_{i-1}(\xi)|^{\alpha_2} \\
&\quad \times |P_{\tau}|^{-1+n/2} |P_{i-1}|^{-n+1} \\
&< c_h^{-1} |P_{i-1}(\xi)|^{1/2+\alpha_1-\alpha_2} |P_{\tau}|^{(2A+n-2)\alpha_1/3} |P_{i-1}|^{(n-1)\alpha_1/3} \\
&\quad \times |P_{\tau}|^{-\alpha_2} |P_{\tau}|^{-1+n/2} |P_{i-1}|^{-n+1}.
\end{align*}
\]
where $\alpha_1$ and $\alpha_2$ are any nonnegative constants. Put

$$
\alpha_1 = \frac{3(n-2)(A-1)}{n-1} + 3,
$$

$$
\alpha_2 = \frac{7}{2} + \frac{3(n-2)(A-1)}{n-1} - \left(\frac{A}{2} - 1\right)(A-1).
$$

(6.9)

It is easy to verify that for $n = 3, \ldots, 7$ the constants $\alpha_1$ and $\alpha_2$ are positive. By (6.9) we have

$$
\frac{1}{2} + \alpha_1 - \alpha_2 = \left(\frac{A}{2} - 1\right)(A-1), \quad \frac{n-1}{3} - \alpha_1 - n + 1 = (n-2)(A-1),
$$

(6.10)

and

$$
\begin{align*}
\frac{2A + n - 2}{3} & \alpha_1 - n\alpha_2 - 1 + \frac{n}{2} \\
& = \frac{n^2A^2 + 3nA^2 - 7n^2A + 7nA - 8A^2 + 12A + 2n^2 - 8n - 2}{2(n-1)} \\
& \equiv \frac{2((3n-5)A^2 - (2n^2 + n - 9)A - n - 3) + (n-1)(n-2)(A-2)(A-1)}{2(n-1)}
\end{align*}
$$

hence by (1.3) we obtain

$$
\frac{2A + n - 2}{3} \alpha_1 - n\alpha_2 - 1 + \frac{n}{2} \\
= \frac{(n-1)(n-2)(A-2)(A-1)}{2(n-1)} = (n-2)\left(\frac{A}{2} - 1\right)(A-1).
$$

(6.11)

Finally, (6.8), (6.10), and (6.11) imply that

$$
|L(\xi)| < c_h^{n-1} |P_{r-1}(\xi)|^{(A/2-1)(A-1)} \\
\times \left|P_{r-1}(\xi)\right|^{(n-2)(A/2-1)(A-1)} \left|P_{r-1}\right|^{(n-2)(A-1)}.
$$

(6.12)

On the other hand, if we raise both sides of (6.5) to the power $-A + 1$ and apply (6.3), we get

$$
|L'(\xi)|^{-A+1} > \left|P_{r-1}(\xi)\right|^{(A/2-1)(A-1)} \left|P_{r-1}(\xi)\right|^{(n-2)(A/2-1)(A-1)} \left|P_{r-1}\right|^{(n-2)(A-1)}
\geq c_h^{(n-2)(A-1)} |P_{r-1}(\xi)|^{(A/2-1)(A-1)} \\
\times \left|P_{r-1}(\xi)\right|^{(n-2)(A/2-1)(A-1)} \left|P_{r-1}\right|^{(n-2)(A-1)}.
$$

Combining this with (6.12), we find that $|L(\xi)| < c_h^{(n-1)A} |L'(\xi)|^{-A+1}$. We now divide both sides of this inequality by $|L'(\xi)|$ and apply (6.6):
\[
\frac{|L(\xi)|}{|L'(\xi)|} < c_h^{(n-1)A}|L'(\xi)|^{-A} < c_h^{(n-1)A}\xi^{-(n-1)A}|L|^{-A},
\]
which gives (6.7). \[\square\]

**Statement 6.3.** Let \(i\) and \(\tau\) be as in Statement 6.2. Let also \(A_1, \ldots, A_n\) be positive numbers such that

(6.13) \[
\prod_{\nu=1}^{n} A_{\nu} \geq n!|P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} |Q_i(\nu)|,
\]

(6.14) \[
A_1 \leq c_p^n\frac{|P_{i-1}(\xi)|}{n} \prod_{\nu=0}^{n-2} \left|Q_i(\nu)\right| |P_\tau|^{-n+1},
\]

