On the distribution of squares of integral quaternions

by

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1. Introduction and statement of the main result. Let $\mathbb{H}$ denote the division ring of Hamilton’s quaternions and let $\mathbb{J}$ denote the Hurwitz subring of integral quaternions. Thus, as sets, $\mathbb{H} = \mathbb{R}^4$ and $\mathbb{J} = \mathbb{Z}^4 \cup (\frac{1}{2} + \mathbb{Z})^4$. (See Section 8 below for some motivating comments on the choice of $\mathbb{J}$.)

We are interested in an asymptotic formula for the number of quaternions $q^2$ with $q \in \mathbb{J}$ and with all four components of $q^2$ lying in the interval $[-X, X]$, where $X$ is a large positive parameter. This question is motivated by H. Müller and W. G. Nowak [8] where (among other things) an analogous problem is investigated for the ring $\mathbb{Z}[i]$ of Gaussian integers. But there is a remarkable difference between squares in $\mathbb{H}$ and squares in $\mathbb{C}$. For instance, the equation $q^2 = -1$ has infinitely many solutions in $\mathbb{H}$ and still six in $\mathbb{J}$.

Now, the main result of the present paper is the following theorem.

**Theorem 1.** For positive real $X$ let

$$A(X) := \# \{ q^2 \mid q \in \mathbb{J} \land q^2 \in [-X, X]^4 \}.$$ 

Then as $X \to \infty$,

$$A(X) = cX^2 - \frac{2\pi}{3} X^{3/2} + O(X^{96/73}(\log X)^{461/146}),$$

where $c = 7.674124\ldots$ is the four-dimensional volume of $\{ q \in \mathbb{R}^4 \mid q^2 \in [-1, 1]^4 \}$.

**Remark.** Clearly, $cX^2$ equals the volume of $\{ q \in \mathbb{R}^4 \mid q^2 \in [-X, X]^4 \} =: K(X)$. The term $\frac{2\pi}{3} X^{3/2}$ occurs because of the exceptional role of the imaginary space $\text{Im} \mathbb{H} := \{0\} \times \mathbb{R}^3$. Actually, $K(X) \cap \mathbb{J} \cap \text{Im} \mathbb{H}$ contains many points but produces only few different squares.

At first sight the error estimate seems rather coarse. Although the domain $K(X)$ is not convex, one might expect that standard methods for convex bodies like Fourier transformation, the Poisson summation formula,

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Stokes’ theorem, etc. could be successful to obtain at least Hlawka’s classical estimate \( O(X^{6/5}) \) (see [3]). Unfortunately, this is not the case. As we will see in Section 5, the error estimate in Theorem 1 can only be improved together with the sharpest-known estimate in the famous divisor problem.

2. Squaring quaternions. As usual, if \( a = (a_0, a_1, a_2, a_3) \in \mathbb{H} \) let \( \overline{a} = (a_0, -a_1, -a_2, -a_3) \) the conjugate of \( a \), \( \text{Re}(a) = a_0 \) the real or scalar part of \( a \), and \( N(a) = a\overline{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2 \) the norm of \( a \). \( \text{Im}\mathbb{H} = \{0\} \times \mathbb{R}^3 = \{ a \in \mathbb{H} \mid \text{Re}(a) = 0 \} = \{ a \in \mathbb{H} \mid a + \overline{a} = 0 \} \) is the imaginary space. Then we have
\[
a^2 = a(2\text{Re}(a) - \overline{a}) = 2\text{Re}(a)a - N(a) = (a_0^2 - a_1^2 - a_2^2 - a_3^2, 2a_0a_1, 2a_0a_2, 2a_0a_3).
\]

Therefore, \( q^2 \in [-X, X]^4 \) iff \( q \in K(X) \), where
\[
K(X) = \{(a_0, a_1, a_2, a_3) \in \mathbb{R}^4 \mid -X \leq a_0^2 - a_1^2 - a_2^2 - a_3^2, 2a_0a_1, 2a_0a_2, 2a_0a_3 \leq X \}.
\]

Define an equivalence relation \( \sim \) on \( \mathbb{H} \) by \( p \sim q \) iff \( p^2 = q^2 \). How do the equivalence classes look like? It is not difficult to see that \([q]_\sim = \{q, -q\} \) if \( q \in \mathbb{H} \setminus \text{Im}\mathbb{H} \), and \([q]_\sim = \{a \in \text{Im}\mathbb{H} \mid N(a) = N(q)\} \) if \( q \in \text{Im}\mathbb{H} \), the latter being infinite if \( q \neq 0 \).

Now let
\[
A(X) = \# \{ q^2 \mid q \in \mathbb{J} \land q^2 \in [-X, X]^4 \}.
\]

Then we have
\[
A(X) = \# \{ [q]_\sim \mid q \in \mathbb{J} \cap K(X) \}
= \# \{ q, -q \mid q \in \mathbb{J} \cap K(X) \setminus \text{Im}\mathbb{H} \}
+ \# \{ N(q) \mid q \in \mathbb{J} \cap \text{Im}\mathbb{H} \land N(q) \leq X \}
= \# \{ q \in \mathbb{J} \cap K(X) \mid \text{Re}(q) > 0 \} + O(X).
\]

Thus our problem is to count (integral and half odd integral) lattice points in a four-dimensional domain.

