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On the distribution of squares of integral quaternions

by

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1. Introduction and statement of the main result. Let $\mathbb{H}$ denote the division ring of Hamilton's quaternions and let $\mathbb{J}$ denote the Hurwitz subring of integral quaternions. Thus, as sets, $\mathbb{H} = \mathbb{R}^4$ and $\mathbb{J} = \mathbb{Z}^4 \cup \left(\frac{1}{2} + \mathbb{Z}\right)^4$. (See Section 8 below for some motivating comments on the choice of $\mathbb{J}$.)

We are interested in an asymptotic formula for the number of quaternions q^2 with $q \in \mathbb{J}$ and with all four components of q^2 lying in the interval $[-X, X]$, where X is a large positive parameter. This question is motivated by H. Müller and W. G. Nowak [8] where (among other things) an analogous problem is investigated for the ring $\mathbb{Z}[\mathrm{i}]$ of Gaussian integers. But there is a remarkable difference between squares in $\mathbb{H}$ and squares in $\mathbb{C}$. For instance, the equation $q^2 = -1$ has infinitely many solutions in $\mathbb{H}$ and still six in $\mathbb{J}$.

Now, the main result of the present paper is the following theorem.

Theorem 1. *For positive real X let*

$$A(X) := \#\{q^2 \mid q \in \mathbb{J} \wedge q^2 \in [-X, X]^4\}.$$

Then as $X \to \infty$,

$$A(X) = cX^2 - \frac{2\pi}{3} X^{3/2} + O(X^{96/73} (\log X)^{461/146}),$$

where $c = 7.674124\ldots$ is the four-dimensional volume of $\{q \in \mathbb{R}^4 \mid q^2 \in [-1, 1]^4\}$.

Remark. Clearly, cX^2 equals the volume of $\{q \in \mathbb{R}^4 \mid q^2 \in [-X, X]^4\} =: K(X)$. The term $\frac{2\pi}{3} X^{3/2}$ occurs because of the exceptional role of the *imaginary space* $\operatorname{Im}\mathbb{H} := \{0\} \times \mathbb{R}^3$. Actually, $K(X) \cap \mathbb{J} \cap \operatorname{Im}\mathbb{H}$ contains many points but produces only few different squares.

At first sight the error estimate seems rather coarse. Although the domain $K(X)$ is not convex, one might expect that standard methods for convex bodies like Fourier transformation, the Poisson summation formula,

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Stokes' theorem, etc. could be successful to obtain at least Hlawka's classical estimate $O(X^{6/5})$ (see [3]). Unfortunately, this is not the case. As we will see in Section 5, the error estimate in Theorem 1 can only be improved together with the sharpest-known estimate in the famous divisor problem.

2. Squaring quaternions. As usual, if $a = (a_0, a_1, a_2, a_3) \in \mathbb{H}$ let $\overline{a} = (a_0, -a_1, -a_2, -a_3)$ the conjugate of a, $\mathrm{Re}(a) = a_0$ the real or scalar part of a, and $N(a) = a\overline{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2$ the norm of a. $\mathrm{Im}\,\mathbb{H} = \{0\} \times \mathbb{R}^3 = \{a \in \mathbb{H} \mid \mathrm{Re}(a) = 0\} = \{a \in \mathbb{H} \mid a + \overline{a} = 0\}$ is the imaginary space. Then we have

$$\begin{aligned} a^2 &= a(2\,\mathrm{Re}(a) - \overline{a}) = 2\,\mathrm{Re}(a)a - N(a) \\ &= (a_0^2 - a_1^2 - a_2^2 - a_3^2, 2a_0a_1, 2a_0a_2, 2a_0a_3). \end{aligned}$$

Therefore, $q^2 \in [-X, X]^4$ iff $q \in K(X)$, where

$$\begin{aligned} K(X) = \{(a_0, a_1, a_2, a_3) \in \mathbb{R}^4 \mid -X \le a_0^2 - a_1^2 - a_2^2 - a_3^2, \\ 2a_0a_1,\ 2a_0a_2,\ 2a_0a_3 \le X\}. \end{aligned}$$

Define an equivalence relation $\sim$ on $\mathbb{H}$ by $p \sim q$ iff $p^2 = q^2$. How do the equivalence classes look like? It is not difficult to see that $[q]_\sim = \{q, -q\}$ if $q \in \mathbb{H} \setminus \mathrm{Im}\,\mathbb{H}$, and $[q]_\sim = \{a \in \mathrm{Im}\,\mathbb{H} \mid N(a) = N(q)\}$ if $q \in \mathrm{Im}\,\mathbb{H}$, the latter being infinite if $q \ne 0$.

Now let

$$A(X) = \#\{q^2 \mid q \in \mathbb{J} \wedge q^2 \in [-X, X]^4\}.$$

Then we have

$$\begin{aligned} A(X) &= \#\{[q]_\sim \mid q \in \mathbb{J} \cap K(X)\} \\ &= \#\{\{q, -q\} \mid q \in \mathbb{J} \cap K(X) \setminus \mathrm{Im}\,\mathbb{H}\} \\ &\quad + \#\{N(q) \mid q \in \mathbb{J} \cap \mathrm{Im}\,\mathbb{H} \wedge N(q) \le X\} \\ &= \#\{q \in \mathbb{J} \cap K(X) \mid \mathrm{Re}(q) > 0\} + O(X). \end{aligned}$$

Thus our problem is to count (integral and half odd integral) lattice points in a four-dimensional domain.