(6.15) \[
\overline{P}_1 \leq A_2 \leq \ldots \leq A_n \leq c_p^{(n-3)/2}|P_{i-1}(\xi)|^{-1/2} |P_1|^{-1+n/2} |P_\tau|^{-n+2}.
\]

Then there exists a nonzero polynomial \(L(x) = c_n x^n + \ldots + c_1 x + c_0\) with integer coefficients which satisfies

(6.16) \[
|L(\xi)| < A_1,
\]

(6.17) \[
|c_{k_{\nu}}| \leq A_{\nu+1} \quad (1 \leq \nu \leq n-1),
\]

(6.18) \[
|L| < \xi^{-n+1} A_n,
\]

where \(\{k_1, \ldots, k_{n-1}\} = \{1, \ldots, n-1\}\).

**Proof.** First we note that by (5.9)(II), (6.3), and (4.1)(ii) we obtain

(6.19) \[
\left|\sum_{\nu=0}^{n-2} Q_i^{(\nu)}(\xi) x_{\nu} + P_{i-1}(\xi) x_{n-1}\right| < A_1,
\]

(6.19) \[
\left|\sum_{\nu=0}^{n-2} a_{k_1}^{(\nu)} x_{\nu} + b_{k_1} x_{n-1}\right| \leq A_2,
\]

(6.19) \[
\left|\sum_{\nu=0}^{n-2} a_{k_{n-1}}^{(\nu)} x_{\nu} + b_{k_{n-1}} x_{n-1}\right| \leq A_n.
\]

We now prove that

(6.20) \[
|\Delta| = \left|\begin{array}{cccc}
Q_i^{(0)}(\xi) & \ldots & Q_i^{(n-2)}(\xi) & P_{i-1}(\xi) \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
a_k(0)_{k_1} & \ldots & a_{k_1}^{(n-2)} & b_{k_1} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
a_{k_{n-1}}(0) & \ldots & a_{k_{n-1}}^{(n-2)} & b_{k_{n-1}}
\end{array}\right| \leq n!|P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} |Q_i(\nu)|.
\]
In fact, it follows from (1) with \( l = 0, \ldots, n - 2 \) that the entries of the first line of the determinant \( \Delta \) are at most \(|P_{i-1}(\xi)|\) in absolute value. On the other hand, (4.1)(ii) and (4.3) imply that any minor of the other \( n - 1 \) lines has absolute value at most \((n - 1)! \prod_{\nu=0}^{n-2} \left|Q_i^{(\nu)}\right|\). This gives (6.20).

Thanks to (6.13), (6.20), and Minkowski’s Theorem on linear forms there exists a nonzero integer solution \((\tilde{x}_0, \ldots, \tilde{x}_{n-1}) \in \mathbb{Z}^n\) of (6.19). Using Remark 5.8, we have \(\{k_1, \ldots, k_{n-1}\} = \{j_1, \ldots, j_{n-1}\}\), where \(j_1, \ldots, j_{n-1}\) are the indices of the \(Q_i\)-system. Therefore we can apply (6.1) to the determinant \(\Delta\). It follows from (1) with \( l = 0, \ldots, n - 2 \), (4.1)(ii), (4.3), (6.1), and (6.15) that the system (6.19) satisfies the conditions of Lemma 3.7. By this lemma and the definition of \(c_p\), we have

\[
|\tilde{x}_\nu| \leq c_p \max \left( \frac{A_1|P_{i-1}|}{|P_{i-1}(\xi)|} \frac{A_n}{|Q_i^{(\nu)}|} \right) \quad (\nu = 0, \ldots, n - 2),
\]

(6.21)

\[
|\tilde{x}_{n-1}| \leq c_p \max \left( \frac{A_1}{|P_{i-1}(\xi)|} \frac{A_n}{|P_{i-1}|} \right).
\]

Put

\[
L(x) = \sum_{\nu=0}^{n-2} Q_i^{(\nu)}(x)\tilde{x}_\nu + P_{i-1}(x)\tilde{x}_{n-1} = c_n x^n + \ldots + c_1 x + c_0.
\]

The polynomials \(Q_i^{(0)}(x), \ldots, Q_i^{(n-2)}(x)\) and \(P_{i-1}(x)\) have integer coefficients and by Statement 6.1 are linearly independent. On the other hand, the solution \((\tilde{x}_0, \ldots, \tilde{x}_{n-1})\) is nonzero and integer. Hence the polynomial \(L(x)\) is nonzero and has integer coefficients as well.