3. Preparation of the proof. It is plain that the domain \( K(X) \) is bounded. More precisely, \( K(X) \) is a subset of the four-dimensional cuboid
\[
\left[ -\sqrt{\frac{3X}{2}}, \sqrt{\frac{3X}{2}} \right] \times \left[ -\sqrt{\frac{(\sqrt{2}+1)X}{2}}, \sqrt{\frac{(\sqrt{2}+1)X}{2}} \right] \times \left[ -\sqrt{\frac{3X}{2}}, \sqrt{\frac{3X}{2}} \right] \times \left[ -\sqrt{\frac{(\sqrt{2}+1)X}{2}}, \sqrt{\frac{(\sqrt{2}+1)X}{2}} \right],
\]

which is the smallest set \( I_0 \times I_1 \times I_2 \times I_3 \) containing \( K(X) \).
We have
\[ A(X) = \sum_{a \in \frac{1}{2}\mathbb{Z}} \#(K_a(X) \cap (a + \mathbb{Z})^3) + O(X), \]
where the three-dimensional domain \( K_a(X) \) is given by
\[ K_a(X) := \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid -X + a^2 \leq a_1^2 + a_2^2 + a_3^2 \leq X + a^2 \wedge |a_1|, |a_2|, |a_3| \leq X/(2a) \}. \]
Let
\[ D_a(X) := \{(x, y, z) \in \mathbb{R}^3 \mid -X + a^2 \leq (x - a)^2 + (y - a)^2 + (z - a)^2 \leq X + a^2 \wedge |x - a|, |y - a|, |z - a| \leq X/(2a) \}. \]
Then we obviously have
\[ \#(K_a(X) \cap (a + \mathbb{Z})^3) = \#(D_a(X) \cap \mathbb{Z}^3) \quad \text{for all } a \in \frac{1}{2}\mathbb{Z}. \]
Therefore our program is counting ordinary lattice points in the three-dimensional domain \( D_a(X) \) for every \( a \in \frac{1}{2}\mathbb{Z} \) and then summing up.

How do the domains \( D_a(X) \) look like? For abbreviation, define the constants
\[ c_2 := \sqrt{\frac{2}{3} - 1}, \quad c_3 := \sqrt{\frac{3}{2} - 1}, \quad c_4 := \sqrt{\frac{1}{2}}, \]
\[ c_6 := \sqrt{\frac{2}{3} + 1}, \quad c_7 := \sqrt{\frac{3}{2} + 1}, \quad c_8 := \sqrt{\frac{3}{2}}. \]
Then we observe that \( D_a(X) \) is a ball with radius \( \sqrt{X + a^2} \) for \( 0 < a \leq c_2\sqrt{X} \), a cube with half the length of an edge equal to \( X/(2a) \) for \( c_4\sqrt{X} \leq a \leq \sqrt{X} \), the intersection of a ball and a cube for \( c_2\sqrt{X} \leq a \leq c_4\sqrt{X} \), a cube minus the interior of a ball contained in the cube for \( \sqrt{X} \leq a \leq c_6\sqrt{X} \), and the intersection of a cube and the complement of the interior of a ball for \( c_6\sqrt{X} \leq a \leq c_8\sqrt{X} \).

In order to count the lattice points in \( D_a(X) \) in the various situations we will count the lattice points in cubes, balls, ball segments, and symmetrical intersections of two segments.

For \( H, R, a \in \mathbb{R} \) define
\[ C_a(H) := \#\{(x, y, z) \in \mathbb{Z}^3 \mid -H \leq x - a, y - a, z - a \leq H \}, \]
\[ B_a(R) := \#\{(x, y, z) \in \mathbb{Z}^3 \mid (x - a)^2 + (y - a)^2 + (z - a)^2 \leq R^2 \}, \]
\[ S_a(R, H) := \#\{(x, y, z) \in \mathbb{Z}^3 \mid z - a > H \wedge (x - a)^2 + (y - a)^2 + (z - a)^2 \leq R^2 \}, \]
\[ S_a^*(R, H) := \#\{(x, y, z) \in \mathbb{Z}^3 \mid x - a, z - a > H \wedge (x - a)^2 + (y - a)^2 + (z - a)^2 \leq R^2 \}. \]
Then $\#(D_a(X) \cap \mathbb{Z}^3) = B_a(\sqrt{X} + a^2)$ for $0 < a \leq c_2\sqrt{X}$, $\#(D_a(X) \cap \mathbb{Z}^3) = C_a(X/(2a))$ for $c_4\sqrt{X} \leq a \leq \sqrt{X}$, and for $a \in \frac{1}{2}\mathbb{Z}$, by symmetry,

$$\#(D_a(X) \cap \mathbb{Z}^3) = B_a(\sqrt{X} + a^2) - 6S_a(\sqrt{X} + a^2, X/(2a))$$

if $c_2\sqrt{X} < a \leq c_3\sqrt{X}$,

$$\#(D_a(X) \cap \mathbb{Z}^3) = B_a(\sqrt{X} + a^2) - 6S_a(\sqrt{X} + a^2, X/(2a)) + 12S_a^*(\sqrt{X} + a^2, X/(2a))$$

if $c_3\sqrt{X} < a \leq c_4\sqrt{X}$,

$$\#(D_a(X) \cap \mathbb{Z}^3) = C_a(X/(2a)) - B_a(\sqrt{a^2 - X}) + O(X^{1/2+\varepsilon})$$