3. Preparation of the proof. It is plain that the domain $K(X)$ is bounded. More precisely, $K(X)$ is a subset of the four-dimensional cuboid

$$\left[-\sqrt{\frac{3X}{2}}, \sqrt{\frac{3X}{2}}\right] \times \left[-\sqrt{\frac{(\sqrt{2}+1)X}{2}}, \sqrt{\frac{(\sqrt{2}+1)X}{2}}\right]^3,$$

which is the smallest set $I_0 \times I_1 \times I_2 \times I_3$ containing $K(X)$.

We have

$$A(X) = \sum_{\substack{a\in\frac{1}{2}\mathbb{Z} \\ 0<a\le\sqrt{3X/2}}} \#(K_a(X)\cap(a+\mathbb{Z})^3) + O(X),$$

where the three-dimensional domain $K_a(X)$ is given by

$$K_a(X) := \{(a_1,a_2,a_3)\in\mathbb{R}^3 \mid -X+a^2 \le a_1^2+a_2^2+a_3^2 \le X+a^2 \wedge |a_1|,|a_2|,|a_3| \le X/(2a)\}.$$

Let

$$D_a(X) := \{(x,y,z)\in\mathbb{R}^3 \mid -X+a^2 \le (x-a)^2+(y-a)^2+(z-a)^2 \le X+a^2 \ \wedge\ |x-a|,|y-a|,|z-a| \le X/(2a)\}.$$

Then we obviously have

$$\#(K_a(X)\cap(a+\mathbb{Z})^3) = \#(D_a(X)\cap\mathbb{Z}^3) \quad \text{for all } a\in\tfrac{1}{2}\mathbb{Z}.$$

Therefore our program is counting ordinary lattice points in the three-dimensional domain $D_a(X)$ for every $a\in\frac{1}{2}\mathbb{Z}$ and then summing up.

How do the domains $D_a(X)$ look like? For abbreviation, define the constants

$$c_2 := \sqrt{\frac{\sqrt{2}-1}{2}}, \quad c_3 := \sqrt{\frac{\sqrt{3}-1}{2}}, \quad c_4 := \sqrt{\frac{1}{2}},$$
$$c_6 := \sqrt{\frac{\sqrt{2}+1}{2}}, \quad c_7 := \sqrt{\frac{\sqrt{3}+1}{2}}, \quad c_8 := \sqrt{\frac{3}{2}}.$$

Then we observe that $D_a(X)$ is a ball with radius $\sqrt{X+a^2}$ for $0<a\le c_2\sqrt{X}$, a cube with half the length of an edge equal to $X/(2a)$ for $c_4\sqrt{X}\le a\le\sqrt{X}$, the intersection of a ball and a cube for $c_2\sqrt{X}\le a\le c_4\sqrt{X}$, a cube minus the interior of a ball contained in the cube for $\sqrt{X}\le a\le c_6\sqrt{X}$, and the intersection of a cube and the complement of the interior of a ball for $c_6\sqrt{X}\le a\le c_8\sqrt{X}$.

In order to count the lattice points in $D_a(X)$ in the various situations we will count the lattice points in cubes, balls, ball segments, and symmetrical intersections of two segments.

For $H,R,a\in\mathbb{R}$ define

$$\mathrm{C}_a(H) := \#\{(x,y,z)\in\mathbb{Z}^3 \mid -H\le x-a,\ y-a,\ z-a\le H\},$$
$$\mathrm{B}_a(R) := \#\{(x,y,z)\in\mathbb{Z}^3 \mid (x-a)^2+(y-a)^2+(z-a)^2\le R^2\},$$
$$\mathrm{S}_a(R,H) := \#\{(x,y,z)\in\mathbb{Z}^3 \mid z-a>H \ \wedge\ (x-a)^2+(y-a)^2+(z-a)^2\le R^2\},$$
$$\mathrm{S}_a^*(R,H) := \#\{(x,y,z)\in\mathbb{Z}^3 \mid x-a,\ z-a>H \ \wedge\ (x-a)^2+(y-a)^2+(z-a)^2\le R^2\}.$$

Then $\#(D_a(X)\cap\mathbb{Z}^3) = \mathrm{B}_a(\sqrt{X+a^2})$ for $0 < a \le c_2\sqrt{X}$, $\#(D_a(X)\cap\mathbb{Z}^3) = \mathrm{C}_a(X/(2a))$ for $c_4\sqrt{X} \le a \le \sqrt{X}$, and for $a \in \frac{1}{2}\mathbb{Z}$, by symmetry,

$$\#(D_a(X)\cap\mathbb{Z}^3) = \mathrm{B}_a(\sqrt{X+a^2}) - 6\mathrm{S}_a(\sqrt{X+a^2}, X/(2a)) \qquad \text{if } c_2\sqrt{X} < a \le c_3\sqrt{X},$$