From (6.19) and (6.22) we deduce (6.16) and (6.17). Let us prove (6.18). We first obtain an upper bound for \(|L(\xi)|\) and \(|L'(\xi)|\).

Applying (6.14) to (6.16) and using (5.15), we find that

\[
|L(\xi)| < |P_{i-1}(\xi)|^{1/2} \left|\frac{A_1}{P_{i-1}(\xi)} \frac{A_n}{Q_i^{(\nu)}}\right|^{-1/2} \left|\frac{P_{i-1}(\xi)}{Q_i^{(\nu)}}\right|^{-n+1}.
\]

Using (6.22), (2.3), (6.21), (1) with \(l = 0, \ldots, n - 2\), (4.1)(ii), and (4.3), we get

\[
|L'(\xi)| \leq \sum_{\nu=0}^{n-2} |Q_i^{(\nu)}(\xi)| \cdot |\tilde{x}_\nu| + |P_{i-1}(\xi)| \cdot |\tilde{x}_{n-1}|
\]

\[
\leq c_p^{-1} c_p \left( \sum_{\nu=0}^{n-2} |Q_i^{(\nu)}(\xi)| \right)^{A} \max \left( \frac{A_1}{|P_{i-1}(\xi)|} \frac{A_n}{|Q_i^{(\nu)}|} \right) + |P_{i-1}(\xi)| \left|\frac{P_{i-1}}{Q_i^{(\nu)}}\right|^{A} \max \left( \frac{A_1}{|P_{i-1}(\xi)|} \frac{A_n}{|P_{i-1}|} \right).
\]

\[
(6.24)
\]

\[
|L'(\xi)| \leq \sum_{\nu=0}^{n-2} |Q_i^{(\nu)}(\xi)| \cdot |\tilde{x}_\nu| + |P_{i-1}(\xi)| \cdot |\tilde{x}_{n-1}|
\]

\[
\leq c_p^{-1} c_p \left( \sum_{\nu=0}^{n-2} |Q_i^{(\nu)}(\xi)| \right)^{A} \max \left( \frac{A_1}{|P_{i-1}(\xi)|} \frac{A_n}{|Q_i^{(\nu)}|} \right) + |P_{i-1}(\xi)| \left|\frac{P_{i-1}}{Q_i^{(\nu)}}\right|^{A} \max \left( \frac{A_1}{|P_{i-1}(\xi)|} \frac{A_n}{|P_{i-1}|} \right).
\]
\[ < c_T^{-1} c_p |P_{i-1}(\xi)| \left( \sum_{\nu=0}^{n-2} Q_i^{(\nu)} |P_{i-1}(\xi)|^{A-1} + |P_{i-1}|^{A-1} \right) \]
\[ \times \max \left( A_1 \frac{|P_{i-1}|}{|P_{i-1}(\xi)|}^{A_n} \right) \]
\[ < nc_T^{-1} c_p |P_{i-1}(\xi)| |Q_i^{(n-2)}| A^{-1} \max \left( A_1 \frac{|P_{i-1}|}{|P_{i-1}(\xi)|}, A_n \right). \]

By (6.14), (5.15), and (6.3) we have
\[ A_1 \frac{|P_{i-1}|}{|P_{i-1}(\xi)|} \leq c_p^n \prod_{\nu=0}^{n-2} \left| Q_i^{(\nu)} \right| P_{i-1}^{-n+1} \]
\[ < |P_{i-1}(\xi)|^{-1/2} |P_T|^{-1+n/2} |P_T|^{-n+2} \]
\[ \leq c_h |P_{i-1}(\xi)|^{-1/2} |P_T|^{-1+n/2} |P_T|^{-n+2}. \]

Substituting (6.15) and (6.25) into (6.24), then using (5.3) and the definitions of \( c_T, c_p \) and \( c_h \), we obtain
\[ |L'(\xi)| < nc_T^{-1} c_p c_h |P_{i-1}(\xi)|^{1/2} |Q_i^{(n-2)}| A^{-1} |P_T|^{-1+n/2} |P_T|^{-n+2} \]
\[ < nc_T^{-1} c_p c_h c_p^{(n-1)(A-1)/2} |P_{i-1}(\xi)|^{1/2} |P_{i-1}(\xi)|^{-(A-1)/2} \]
\[ \times |P_T|^{1-n/2}(A-1) |P_T|^{-1+n/2} |P_T|^{-n+2} \]
\[ < |P_{i-1}(\xi)|^{1-A/2} |P_T|^{(n-2)(1-A/2)} |P_T|^{-n+2}. \]