if $\sqrt{X} < a \leq c_6\sqrt{X}$,

$$\#(D_a(X) \cap \mathbb{Z}^3) = C_a(X/(2a)) - B_a(\sqrt{a^2 - X}) + 6S_a(\sqrt{a^2 - X}, X/(2a)) + O(X^{1/2+\varepsilon})$$

if $c_6\sqrt{X} < a \leq c_7\sqrt{X}$,

$$\#(D_a(X) \cap \mathbb{Z}^3) = C_a(X/(2a)) - B_a(\sqrt{a^2 - X}) + 6S_a(\sqrt{a^2 - X}, X/(2a)) - 12S_a^*(\sqrt{a^2 - X}, X/(2a)) + O(X^{1/2+\varepsilon})$$

if $c_7\sqrt{X} < a \leq c_8\sqrt{X}$.

The $O$-terms arise since the points on the surface of the ball with radius $R = \sqrt{a^2 - X}$ are counted irregularly. In fact, if $a \in \frac{1}{2}\mathbb{Z}$ then

$$\#\{(x, y, z) \in \mathbb{Z}^3 \mid (x - a)^2 + (y - a)^2 + (z - a)^2 = R^2\} \leq \#\{(x, y, z) \in \mathbb{Z}^3 \mid x^2 + y^2 + z^2 = 4R^2\} \ll R^{1+\varepsilon},$$

since $r_3(n) \ll n^{1/2+\varepsilon}$.

We collect similar terms and write

$$A(X) = \Sigma_1(X) + \Sigma_2(X) - \Sigma_3(X) - 6\Sigma_4(X)$$

$$+ 12\Sigma_5(X) + 6\Sigma_6(X) - 12\Sigma_7(X) + O(X^{1+\varepsilon}),$$

where, with the summation index $a$ always running through $\frac{1}{2}\mathbb{Z}$,

$$\Sigma_1(X) := \sum_{0 < a \leq c_2\sqrt{X}} B_a(\sqrt{X} + a^2),$$

$$\Sigma_2(X) := \sum_{c_5\sqrt{X} < a \leq c_6\sqrt{X}} C_a(X/(2a)),$$

$$\Sigma_3(X) := \sum_{\sqrt{X} < a \leq c_6\sqrt{X}} B_a(\sqrt{a^2 - X}),$$

$$\Sigma_4(X) := \sum_{c_2\sqrt{X} < a \leq c_4\sqrt{X}} S_a(\sqrt{X} + a^2, X/(2a)),$$

$$\Sigma_5(X) := \sum_{c_3\sqrt{X} < a \leq c_4\sqrt{X}} S_a^*(\sqrt{X} + a^2, X/(2a)),$$

$$\Sigma_6(X) := \sum_{c_4\sqrt{X} < a \leq c_5\sqrt{X}} S_a(\sqrt{X} + a^2, X/(2a)),$$

$$\Sigma_7(X) := \sum_{c_5\sqrt{X} < a \leq \sqrt{X}} S_a^*(\sqrt{X} + a^2, X/(2a)),$$
\[ \sum_{a \sqrt{X} < a \leq c_6 \sqrt{X}} S_a(\sqrt{a^2 - X}, X/(2a)), \]
\[ \sum_{c_7 \sqrt{X} < a \leq c_8 \sqrt{X}} S^*_a(\sqrt{a^2 - X}, X/(2a)). \]

4. Two estimates of rounding error sums. Let the rounding error function \( \psi \) be defined by
\[
\psi(z) = z - \lfloor z \rfloor - 1/2 \quad (z \in \mathbb{R})
\]
throughout the paper. ([ ] are the Gauss brackets.)

Note that for every \( z \), \( \psi(z+a) = \psi(z) \) if \( a \in \mathbb{Z} \), and \( \psi(z+a) = \psi(z+1/2) \) if \( a \in 1/2 + \mathbb{Z} \).

For the proof of Theorem 1 we will need estimates of two \( \psi \)-sums which are variants of \( \psi \)-sums occurring in the divisor problem and the circle problem. To obtain these estimates the discrete Hardy–Littlewood method is required. (See Huxley [5] for a profound presentation of the method and its various applications to important problems of geometry and analytic number theory.)

**Lemma 1.** Let \( C \geq 1 \) be an absolute constant. Then as \( X \to \infty \),
\[
\sum_{\alpha \leq n \leq \beta} \psi\left( \frac{X}{n} + \frac{n}{2} \right) \ll X^{23/73} (\log X)^{461/146}
\]
uniformly in \( 1 \leq \alpha \leq \beta \leq C \sqrt{X} \).

**Proof.** Split the sum into
\[
\sum_{\alpha/2 \leq m \leq \beta/2} \psi\left( \frac{X}{2m} \right) + \sum_{(\alpha-1)/2 \leq m \leq (\beta-1)/2} \psi\left( \frac{X}{2m + 1 + \frac{1}{2}} \right)
\]
and apply [5], Theorem 18.2.3, with \( T = X \) to every part of a dyadic division of the first and second sum, respectively, where \( F(x) = 1/(2x) \) is taken in the first case and \( F(x) = 1/(2x + 1/M) + M/(2T) \) in the second.