$$\#(D_a(X)\cap\mathbb{Z}^3) = \mathrm{B}_a(\sqrt{X+a^2}) - 6\mathrm{S}_a(\sqrt{X+a^2}, X/(2a)) + 12\mathrm{S}_a^*(\sqrt{X+a^2}, X/(2a)) \qquad \text{if } c_3\sqrt{X} < a \le c_4\sqrt{X},$$

$$\#(D_a(X)\cap\mathbb{Z}^3) = \mathrm{C}_a(X/(2a)) - \mathrm{B}_a(\sqrt{a^2-X}) + O(X^{1/2+\varepsilon}) \qquad \text{if } \sqrt{X} < a \le c_6\sqrt{X},$$

$$\#(D_a(X)\cap\mathbb{Z}^3) = \mathrm{C}_a(X/(2a)) - \mathrm{B}_a(\sqrt{a^2-X}) + 6\mathrm{S}_a(\sqrt{a^2-X}, X/(2a)) + O(X^{1/2+\varepsilon}) \qquad \text{if } c_6\sqrt{X} < a \le c_7\sqrt{X},$$

$$\#(D_a(X)\cap\mathbb{Z}^3) = \mathrm{C}_a(X/(2a)) - \mathrm{B}_a(\sqrt{a^2-X}) + 6\mathrm{S}_a(\sqrt{a^2-X}, X/(2a)) - 12\mathrm{S}_a^*(\sqrt{a^2-X}, X/(2a)) + O(X^{1/2+\varepsilon}) \qquad \text{if } c_7\sqrt{X} < a \le c_8\sqrt{X}.$$

The O-terms arise since the points on the surface of the ball with radius $R = \sqrt{a^2-X}$ are counted irregularly. In fact, if $a \in \frac{1}{2}\mathbb{Z}$ then

$$\#\{(x,y,z)\in\mathbb{Z}^3 \mid (x-a)^2+(y-a)^2+(z-a)^2 = R^2\} \le \#\{(x,y,z)\in\mathbb{Z}^3 \mid x^2+y^2+z^2 = 4R^2\} \ll R^{1+\varepsilon},$$

since $r_3(n) \ll n^{1/2+\varepsilon}$.

We collect similar terms and write

$$A(X) = \Sigma_1(X) + \Sigma_2(X) - \Sigma_3(X) - 6\Sigma_4(X) + 12\Sigma_5(X) + 6\Sigma_6(X) - 12\Sigma_7(X) + O(X^{1+\varepsilon}),$$

where, with the summation index a always running through $\frac{1}{2}\mathbb{Z}$,

$$\Sigma_1(X) := \sum_{0<a\le c_4\sqrt{X}} \mathrm{B}_a(\sqrt{X+a^2}),$$

$$\Sigma_2(X) := \sum_{c_4\sqrt{X}<a\le c_8\sqrt{X}} \mathrm{C}_a(X/(2a)),$$

$$\Sigma_3(X) := \sum_{\sqrt{X}<a\le c_8\sqrt{X}} \mathrm{B}_a(\sqrt{a^2-X}),$$

$$\Sigma_4(X) := \sum_{c_2\sqrt{X}<a\le c_4\sqrt{X}} \mathrm{S}_a(\sqrt{X+a^2}, X/(2a)),$$

$$\Sigma_5(X) := \sum_{c_3\sqrt{X}<a\le c_4\sqrt{X}} \mathrm{S}_a^*(\sqrt{X+a^2}, X/(2a)),$$

$$\Sigma_6(X) := \sum_{c_6\sqrt{X}<a\le c_8\sqrt{X}} \mathrm{S}_a(\sqrt{a^2-X}, X/(2a)),$$
$$\Sigma_7(X) := \sum_{c_7\sqrt{X}<a\le c_8\sqrt{X}} \mathrm{S}_a^*(\sqrt{a^2-X}, X/(2a)).$$

4. Two estimates of rounding error sums. Let the rounding error function ψ be defined by

$$\psi(z) = z - [z] - 1/2 \qquad (z \in \mathbb{R})$$

throughout the paper. ([ ] are the Gauss brackets.)

Note that for every z, $\psi(z+a) = \psi(z)$ if $a \in \mathbb{Z}$, and $\psi(z+a) = \psi(z+1/2)$ if $a \in 1/2 + \mathbb{Z}$.

For the proof of Theorem 1 we will need estimates of two ψ-sums which are variants of ψ-sums occurring in the divisor problem and the circle problem. To obtain these estimates the discrete Hardy–Littlewood method is required. (See Huxley [5] for a profound presentation of the method and its various applications to important problems of geometry and analytic number theory.)

LEMMA 1. *Let $C \ge 1$ be an absolute constant. Then as $X \to \infty$,*

$$\sum_{\alpha\le n\le\beta} \psi\left(\frac{X}{n} + \frac{n}{2}\right) \ll X^{23/73}(\log X)^{461/146}$$

uniformly in $1 \le \alpha \le \beta \le C\sqrt{X}$.

Proof. Split the sum into

$$\sum_{\alpha/2\le m\le\beta/2} \psi\left(\frac{X}{2m}\right) + \sum_{(\alpha-1)/2\le m\le(\beta-1)/2} \psi\left(\frac{X}{2m+1} + \frac{1}{2}\right)$$

and apply [5], Theorem 18.2.3, with $T = X$ to every part of a dyadic division of the first and second sum, respectively, where $F(x) = 1/(2x)$ is taken in the first case and $F(x) = 1/(2x + 1/M) + M/(2T)$ in the second.