Now we can complete the proof of (6.18). Assume that \( |L| \geq \xi^{-n+1} A_n \). Hence by (6.15) and (6.17) we have \( |c_{k_\nu}| \leq A_{\nu+1} \leq A_n \leq \xi^{n-1} |L|, \nu = 1, \ldots, n - 1 \). Therefore \( L(x) \) satisfies the conditions of Lemma 3.2. Thus \( |L| < \xi^{-n+1} |L'(\xi)| \). Hence by (6.23) and (6.26) the polynomial \( L(x) \) satisfies the conditions of Statement 6.2. It follows that
\[ \frac{|L(\xi)|}{|L'(\xi)|} < (c_h \xi^{-1})^{(n-1)A} |L|^{-A}. \]

Since \( |L| \geq \xi^{-n+1} A_n > \frac{1}{P_T} \) and \( c_T > (c_h \xi^{-1})^{(n-1)A} \), we obtain a contradiction with (2.3). Hence \( |L| < \xi^{-n+1} A_n \). 

**Corollary 6.4.** For any natural \( i > 2 \) we have
\[ |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} Q_i^{(\nu)} < n! c_p^n |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} Q_i^{(\nu)}, \]
where \( P_{i-1}(x) \) is the polynomial from (4.1) with \( |P_{i-1}| \leq c_h |P_T|, 1 < \tau \leq i - 1 \).
Proof. Suppose that

\[(6.28) \quad |P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} Q_{i}^{(\nu)} \geq n! c_{p}^{n} |P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} Q_{i}^{(\nu)} \]

for some natural \(i > 2\). Put

\[(6.29) \quad A_{1} = \min \left( c_{p}^{-1} |P_{\tau-1}(\xi)|, c_{p}^{n} |P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} Q_{i}^{(\nu)} \prod_{\nu=0}^{\nu-1} \prod_{i}^{n+1} \right),\]

\[(6.30) \quad A_{\nu} = c_{p}^{-1} Q_{i}^{(\nu-2)} \quad (2 \leq \nu \leq n).\]

We now prove that \(A_{1}, \ldots, A_{n}\) satisfy the conditions of Statement 6.3.

In fact, if \(A_{1} = c_{p}^{-1} |P_{\tau-1}(\xi)|\), then by (6.28)–(6.30) we get

\[\prod_{\nu=1}^{n} A_{\nu} = c_{p}^{-1} |P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} Q_{i}^{(\nu)} \geq c_{p}^{-n} n! c_{p}^{n} |P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} Q_{i}^{(\nu)} = n! |P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} Q_{i}^{(\nu)}.\]

Similarly, if \(A_{1} = c_{p}^{n} |P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} Q_{i}^{(\nu)} \prod_{\nu=0}^{\nu-1} \prod_{i}^{n+1}\), then by (6.30), (4.3), and the definition of \(c_{p}\),

\[\prod_{\nu=1}^{n} A_{\nu} = c_{p}^{n} |P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} Q_{i}^{(\nu)} \prod_{\nu=0}^{n-2} Q_{i}^{(\nu)} > n! |P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} Q_{i}^{(\nu)} \]

By (6.29) we have

\[A_{1} \leq c_{p}^{n} |P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} Q_{i}^{(\nu)} |P_{\tau}|^{-n+1}.\]

Finally, by (6.30) and (5.3) we obtain

\[(6.31) \quad A_{n} = c_{p}^{-1} Q_{i}^{(n-2)} \leq c_{p}^{(n-3)/2} |P_{\tau-1}(\xi)|^{-1/2} |P_{\tau}|^{-n/2} = c_{p}^{(n-3)/2} |P_{\tau-1}(\xi)|^{-1/2} \frac{|P_{\tau-1}(\xi)|^{-1/2}}{|P_{\tau-1}(\xi)|^{-1/2}} \times |P_{\tau}|^{-1+1/2} |P_{\tau}|^{-1+1/2} |P_{\tau}|^{-n/2}.\]