**Lemma 2.** Let \( \tau \) be an absolute constant, \( 0 < \tau < 1 \). Then as \( r \to \infty \),
\[
\sum_{\delta + h < n \leq \delta + r} \psi(\delta + \sqrt{r^2 - (n - \delta)^2}) \ll r^{67/73} (\log r)^{315/146}
\]
uniformly in \( 0 \leq \delta \leq 1 \) and \( \tau r \leq h \leq r \).

**Proof.** Let \( g(t) = \delta + \sqrt{r^2 - (t - \delta)^2} \) and fix \( M_0 = [r^{67/73}] \). Since for \( r - h \leq M_0 \) the estimate is trivial, assume \( r - h > M_0 \) and choose \( J \in \mathbb{N} \) with \( 2^{J-1} M_0 \leq r - h < 2^J M_0 \). Define a dyadic sequence \( M_j = 2^j M_0 \) \( (j < J) \) and
put $M_j = [r - h]$. Then
\[
\sum_{\delta + h < n \leq \delta + r} \psi(g(n)) = \sum_{j=0}^{J-1} \sum_{M_j < m < M_{j+1}} \psi(f(m)) + O(M_0),
\]
with $f(u) := g([r] - u)$. Now, apply [5], Theorem 18.2.3, to each of the inner sums by setting $M = M_j$, $M' = M_{j+1} - 1$, $C = 1$, $T = M^{3/2}r^{1/2}$, and $F(u) = Mf(Mu)/T$.

**Remark.** It is important to fix $\tau > 0$ in Lemma 2 since otherwise the odd derivatives of $f$ destroy the proof.

5. Lattice points in cubes, balls, and ball segments

**Proposition 1.** For $H > 0$ and $a \in \frac{1}{2} \mathbb{Z}$,
\[ C_a(H) = 8H^3 - 24H^2 \psi(H + a) + O(H). \]

**Proof.** Obviously, $C_a(H) = (2[H] + 1)^3$ if $a \in \mathbb{Z}$, and $C_a(H) = (2[H + 1/2])^3$ if $a \in 1/2 + \mathbb{Z}$.

What is the sharpest estimate of the error that inevitably arises when we sum up the cubes? The summation interval for the cubes is $c_4 \sqrt{X} < a \leq c_5 \sqrt{X}$, at least it contains the interval $c_4 \sqrt{X} < a \leq \sqrt{X}$ where the points in whole cubes are to be counted. Thus, by substituting $a \in \frac{1}{2} \mathbb{Z}$ by $n/2$ with $n \in \mathbb{Z}$, we have to estimate
\[
\sum_{\sqrt{X} \leq n \leq \sqrt{X}} \left( \frac{X}{n} \right)^2 \left( \frac{X}{n} + \frac{n}{2} \right). 
\]
The best estimate of this weighted $\psi$-sum is obtained by Abelian summation combined with the sharpest-known estimate of the unweighted $\psi$-sum (Lemma 1). This yields an error not better than $O(X^{96/73}(\log X)^{461/146})$, which should be taken into consideration when we count the points in the other domains.

Next we consider balls. Obviously, $B_a(R) = B_0(R)$ for $a \in \mathbb{Z}$, and $B_a(R) = B_{1/2}(R)$ for $a \in 1/2 + \mathbb{Z}$. Quite recently, improving Vinogradov’s classical estimate ([9], Theorem 2), Chamizo and Iwaniec [1] and Heath-Brown [2] showed that
\[
(*) \quad B_0(R) = \frac{4\pi}{3} R^3 + O_{\varepsilon}(R^{21/16 + \varepsilon}) \quad (R \to \infty).
\]
In order to obtain a formula for $B_{1/2}(R)$ as well, we write
\[
B_{1/2}(R) = \# \{(x_1, x_2, x_3) \in (1/2 + \mathbb{Z})^3 \mid x_1^3 + x_2^3 + x_3^3 \leq R^2 \}.
\]
The grid $(1/2 + \mathbb{Z})^3$ has the same symmetry as $\mathbb{Z}^3$ but it contains no points lying in a coordinate plane. Fortunately, we can adapt Vinogradov’s proof
[9] for the number of integral points in the sphere to half odd lattice points because each of the 48 pyramids \(0 \leq \delta_i x_i \leq \delta_j x_j \leq \delta_k x_k \ (\{i,j,k\} = \{1,2,3\}, \delta_i, \delta_j, \delta_k \in \{-1,1\})\) has exactly one face lying in a coordinate plane. Additionally, we correct the sloppy estimate \(\sum (\xi(z))^2 \ll M^2 (\ln a)^3\) in [9], p. 320, l. 24, by using the upper bound \(M^2 (\ln a)\). Altogether we obtain

\[
B_{1/2}(R) = \frac{4\pi}{3} R^3 + O(R^{4/3}(\log R)^{10/4}) \quad (R \to \infty).
\]

But we will use (\(*\)) and (\(**\)) only in Section 9. To reach our goal it suffices to allow the coarser error \(O(R^{119/73}(\log R)^{315/146})\), which follows immediately from the next proposition.