LEMMA 2. *Let τ be an absolute constant*, $0 < \tau < 1$. *Then as $r \to \infty$,*

$$\sum_{\delta+h<n\le\delta+r} \psi(\delta + \sqrt{r^2 - (n-\delta)^2}) \ll r^{46/73}(\log r)^{315/146}$$

uniformly in $0 \le \delta \le 1$ and $\tau r \le h \le r$.

Proof. Let $g(t) = \delta + \sqrt{r^2 - (t-\delta)^2}$ and fix $M_0 = [r^{46/73}]$. Since for $r - h \le M_0$ the estimate is trivial, assume $r - h > M_0$ and choose $J \in \mathbb{N}$ with $2^{J-1}M_0 \le r - h < 2^J M_0$. Define a dyadic sequence $M_j = 2^j M_0$ $(j < J)$ and put $M_J = [r - h]$. Then

$$\sum_{\delta+h<n\le\delta+r} \psi(g(n)) = \sum_{j=0}^{J-1} \sum_{M_j\le m<M_{j+1}} \psi(f(m)) + O(M_0),$$

with $f(u) := g([r]-u)$. Now, apply [5], Theorem 18.2.3, to each of the inner sums by setting $M = M_j$, $M' = M_{j+1} - 1$, $C = 1$, $T = M^{3/2}r^{1/2}$, and $F(u) = Mf(Mu)/T$.

REMARK. It is important to fix $\tau > 0$ in Lemma 2 since otherwise the odd derivatives of f destroy the proof.

5. Lattice points in cubes, balls, and ball segments

PROPOSITION 1. *For $H > 0$ and $a \in \frac{1}{2}\mathbb{Z}$,*

$$\mathrm{C}_a(H) = 8H^3 - 24H^2\psi(H+a) + O(H).$$

Proof. Obviously, $\mathrm{C}_a(H) = (2[H]+1)^3$ if $a \in \mathbb{Z}$, and $\mathrm{C}_a(H) = (2[H+1/2])^3$ if $a \in 1/2 + \mathbb{Z}$.

What is the sharpest estimate of the error that inevitably arises when we sum up the cubes? The summation interval for the cubes is $c_4\sqrt{X} < a \le c_8\sqrt{X}$, at least it contains the interval $c_4\sqrt{X} < a \le \sqrt{X}$ where the points in whole cubes are to be counted. Thus, by substituting $a \in \frac{1}{2}\mathbb{Z}$ by $n/2$ with $n \in \mathbb{Z}$, we have to estimate

$$\sum_{\sqrt{X}\ll n\ll\sqrt{X}} \left(\frac{X}{n}\right)^2 \psi\left(\frac{X}{n} + \frac{n}{2}\right).$$

The best estimate of this weighted ψ-sum is obtained by Abelian summation combined with the sharpest-known estimate of the unweighted ψ-sum (Lemma 1). This yields an error not better than $O(X^{96/73}(\log X)^{461/146})$, which should be taken into consideration when we count the points in the other domains.

Next we consider balls. Obviously, $\mathrm{B}_a(R) = \mathrm{B}_0(R)$ for $a \in \mathbb{Z}$, and $\mathrm{B}_a(R) = \mathrm{B}_{1/2}(R)$ for $a \in 1/2 + \mathbb{Z}$. Quite recently, improving Vinogradov's classical estimate ([9], Theorem 2), Chamizo and Iwaniec [1] and Heath-Brown [2] showed that

$$(*) \qquad \mathrm{B}_0(R) = \frac{4\pi}{3}R^3 + O_\varepsilon(R^{21/16+\varepsilon}) \quad (R\to\infty).$$

In order to obtain a formula for $\mathrm{B}_{1/2}(R)$ as well, we write

$$\mathrm{B}_{1/2}(R) = \#\{(x_1,x_2,x_3) \in (1/2+\mathbb{Z})^3 \mid x_1^2 + x_2^2 + x_3^2 \le R^2\}.$$

The grid $(1/2+\mathbb{Z})^3$ has the same symmetry as $\mathbb{Z}^3$ but it contains no points lying in a coordinate plane. Fortunately, we can adapt Vinogradov's proof

[9] for the number of integral points in the sphere to half odd lattice points because each of the 48 pyramids $0 \le \delta_i x_i \le \delta_j x_j \le \delta_k x_k$ ($\{i,j,k\} = \{1,2,3\}$, $\delta_i, \delta_j, \delta_k \in \{-1,1\}$) has exactly one face lying in a coordinate plane. Additionally, we correct the sloppy estimate $\sum(\xi(z))^2 \ll M^2(\ln a)^3$ in [9], p. 320, l. 24, by using the upper bound $M^2(\ln a)$. Altogether we obtain

$$(**) \qquad \mathrm{B}_{1/2}(R) = \frac{4\pi}{3}R^3 + O(R^{4/3}(\log R)^{19/4}) \qquad (R \to \infty).$$

But we will use $(*)$ and $(**)$ only in Section 9. To reach our goal it suffices to allow the coarser error $O(R^{119/73}(\log R)^{315/146})$, which follows immediately from the next proposition.