Since \(\tau \leq i-1\), from (6.31), (4.1)(i), and (4.1)(ii) we deduce

\[A_{n} < c_{p}^{(n-3)/2} |P_{\tau-1}(\xi)|^{-1/2} |P_{\tau}|^{-1+1/2} |P_{\tau}|^{-n+2},\]

hence by (6.30) and (4.3) we get

\[|P_{1}| \leq A_{2} \leq \ldots \leq A_{n} \leq c_{p}^{(n-3)/2} |P_{\tau-1}(\xi)|^{-1/2} |P_{\tau}|^{-1+1/2} |P_{\tau}|^{-n+2}.\]
Thus, $A_1, \ldots, A_n$ satisfy the conditions of Statement 6.3. Hence there exists a nonzero polynomial $L(x) = c_n x^n + \ldots + c_1 x + c_0$ with integer coefficients which satisfies

$$|L(\xi)| < A_1 \leq c_p^{-1}|P_{\tau-1}(\xi)|,$$

$$|c_{j_\nu}| \leq c_p^{-1}[Q^{(\nu-1)}_\tau] \quad (\nu = 1, \ldots, n - 1),$$

$$|L| < \xi^{-n+1} A_n = \xi^{-n+1} c_p^{-1}[Q^{(n-2)}_\tau] < [Q^{(n-2)}_\tau] \leq [Q^{(n-1)}_\tau],$$

where $j_1, \ldots, j_{n-1}$ are the indices of the $Q_\tau$-system. We obtain a contradiction with the minimality property of $Q^{(n-1)}_\tau(x)$. This contradiction proves Corollary 6.4.

7. Proof of the Theorem. We consider a sequence of natural numbers $1 = m_1 < m_2 < \ldots$ such that

$$[P_{m_{k+1}}] \leq \max(c_h[P_{m_k}], [P_{m_k+1}]) < [P_{m_{k+1}+1}].$$

We have

$$c_h[P_{m_{k+1}+1}] \leq c_h[P_{m_k}] \leq \max(c_h[P_{m_k}], [P_{m_k+1}]) < [P_{m_{k+1}+1}],$$

hence

$$[P_{m_{k+1}+1}]^{-1} < c_h^{-1} [P_{m_{k+1}+1}]^{-1},$$

for any natural $k > 1$. If we multiply these inequalities together for all $1 < k \leq l$, we obtain

$$(7.1) \quad [P_{m_{l+1}+1}]^{-1} < c_h^{-\ell/2} [P_{2}]^{-1},$$

where $l$ is even. It follows from Corollary 6.4 that for any $k \in \mathbb{N},$

$$|P_{m_k}(\xi)| \prod_{\nu=0}^{n-2} [Q^\nu_{m_k+1}] < n!c_p^n |P_{m_{k+1}}(\xi)| \prod_{\nu=0}^{n-2} [Q^\nu_{m_{k+1}+1}].$$

If we multiply these inequalities together for all $1 \leq k \leq l$, we obtain

$$(7.2) \quad |P_{l}(\xi)| < (n!c_p^n) |P_{m_{l+1}}(\xi)| \prod_{\nu=0}^{n-2} [Q^\nu_{m_{l+1}+1}],$$

for any $l \in \mathbb{N}$. Hence

Let $l$ be even. We substitute (5.15) into (7.2) and apply first (5.9)(II), then
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(7.1) and the definition of $c_h$:

$$|P_1(\xi)| < (n!c_p^n)^l |P_{m+1}(\xi)|^{1/2} \left[ \frac{1}{P_{m+1}+1} \right]^{-1-n/2} < (n!c_p^n)^l \left[ \frac{1}{P_{m+1}+1} \right]^{-1}$$

$$< (n!c_p^n)^l c_h^{-l/2} |P_2|^{-1} < (n!c_p^n)^l c_h^{-l/2} = \left( \frac{1}{2} \right)^l.$$ 

Letting $l \to \infty$ we come to a contradiction with the boundedness of $|P_1(\xi)|$. Thus, the assumption

$$\exists \tilde{H}_0 > 0 \forall Q(x) \in \mathbb{Z}[x], \deg Q(x) \leq n, \frac{|Q(\xi)|}{|Q'(\xi)|} > c_T \sqrt[3]{|Q|}^{-A},$$

cannot be true. So neither can (2.1). Hence for any real number $0 < \xi < 1/4$ which is not an algebraic number of degree $\leq n$, we have

$$\exists c > 0 \forall \tilde{H}_0 > 0 \exists \alpha \in \mathbb{A}_n, H(\alpha) > \tilde{H}_0, \quad |\xi - \alpha| \leq cH(\alpha)^{-A},$$

and this completes the proof of the Theorem.

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