** Proposition 2.** As \(R \to \infty\),

\[
S_a(R,H) = \frac{2\pi}{3} R^3 - \pi R^2 H + \frac{\pi}{3} H^3 + \pi (R^2 - H^2) \psi(H + a) + O(R^{119/73}(\log R)^{315/146})
\]

uniformly in \(a \in \mathbb{R}\) and \(-R \leq H \leq R\).

** Proof.** We count the points in level disks by making use of Huxley’s deep estimate in the circle problem. Obviously,

\[
S_a(R,H) = \sum_{a + H < z \leq a + R} \# \{(x,y) \in \mathbb{Z}^2 \mid (x-a)^2 + (y-a)^2 \leq R^2 - (z-a)^2\}.
\]

In the circle problem there is no difficulty concerning the center of the circle. It follows from Huxley [5], Theorem 18.3.2, that uniformly in \((\alpha,\beta) \in \mathbb{R}^2\),

\[
\# \{(x,y) \in \mathbb{Z}^2 \mid (x-\alpha)^2 + (y-\beta)^2 \leq T\} = \pi T + O(T^{23/73}(\log T)^{315/146}).
\]

Consequently,

\[
S_a(R,H) = \pi \sum_{a + H < z \leq a + R} (R^2 - (z-a)^2) + O(R^{119/73}(\log R)^{315/146}).
\]

Now we apply the Euler summation formula (cf. Krätzel [6], Theorem 1.3) to the sum. The main integral yields the main term, which clearly equals the volume of the segment \(\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq H \land x_1^2 + x_2^2 + x_3^2 \leq R^2\}\), and the \(\psi\)-integral is \(\ll R\) by the second mean-value theorem. This concludes the proof of Proposition 2.

6. Counting in intersections of ball segments. For \(0 \leq H \leq R/\sqrt{2}\) let \(V(R,H)\) denote the volume of the domain

\[
\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geq H \land x_3 \geq H \land x_1^2 + x_2^2 + x_3^2 \leq R^2\}.
\]
We compute

\[ V(R, H) = \frac{4}{3} R^3 \arctan \left( \sqrt{1 - \frac{2H^2}{R^2}} \right) + \frac{2}{3} H^2 \sqrt{R^2 - 2H^2} \]

\[ - \left( 2R^2H - \frac{2}{3} H^3 \right) \arctan \left( \sqrt{\frac{R^2}{H^2} - 2} \right). \]

Further, for \(0 \leq H \leq r\) let

\[ \varphi(r, H) = r^2 \arccos \left( \frac{H}{r} \right) - H \sqrt{r^2 - H^2} \]

denote the area of the circle segment \(\{(x, y) \in \mathbb{R}^2 \mid x \geq H \land x^2 + y^2 \leq r^2\}\).

**Proposition 3.** Suppose that \(a \in \frac{1}{2} \mathbb{Z}\). Then as \(R \to \infty\),

\[ S_a^*(R, H) = V(R, H) + 2\psi(H + a) \varphi(\sqrt{R^2 - H^2}, H) \]

\[ + O(R^{119/73} (\log R)^{315/146}) \]

uniformly in \(R/\sqrt{3} \leq H \leq R/\sqrt{2}\).

**Proof.** We write

\[ S_a(R, H) = \sum_{a + H < z \leq a + \sqrt{R^2 - H^2}} \sigma_a(\sqrt{R^2 - (z - a)^2}, H), \]

where

\[ \sigma_a(r, H) = \# \{(x, y) \in \mathbb{Z}^2 \mid x - a > H \land (x - a)^2 + (y - a)^2 \leq r^2\}. \]

First we count the lattice points in circle segments. We have

\[ \frac{1}{2} \sigma_a(r, H) = \sum_{a + H < x \leq a + r} \sqrt{r^2 - (x - a)^2} - \sum_{a + H < x \leq a + r} \psi(a + \sqrt{r^2 - (x - a)^2}), \]

since \([a + b] - [a] - \psi(a) = b - \psi(a + b)\). In view of

\[ \int_H^{r-1} \frac{t}{\sqrt{r^2 - t^2}} \psi(t + a) \, dt \ll \frac{r - 1}{\sqrt{r^2 - (r - 1)^2}} \ll \sqrt{r} \quad (H \leq r - 1) \]

and

\[ \left| \int_{r-1}^r \frac{t}{\sqrt{r^2 - t^2}} \psi(t + a) \, dt \right| \leq \int_{r-1}^r \frac{t}{\sqrt{r^2 - t^2}} \, dt = \sqrt{2r - 1} \ll \sqrt{r} \]

we obtain, by applying the Euler summation formula to the first sum and Lemma 2 (with \(\delta = a - [a]\)) to the second,

\[ \sigma_a(r, H) = \varphi(r, H) + 2\psi(H + a) \sqrt{r^2 - H^2} + O(r^{46/73} (\log r)^{315/146}) \]

uniformly in \(r/\sqrt{2} \leq H \leq r\).
We insert this formula into the sum which we started from and get
\[
S_a(R, H) = \sum_{a+H < z \leq a+\sqrt{R^2-H^2}} \varphi(\sqrt{R^2-(z-a)^2}, H)
+ 2\psi(a+H) \sum_{a+H < z \leq a+\sqrt{R^2-H^2}} \sqrt{R^2-H^2-(z-a)^2}
+ O(R^{119/73}(\log R)^{315/146}).
\]
Again by the Euler summation formula, the second sum equals
\[
\frac{1}{2} \varphi(\sqrt{R^2-H^2}, H) + \psi(H+a)\sqrt{R^2-2H^2} + O(\sqrt{R})
\]
and the first equals
\[
\int_{H}^{\sqrt{R^2-H^2}} \varphi(\sqrt{R^2-t^2}, H) dt + \psi(H+a)\varphi(\sqrt{R^2-H^2}, H)
- 2 \int_{H}^{\sqrt{R^2-H^2}} t \arccos \left( \frac{H}{\sqrt{R^2-t^2}} \right) \psi(t+a) dt.
\]
The main integral is, by the Cavalieri principle, equal to \(V(H,R)\), and the \(\psi\)-integral is, by the second mean-value theorem, \(\ll R\). This concludes the proof of Proposition 3.