PROPOSITION 2. *As* $R \to \infty$,

$$\mathrm{S}_a(R,H) = \frac{2\pi}{3}R^3 - \pi R^2 H + \frac{\pi}{3}H^3 + \pi(R^2 - H^2)\,\psi(H+a) + O(R^{119/73}(\log R)^{315/146})$$

uniformly in $a \in \mathbb{R}$ *and* $-R \le H \le R$.

P r o o f. We count the points in level disks by making use of Huxley's deep estimate in the circle problem. Obviously,

$$\mathrm{S}_a(R,H) = \sum_{a+H<z\le a+R} \#\{(x,y) \in \mathbb{Z}^2 \mid (x-a)^2 + (y-a)^2 \le R^2 - (z-a)^2\}.$$

In the circle problem there is no difficulty concerning the center of the circle. It follows from Huxley [5], Theorem 18.3.2, that uniformly in $(\alpha,\beta) \in \mathbb{R}^2$,

$$\#\{(x,y) \in \mathbb{Z}^2 \mid (x-\alpha)^2 + (y-\beta)^2 \le T\} = \pi T + O(T^{23/73}(\log T)^{315/146}).$$

Consequently,

$$\mathrm{S}_a(R,H) = \pi \sum_{a+H<z\le a+R} (R^2 - (z-a)^2) + O(R^{119/73}(\log R)^{315/146}).$$

Now we apply the Euler summation formula (cf. Krätzel [6], Theorem 1.3) to the sum. The main integral yields the main term, which clearly equals the volume of the segment $\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_3 \ge H \wedge x_1^2+x_2^2+x_3^2 \le R^2\}$, and the ψ-integral is $\ll R$ by the second mean-value theorem. This concludes the proof of Proposition 2.

6. Counting in intersections of ball segments. For $0 \le H \le R/\sqrt{2}$ let $V(R,H)$ denote the volume of the domain

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1 \ge H \wedge x_3 \ge H \wedge x_1^2 + x_2^2 + x_3^2 \le R^2\}.$$

We compute

$$V(R,H)=\frac{4}{3}R^3\arctan\left(\sqrt{1-\frac{2H^2}{R^2}}\right)+\frac{2}{3}H^2\sqrt{R^2-2H^2}$$
$$-\left(2R^2H-\frac{2}{3}H^3\right)\arctan\left(\sqrt{\frac{R^2}{H^2}-2}\right).$$

Further, for $0\le H\le r$ let

$$\varphi(r,H)=r^2\arccos\left(\frac{H}{r}\right)-H\sqrt{r^2-H^2}$$

denote the area of the circle segment $\{(x,y)\in\mathbb{R}^2\mid x\ge H\wedge x^2+y^2\le r^2\}$.

Proposition 3. *Suppose that* $a\in\frac{1}{2}\mathbb{Z}$. *Then as* $R\to\infty$,

$$\mathrm{S}_a^*(R,H)=V(R,H)+2\psi(H+a)\varphi(\sqrt{R^2-H^2},H)$$
$$+O(R^{119/73}(\log R)^{315/146})$$

uniformly in $R/\sqrt{3}\le H\le R/\sqrt{2}$.

Proof. We write

$$\mathrm{S}_a(R,H)=\sum_{a+H<z\le a+\sqrt{R^2-H^2}}\sigma_a(\sqrt{R^2-(z-a)^2},H),$$

where

$$\sigma_a(r,H)=\#\{(x,y)\in\mathbb{Z}^2\mid x-a>H\wedge(x-a)^2+(y-a)^2\le r^2\}.$$

First we count the lattice points in circle segments. We have

$$\frac{1}{2}\sigma_a(r,H)=\sum_{a+H<x\le a+r}\sqrt{r^2-(x-a)^2}-\sum_{a+H<x\le a+r}\psi(a+\sqrt{r^2-(x-a)^2}),$$

since $[a+b]-[a]-\psi(a)=b-\psi(a+b)$. In view of

$$\int_H^{r-1}\frac{t}{\sqrt{r^2-t^2}}\psi(t+a)\,dt\ll\frac{r-1}{\sqrt{r^2-(r-1)^2}}\ll\sqrt{r}\qquad(H\le r-1)$$

and

$$\left|\int_{r-1}^{r}\frac{t}{\sqrt{r^2-t^2}}\psi(t+a)\,dt\right|\le\int_{r-1}^{r}\frac{t}{\sqrt{r^2-t^2}}\,dt=\sqrt{2r-1}\ll\sqrt{r}$$

we obtain, by applying the Euler summation formula to the first sum and Lemma 2 (with $\delta=a-[a]$) to the second,

$$\sigma_a(r,H)=\varphi(r,H)+2\psi(H+a)\sqrt{r^2-H^2}+O(r^{46/73}(\log r)^{315/146})$$

uniformly in $r/\sqrt{2}\le H\le r$.

We insert this formula into the sum which we started from and get

$$\begin{aligned}\mathrm{S}_a(R,H) = &\sum_{a+H<z\le a+\sqrt{R^2-H^2}} \varphi(\sqrt{R^2-(z-a)^2},H)\\ &+2\psi(a+H)\sum_{a+H<z\le a+\sqrt{R^2-H^2}} \sqrt{R^2-H^2-(z-a)^2}\\ &+O(R^{119/73}(\log R)^{315/146}).\end{aligned}$$

Again by the Euler summation formula, the second sum equals

$$\frac{1}{2}\varphi(\sqrt{R^2-H^2},H)+\psi(H+a)\sqrt{R^2-2H^2}+O(\sqrt{R})$$

and the first equals

$$\int\limits_H^{\sqrt{R^2-H^2}} \varphi(\sqrt{R^2-t^2},H)\,dt+\psi(H+a)\varphi(\sqrt{R^2-H^2},H)$$
$$-2\int\limits_H^{\sqrt{R^2-H^2}} t\arccos\left(\frac{H}{\sqrt{R^2-t^2}}\right)\psi(t+a)\,dt.$$

The main integral is, by the Cavalieri principle, equal to $V(H,R)$, and the ψ-integral is, by the second mean-value theorem, $\ll R$. This concludes the proof of Proposition 3.

7. Proof of Theorem 1. First we substitute the summation index $a\in\frac{1}{2}\mathbb{Z}$ by $n/2$ with $n\in\mathbb{Z}$. Then we insert the formulas given in Propositions 1–3, and the formula

$$\mathrm{B}_a(R)=\frac{4\pi}{3}R^3+O(R^{119/73}(\log R)^{315/146})$$

into the seven terms $\Sigma_i(X)$ $(1\le i\le 7)$ from Section 3.

For abbreviation, let α_i,β_i $(1\le i\le 7)$ be defined by the following table.

i	1	2	3	4	5	6	7
α_i	0	$2c_4$	2	$2c_2$	$2c_3$	$2c_6$	$2c_7$
β_i	$2c_4$	$2c_8$	$2c_8$	$2c_4$	$2c_4$	$2c_8$	$2c_8$

Let

$$F_1(X,t):=\frac{4\pi}{3}\left(X+\left(\frac{t}{2}\right)^2\right)^{3/2},\qquad F_2(X,t):=8\left(\frac{X}{t}\right)^3,$$
$$F_3(X,t):=-\frac{4\pi}{3}\left(\left(\frac{t}{2}\right)^2-X\right)^{3/2},$$

$$F_4(X,t) := -6\pi\left(\frac{2}{3}\left(X+\left(\frac{t}{2}\right)^2\right)^{3/2} - \left(X+\left(\frac{t}{2}\right)^2\right)\left(\frac{X}{t}\right) + \frac{1}{3}\left(\frac{X}{t}\right)^3\right),$$

$$F_5(X,t) := 12V\left(\sqrt{X+\left(\frac{t}{2}\right)^2}, \frac{X}{t}\right),$$

$$F_6(X,t) := 6\pi\left(\frac{2}{3}\left(\left(\frac{t}{2}\right)^2 - X\right)^{3/2} - \left(\left(\frac{t}{2}\right)^2 - X\right)\left(\frac{X}{t}\right) + \frac{1}{3}\left(\frac{X}{t}\right)^3\right),$$

$$F_7(X,t) := -12V\left(\sqrt{\left(\frac{t}{2}\right)^2 - X}, \frac{X}{t}\right),$$

so that

$$\begin{aligned} F_{6\pm1}(X,t) = &\mp 2(t^2 \mp 4X)^{3/2} \arctan\sqrt{\frac{t^4 \mp 4Xt^2 - 8X^2}{t^4 \mp 4Xt^2}} \\ &\mp \frac{4X^2}{t^3}\sqrt{t^4 \mp 4Xt^2 - 8X^2} \\ &\pm \frac{2X}{t^3}(3t^4 \mp 12t^2X - 4X^2)\arctan\left(\frac{1}{2X}\sqrt{t^4 \mp 4Xt^2 - 8X^2}\right). \end{aligned}$$

Then we have

$$A(X) = \sum_{i=1}^{7}(S_i(X) + \Psi_i(X)) + O(X^{96/73}(\log X)^{315/146}),$$

where

$$S_i(X) := \sum_{\alpha_i\sqrt{X} < n \le \beta_i\sqrt{X}} F_i(X,n) \quad (1 \le i \le 7),$$

and $\Psi_i(X)$ are weighted ψ-sums,

$$\Psi_i(X) = \sum_{\alpha_i\sqrt{X} < n \le \beta_i\sqrt{X}} G_i(X,n)\,\psi\left(\frac{X}{n} + \frac{n}{2}\right) \quad (1 \le i \le 7),$$

with $G_1(X,t) = G_3(X,t) = 0$, and the other weight functions $G_i(X,t)$ being monotonic and $\ll X$. (Note that $\varphi(r_1,H_1) < \varphi(r_2,H_2)$ if $r_1 < r_2$ and $H_1 > H_2$.) We estimate these weighted ψ-sums by Abelian summation combined with Lemma 1 and obtain

$$A(X) = \sum_{i=1}^{7} S_i(X) + O(X^{96/73}(\log X)^{461/146}).$$

Applying the Euler summation formula to each of the seven sums $S_i(X)$, we derive