**7. Proof of Theorem 1.** First we substitute the summation index \(a \in \frac{1}{2}\mathbb{Z}\) by \(n/2\) with \(n \in \mathbb{Z}\). Then we insert the formulas given in Propositions 1–3, and the formula
\[
B_a(R) = \frac{4\pi}{3} R^3 + O(R^{119/73}(\log R)^{315/146})
\]
into the seven terms \(\Sigma_i(X)\) (1 \(\leq i \leq 7\)) from Section 3.

For abbreviation, let \(\alpha_i, \beta_i\) (1 \(\leq i \leq 7\)) be defined by the following table.

<table>
<thead>
<tr>
<th>(i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha_i)</td>
<td>0</td>
<td>2c4</td>
<td>2</td>
<td>2c3</td>
<td>2c6</td>
<td>2c7</td>
<td></td>
</tr>
<tr>
<td>(\beta_i)</td>
<td>2c4</td>
<td>2c8</td>
<td>2c4</td>
<td>2c4</td>
<td>2c8</td>
<td>2c8</td>
<td>2c8</td>
</tr>
</tbody>
</table>

Let
\[
F_1(X,t) := \frac{4\pi}{3} \left( X + \left( \frac{t}{2} \right)^2 \right)^{3/2}, \quad F_2(X,t) := 8 \left( \frac{X}{t} \right)^3,
F_3(X,t) := -\frac{4\pi}{3} \left( \left( \frac{t}{2} \right)^2 - X \right)^{3/2},
\]
\[ F_4(X, t) := -6\pi \left( \frac{2}{3} \left( X + \left( \frac{t}{2} \right)^2 \right)^{3/2} - \left( X + \left( \frac{t}{2} \right)^2 \right) \left( \frac{X}{t} \right) + \frac{1}{3} \left( \frac{X}{t} \right)^3 \right), \]

\[ F_5(X, t) := 12V \left( \sqrt{X + \left( \frac{t}{2} \right)^2} \frac{X}{t} \right), \]

\[ F_6(X, t) := 6\pi \left( \frac{2}{3} \left( \left( \frac{t}{2} \right)^2 - X \right)^{3/2} - \left( \left( \frac{t}{2} \right)^2 - X \right) \left( \frac{X}{t} \right) + \frac{1}{3} \left( \frac{X}{t} \right)^3 \right), \]

\[ F_7(X, t) := -12V \left( \sqrt{\left( \frac{t}{2} \right)^2 - X} \frac{X}{t} \right), \]

so that

\[ F_{6\pm 1}(X, t) = \mp 2(t^2 \mp 4X)^{3/2} \arctan \sqrt{\frac{t^4 \mp 4Xt^2 - 8X^2}{t^4 + 4Xt^2}} \]

\[ \pm \frac{4X^2}{t^3} \sqrt{t^4 \mp 4Xt^2 - 8X^2} \]

\[ \pm \frac{2X}{t^3} (3t^4 \mp 12t^2X - 4X^2) \arctan \left( \frac{1}{2X} \sqrt{t^4 \mp 4Xt^2 - 8X^2} \right). \]

Then we have

\[ A(X) = \sum_{i=1}^{7} (S_i(X) + \Psi_i(X)) + O(X^{96/73}(\log X)^{315/146}), \]

where

\[ S_i(X) := \sum_{\alpha_i \sqrt{X} < n \leq \beta_i \sqrt{X}} F_i(X, n) \quad (1 \leq i \leq 7), \]

and \( \Psi_i(X) \) are weighted \( \psi \)-sums,

\[ \Psi_i(X) = \sum_{\alpha_i \sqrt{X} < n \leq \beta_i \sqrt{X}} G_i(X, n) \psi \left( \frac{X}{n} + \frac{n}{2} \right) \quad (1 \leq i \leq 7), \]

with \( G_1(X, t) = G_3(X, t) = 0 \), and the other weight functions \( G_i(X, t) \) being monotonic and \( \ll X \). (Note that \( \varphi(r_1, H_1) < \varphi(r_2, H_2) \) if \( r_1 < r_2 \) and \( H_1 > H_2 \).) We estimate these weighted \( \psi \)-sums by Abelian summation combined with Lemma 1 and obtain

\[ A(X) = \sum_{i=1}^{7} S_i(X) + O(X^{96/73}(\log X)^{461/146}). \]