$$\sum_{i=1}^{7} S_i(X) = \sum_{i=1}^{7} \int\limits_{\alpha_i\sqrt{X}}^{\beta_i\sqrt{X}} F_i(X,t)\,dt - \frac{2\pi}{3}X^{3/2} + \sum_{i=1}^{7} \int\limits_{\alpha_i\sqrt{X}}^{\beta_i\sqrt{X}} \Big(\frac{d}{dt}F_i(X,t)\Big)\psi(t)\,dt,$$

since

$$-F_1(X,2c_4\sqrt{X}) + F_2(X,2c_4\sqrt{X}) - F_4(X,2c_4\sqrt{X}) - F_5(X,2c_4\sqrt{X}) = 0,$$
$$-F_2(X,2c_8\sqrt{X}) - F_3(X,2c_8\sqrt{X}) - F_6(X,2c_8\sqrt{X}) - F_7(X,2c_8\sqrt{X}) = 0,$$
$$F_3(X,2\sqrt{X}) = F_4(X,2c_2\sqrt{X}) = F_5(X,2c_3\sqrt{X}) = F_6(X,2c_6\sqrt{X}) = F_7(X,2c_7\sqrt{X}) = 0,$$

and

$$-\frac{1}{2}F_1(X,0) = -\frac{2\pi}{3}X^{3/2}.$$

First we estimate the ψ-integrals. Let

$$\frac{d}{dt}F_i(X,t) =: D_i(X,t) \quad (1 \le i \le 7).$$

Obviously, for $1 \le i \le 7$,

$$F_i(X,u\sqrt{X}) = X^{3/2}F_i(1,u) \quad (\alpha_i \le u \le \beta_i).$$

Thus, for $1 \le i \le 7$,

$$D_i(X,t) = XD_i(1,t/\sqrt{X}) \quad (\alpha_i\sqrt{X} \le t \le \beta_i\sqrt{X}).$$

We compute

$$D_2(1,u) = -\frac{24}{u^4}, \qquad D_{2\pm1}(1,u) = \mp\frac{\pi}{2}u\sqrt{u^2 \mp 4},$$
$$D_{5\pm1}(1,u) = \pm\frac{3\pi}{2}\Big(u\sqrt{u^2\mp 4} - 1 \mp \frac{4}{u^2} - \frac{4}{u^4}\Big),$$
$$D_{6\pm1}(1,u) = \mp 6f_\pm(u) \pm 6g_\pm(u)h_\pm(u) \pm 12\widetilde{f}_\pm(u),$$

where

$$f_\pm(u) = u\sqrt{u^2\mp 4}\arctan\bigg(\sqrt{\frac{u^4 \mp 4u^2 - 8}{u^4 \mp 4u^2}}\bigg),$$
$$g_\pm(u) = \Big(1 \pm \frac{2}{u^2}\Big)^2, \qquad h_\pm(u) = \arctan\Big(\frac{1}{2}\sqrt{u^4 \mp 4u^2 - 8}\Big),$$
$$\widetilde{f}_\pm(u) = \frac{1}{u^4}\sqrt{u^4 \mp 4u^2 - 8}.$$

We observe that, if $i = 1,2,3,4,6$, $D_i(1,u)$ is monotonic on $\alpha_i \le u \le \beta_i$. Consequently, if $i = 1,2,3,4,6$ then $D_i(X,t)$ is monotonic on $\alpha_i\sqrt{X} \le t \le \beta_i\sqrt{X}$. Hence, by $|\int_a^b \psi(t)\,dt| \le 1/8$ and the second mean-value theorem,

$$\Big|\int\limits_{\alpha_i\sqrt{X}}^{\beta_i\sqrt{X}} D_i(X,t)\,\psi(t)\,dt\Big| \ll \frac{1}{4}X \max_{\alpha_i\le u\le\beta_i} |D_i(1,u)| \ll X \qquad (i = 1,2,3,4,6).$$

Furthermore, $f_\pm(u)$, $g_\pm(u)$, $h_\pm(u)$, $\widetilde{f}_\pm(u)$ are monotonic on $\alpha_{6\pm1} \le u \le \beta_{6\pm1}$. Hence, with the maxima to be taken over $\alpha_{6\pm1} \le u \le \beta_{6\pm1}$,

$$\Big|\int\limits_{\alpha_{6\pm1}\sqrt{X}}^{\beta_{6\pm1}\sqrt{X}} D_i(X,t)\,\psi(t)\,dt\Big|$$
$$\le 3X(\max|f_\pm(u)| + (\max|g_\pm(u)|)(\max|h_\pm(u)|) + \max|\widetilde{f}_\pm(u)|) \ll X.$$

It remains to calculate the integrals $\int_{\alpha_i\sqrt{X}}^{\beta_i\sqrt{X}} F_i(X,t)\,dt$. We replace t by $u\sqrt{X}$ to get

$$\int\limits_{\alpha_i\sqrt{X}}^{\beta_i\sqrt{X}} F_i(X,t)\,dt = X^2 \int\limits_{\alpha_i}^{\beta_i} F_i(1,u)\,du \qquad (1 \le i \le 7).$$

Since the functions F_5 and F_7 can be integrated only numerically we abstain from integrating the other five functions in closed form. With electronic support,

$$\sum_{i=1}^{7} \int\limits_{\alpha_i}^{\beta_i} F_i(1,u)\,du = 7.674124222443732\ldots$$

From the preparation of the problem it is clear that $7.67412\ldots X^2$ equals the volume of the domain $K(X)$, and this concludes the proof of Theorem 1.