Applying the Euler summation formula to each of the seven sums \( S_i(X), \)
we derive
\[
\sum_{i=1}^{7} S_i(X) = \sum_{i=1}^{7} \int_{\alpha_i \sqrt{X}}^{|\beta_i \sqrt{X}} F_i(X, t) \, dt - \frac{2\pi}{3} X^{3/2}
\]
\[
+ \sum_{i=1}^{7} \int_{\alpha_i \sqrt{X}}^{|\beta_i \sqrt{X}} \left( \frac{d}{dt} F_i(X, t) \right) \psi(t) \, dt,
\]

since
\[
-F_1(X, 2c_4 \sqrt{X}) + F_2(X, 2c_4 \sqrt{X}) - F_4(X, 2c_4 \sqrt{X}) - F_5(X, 2c_4 \sqrt{X}) = 0,
\]
\[
-F_2(X, 2c_8 \sqrt{X}) - F_3(X, 2c_8 \sqrt{X}) - F_6(X, 2c_8 \sqrt{X}) - F_7(X, 2c_8 \sqrt{X}) = 0,
\]
\[
F_3(X, 2\sqrt{X}) = F_4(X, 2c_2 \sqrt{X}) = F_5(X, 2c_3 \sqrt{X}) = F_6(X, 2c_6 \sqrt{X}) = F_7(X, 2c_7 \sqrt{X}) = 0,
\]

and
\[
-\frac{1}{2} F_1(X, 0) = -\frac{2\pi}{3} X^{3/2}.
\]

First we estimate the \(\psi\)-integrals. Let
\[
\frac{d}{dt} F_i(X, t) =: D_i(X, t) \quad (1 \leq i \leq 7).
\]

Obviously, for \(1 \leq i \leq 7\),
\[
F_i(X, u \sqrt{X}) = X^{3/2} F_i(1, u) \quad (\alpha_i \leq u \leq \beta_i).
\]

Thus, for \(1 \leq i \leq 7\),
\[
D_i(X, t) = X D_i(1, t/\sqrt{X}) \quad (\alpha_i \sqrt{X} \leq t \leq \beta_i \sqrt{X}).
\]

We compute
\[
D_2(1, u) = -\frac{24}{u^2}, \quad D_{2 \pm 1}(1, u) = \mp \frac{\pi}{2} u \sqrt{u^2 + 4},
\]
\[
D_{5 \pm 1}(1, u) = \pm \frac{3\pi}{2} \left( u \sqrt{u^2 + 4} - 1 \mp 4 \frac{u}{u^2} - 4 \frac{u^2}{u^4} \right),
\]
\[
D_{6 \pm 1}(1, u) = \mp 6 f_{\pm}(u) \pm 6 g_{\pm}(u) h_{\pm}(u) \pm 12 \tilde{f}_{\pm}(u),
\]

where
\[
f_{\pm}(u) = u \sqrt{u^2 + 4} \arctan \left( \sqrt{\frac{u^4 + 4u^2 - 8}{u^4 + 4u^2}} \right),
\]
\[
g_{\pm}(u) = \left( 1 \pm \frac{2}{u^2} \right)^2, \quad h_{\pm}(u) = \arctan \left( \frac{1}{2} \sqrt{u^4 + 4u^2 - 8} \right),
\]
\[
\tilde{f}_{\pm}(u) = \frac{1}{u^2} \sqrt{u^4 + 4u^2 - 8}.
\]
We observe that, if \( i = 1, 2, 3, 4, 6 \), \( D_i(1, u) \) is monotonic on \( \alpha_i \leq u \leq \beta_i \). Consequently, if \( i = 1, 2, 3, 4, 6 \) then \( D_i(X, t) \) is monotonic on \( \alpha_i \sqrt{X} \leq t \leq \beta_i \sqrt{X} \). Hence, by \( \left| \int_{\alpha_i}^{\beta_i} \psi(t) \, dt \right| \leq 1/8 \) and the second mean-value theorem,

\[
\left| \int_{\alpha_i}^{\beta_i} D_i(X, t) \psi(t) \, dt \right| \leq \frac{1}{4} X \max_{\alpha_i \leq u \leq \beta_i} |D_i(1, u)| \ll X \quad (i = 1, 2, 3, 4, 6).
\]

Furthermore, \( f_{\pm}(u) \), \( g_{\pm}(u) \), \( h_{\pm}(u) \), \( \tilde{f}_{\pm}(u) \) are monotonic on \( \alpha_{6\pm1} \leq u \leq \beta_{6\pm1} \). Hence, with the maxima to be taken over \( \alpha_{6\pm1} \leq u \leq \beta_{6\pm1} \),

\[
\int_{\alpha_{6\pm1}}^{\beta_{6\pm1}} D_i(X, t) \psi(t) \, dt \leq 3X \left( \max |f_{\pm}(u)| \right) \left( \max |g_{\pm}(u)| \right) \left( \max |h_{\pm}(u)| \right) \leq X.
\]

It remains to calculate the integrals \( \int_{\alpha_{\sqrt{X}}}^{\beta_{\sqrt{X}}} F_i(X, t) \, dt \). We replace \( t \) by \( u \sqrt{X} \) to get

\[
\int_{\alpha_{\sqrt{X}}}^{\beta_{\sqrt{X}}} F_i(X, t) \, dt = X^2 \int_{\alpha_i}^{\beta_i} F_i(1, u) \, du \quad (1 \leq i \leq 7).
\]

Since the functions \( F_5 \) and \( F_7 \) can be integrated only numerically we abstain from integrating the other five functions in closed form. With electronic support,

\[
\sum_{i=1}^{7} \int_{\alpha_i}^{\beta_i} F_i(1, u) \, du = 7.674124222443732 \ldots
\]

From the preparation of the problem it is clear that 7.67412... \( X^2 \) equals the volume of the domain \( K(X) \), and this concludes the proof of Theorem 1.

8. On squares of Lipschitz integral quaternions. Historically, the ring \( \mathbb{J} \) does not stand at the beginning of the number theory of quaternions. It is not surprising that the first investigated discrete subring of \( \mathbb{H} \) is \( \mathbb{J}_0 := \mathbb{Z}^4 \). The “integral” quaternions due to Lipschitz are exactly the elements of \( \mathbb{J}_0 \) (cf. [4]). It turned out that \( \mathbb{J}_0 \) is too small to have interesting arithmetic properties. The main arithmetical difference between \( \mathbb{J}_0 \) and \( \mathbb{J} \) is that the Euclidian division algorithm works in \( \mathbb{J} \) but fails in \( \mathbb{J}_0 \). Nevertheless it may be interesting to ask for the distribution of squares of elements in \( \mathbb{J}_0 \). Let us also consider the grid \((1/2 + \mathbb{Z})^4 = \mathbb{J} \setminus \mathbb{J}_0 := \mathbb{J}_{1/2} \) which of course is neither closed under addition nor under multiplication but which is closed under squaring. Then, by adapting the proof of Theorem 1 in a natural way, we obtain
Theorem 2. For positive real $X$ let
\[ A_\nu(X) := \# \{ q^2 \mid q \in J_\nu \land q^2 \in [-X, X]^4 \} \quad (\nu \in \{0, 1/2\}). \]

Then as $X \to \infty$,
\begin{align*}
A_0(X) &= \frac{c}{2} X^2 - \frac{2\pi}{3} X^{3/2} + O(X^{96/73} (\log X)^{461/146}), \\
A_{1/2}(X) &= \frac{c}{2} X^2 + O(X^{96/73} (\log X)^{461/146}),
\end{align*}
where $c$ is the constant in Theorem 1.

Clearly, the term $\frac{2\pi}{3} X^{3/2}$ does not occur in the second formula since $J_{1/2} \cap \text{Im } H = \emptyset$.

9. A variation of the problem. There is another generalization of the problem in Müller and Nowak [8] to quaternions, which can be handled in a very easy way.

Let $\text{Im}(q) := (q_2, q_3)$ denote the imaginary or vector part of the quaternion $q = (q_0, q_1, q_2, q_3)$. Then for $\nu \in \{0, 1/2, 1\}$ let
\[ \tilde{A}_\nu(X) := \# \{ q^2 \mid q \in J_\nu \land |\text{Re}(q^2)|, |\text{Im}(q^2)| \leq X \} \quad (\nu \in \{0, 1/2, 1\}), \]
where $J_1 := J$ and $| \cdot |$ is the Euclidian norm. Then, before summing up over the first component again, we have to count lattice points in the three-dimensional domain
\[ \tilde{K}_a(X) := \{ (a_1, a_2, a_3) \in \mathbb{R}^3 \mid \sqrt{X} + a^2 \leq a_1^2 + a_2^2 + a_3^2 \leq \min \{ X + a^2, X^2/(4a^2) \} \}, \]
which is always a ball with another (possibly empty) concentric ball removed.

Taking into account the exceptional role of the imaginary space and the counting on the surface of the smaller ball, we have
\begin{align*}
\tilde{A}_\nu(X) &= \sum_{0 < a \leq \sqrt{(\sqrt{2}-1)X/2}} B_a(\sqrt{X + a^2}) \\
&\quad + \sum_{\sqrt{(\sqrt{2}-1)X/2} < a \leq \sqrt{(\sqrt{2}+1)X/2}} B_a \left( \frac{X}{2a} \right) \\
&\quad - \sum_{\sqrt{X} < a \leq \sqrt{(\sqrt{2}+1)X/2}} B_a(\sqrt{a^2 - X}) + O(X^{1+\varepsilon}),
\end{align*}
where the summation index $a$ runs through $\mathbb{Z}$ for $\nu = 0$, through $1/2 + \mathbb{Z}$ for $\nu = 1/2$, and through $1/2 \mathbb{Z}$ for $\nu = 1$.

Then, by $(*)$ and $(***)$ in Section 5, it is straightforward to verify
Theorem 3. As $X \to \infty$,

\[
\tilde{A}_0(X) = \pi X^2 - \frac{2\pi}{3} X^{3/2} + O(\varepsilon (X^{37/32} + \varepsilon)),
\]

\[
\tilde{A}_{1/2}(X) = \pi X^2 + O(X^{7/6}(\log X)^{19/4}),
\]

\[
\tilde{A}_1(X) = 2\pi X^2 - \frac{2\pi}{3} X^{3/2} + O(X^{7/6}(\log X)^{19/4}).
\]

Note that now the $O$-terms are sharper than Hlawka's bound $O(X^{6/5})$ for the lattice rest of a four-dimensional convex body. Furthermore, the $O$-terms are also sharper than the bound $O(X^{13/11}(\log X)^{5/11})$ given by Krätzel and Nowak [7].

References


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