8. On squares of Lipschitz integral quaternions. Historically, the ring $\mathbb{J}$ does not stand at the beginning of the number theory of quaternions. It is not surprising that the first investigated discrete subring of $\mathbb{H}$ is $\mathbb{J}_0 := \mathbb{Z}^4$. The "integral" quaternions due to Lipschitz are exactly the elements of $\mathbb{J}_0$ (cf. [4]). It turned out that $\mathbb{J}_0$ is too small to have interesting arithmetic properties. The main arithmetical difference between $\mathbb{J}_0$ and $\mathbb{J}$ is that the Euclidian division algorithm works in $\mathbb{J}$ but fails in $\mathbb{J}_0$. Nevertheless it may be interesting to ask for the distribution of squares of elements in $\mathbb{J}_0$. Let us also consider the grid $(1/2+\mathbb{Z})^4 = \mathbb{J}\setminus\mathbb{J}_0 =: \mathbb{J}_{1/2}$ which of course is neither closed under addition nor under multiplication but which is closed under squaring. Then, by adapting the proof of Theorem 1 in a natural way, we obtain

THEOREM 2. *For positive real X let*

$$A_\nu(X) := \#\{q^2 \mid q \in \mathbb{J}_\nu \wedge q^2 \in [-X,X]^4\} \qquad (\nu \in \{0,1/2\}).$$

Then as $X \to \infty$,

$$A_0(X) = \frac{c}{2}X^2 - \frac{2\pi}{3}X^{3/2} + O(X^{96/73}(\log X)^{461/146}),$$
$$A_{1/2}(X) = \frac{c}{2}X^2 + O(X^{96/73}(\log X)^{461/146}),$$

where c is the constant in Theorem 1.

Clearly, the term $\frac{2\pi}{3}X^{3/2}$ does not occur in the second formula since $\mathbb{J}_{1/2} \cap \operatorname{Im}\mathbb{H} = \emptyset$.

9. A variation of the problem. There is another generalization of the problem in Müller and Nowak [8] to quaternions, which can be handled in a very easy way.

Let $\operatorname{Im}(q) := (q_1, q_2, q_3)$ denote the imaginary or vector part of the quaternion $q = (q_0, q_1, q_2, q_3)$. Then for $\nu \in \{0, 1/2, 1\}$ let

$$\widetilde{A}_\nu(X) := \#\{q^2 \mid q \in \mathbb{J}_\nu \wedge |\operatorname{Re}(q^2)|, |\operatorname{Im}(q^2)| \le X\} \qquad (\nu \in \{0,1/2,1\}),$$

where $\mathbb{J}_1 := \mathbb{J}$ and $|\cdot|$ is the Euclidian norm. Then, before summing up over the first component again, we have to count lattice points in the three-dimensional domain

$$\widetilde{K}_a(X) := \{(a_1,a_2,a_3) \in \mathbb{R}^3 \mid -X + a^2 \le a_1^2 + a_2^2 + a_3^2 \le \min\{X + a^2, X^2/(4a^2)\}\},$$

which is always a ball with another (possibly empty) concentric ball removed.

Taking into account the exceptional role of the imaginary space and the counting on the surface of the smaller ball, we have

$$\widetilde{A}_\nu(X) = \sum_{0<a\le\sqrt{(\sqrt{2}-1)X/2}} \mathrm{B}_a(\sqrt{X+a^2}) + \sum_{\sqrt{(\sqrt{2}-1)X/2}<a\le\sqrt{(\sqrt{2}+1)X/2}} \mathrm{B}_a\left(\frac{X}{2a}\right) - \sum_{\sqrt{X}<a\le\sqrt{(\sqrt{2}+1)X/2}} \mathrm{B}_a(\sqrt{a^2-X}) + O(X^{1+\varepsilon}),$$

where the summation index a runs through $\mathbb{Z}$ for $\nu = 0$, through $1/2 + \mathbb{Z}$ for $\nu = 1/2$, and through $\frac{1}{2}\mathbb{Z}$ for $\nu = 1$.

Then, by $(*)$ and $(**)$ in Section 5, it is straightforward to verify

THEOREM 3. *As* $X \to \infty$,

$$\widetilde{A}_0(X) = \pi X^2 - \frac{2\pi}{3} X^{3/2} + O_\varepsilon(X^{37/32+\varepsilon}),$$
$$\widetilde{A}_{1/2}(X) = \pi X^2 + O(X^{7/6}(\log X)^{19/4}),$$
$$\widetilde{A}_1(X) = 2\pi X^2 - \frac{2\pi}{3} X^{3/2} + O(X^{7/6}(\log X)^{19/4}).$$

Note that now the O-terms are sharper than Hlawka's bound $O(X^{6/5})$ for the lattice rest of a four-dimensional convex body. Furthermore, the O-terms are also sharper than the bound $O(X^{13/11}(\log X)^{5/11})$ given by Krätzel and Nowak [7].

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