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# The Fourier coefficients of modular forms and Niebur modular integrals having small positive weight, II

by

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**1. Introduction.** We introduce and investigate a family of functions called nonanalytic "pseudo-Poincaré series". These functions are inspired by Douglas Niebur's work [5, 6] on automorphic forms and integrals of negative weight. We prove that an arbitrary Niebur modular integral (including a modular form) on the full modular group, $\Gamma(1)$, of weight $k$, $0 < k < 1$, can be decomposed uniquely as a sum of a cusp form and a finite linear combination of (special values of) pseudo-Poincaré series. We derive exact formulas, as convergent infinite series, for the Fourier coefficients of these pseudo-Poincaré series. In the weight range $0 < k < 2/3$, the formulas we produce for these series have precisely the same structure as the well-known expressions for negative weights found by Rademacher and Zuckerman [10]; both involve the modified Bessel function of the first kind and generalized Kloosterman sums. In the weight range $2/3 \le k < 1$, however, the formulas we discover are not as satisfying because they contain Selberg's Kloosterman zeta-function evaluated outside of its known range of convergence. In our prequel [9] we already found expressions, which contain residues of the zeta-function just mentioned, for the Fourier coefficients of small positive powers (between 0 and 2) of the Dedekind eta-function. So our decomposition theorem implies that we possess the Fourier expansions of all Niebur modular integrals on $\Gamma(1)$ of weight $k$, $0 < k < 1$.

The results established here mirror those presented in our first paper [9], which focused on Niebur modular integrals in the weight range $1 < k < 2$. Recall that this previous paper provided an extension of Knopp's work [4] on the Fourier coefficients of modular forms of weight $k$, $4/3 < k < 2$. Several of the tools we employ here are the same as those used in [4] and

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[9]. These include Poisson summation and work pertaining to the analyticity and growth behaviour of Selberg's Kloosterman zeta-function [2, 11, 12]. However, there are some interesting differences here as well. Our point of departure is *not* to invoke (nonanalytic) Poincaré series, which converge absolutely for $\mathrm{Re}(s) > 2-k$, but rather to explore (nonanalytic) pseudo-Poincaré series (mentioned above), which converge absolutely for $\mathrm{Re}(s) > k$. This favorable convergence property (recall that here $0 < k < 1$) is created by the subtracted "Rademacher convergence summand" present in the pseudo-Poincaré series. Ironically, it is this difference which ultimately permits us the use of the same results on Selberg's Kloosterman zeta-function that we invoked in [9]. Another major difference here is the use of a two-variable summation formula [7, 8], one which is needed to handle a certain series that arises in the period of the modular relation for our pseudo-Poincaré series. In fact, the study of this series provides us with a rediscovery of the known formulas [4, 9] for the Fourier coefficients of arbitrary cusp forms of weight $k$, $1 < k < 2$. Lastly, by summoning properties of cusp forms on the full modular group, we prove that "most" of the Niebur modular integrals studied here are actually modular forms.

**2. Definitions and notation.** The basic notation is the same as in [9]. The *full modular group*, $\Gamma(1)$, refers to both $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{PSL}(2,\mathbb{Z})$. This abuse is innocuous. We define the real power of a nonzero complex number by $z^r = e^{r(\log|z|+i\arg z)}$, where $-\pi \le \arg z < \pi$ and $\log|z|$ is real. Throughout, $k$ is a real number and $v$ is a *multiplier system* (*MS*) *for* $\Gamma(1)$ *in the weight* $k$. Specifically, $v$ is a function from the *matrix group* $\Gamma(1)$ into the unit circle which, for $\tau \in \mathcal{H}$, the upper half-plane, satisfies the "consistency condition"

$$v(M_1M_2)(c_3\tau+d_3)^k = v(M_1)v(M_2)(c_1M_2\tau+d_1)^k(c_2\tau+d_2)^k,$$

where $M_j = \begin{pmatrix} * & * \\ c_j & d_j \end{pmatrix} \in \Gamma(1)$, for $j = 1,2$, and $M_1M_2 = \begin{pmatrix} * & * \\ c_3 & d_3 \end{pmatrix}$. Connected with this MS is a *parameter* $\kappa$, which is determined from $v$ by

$$v(S) = e^{2\pi i\kappa}, \quad 0 \le \kappa < 1,$$

where $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Observe that the conjugate function $\overline{v}$ is a MS for $\Gamma(1)$ in the weight $2-k$.

A *Niebur modular integral on* $\Gamma(1)$ *of weight* $k$ *and MS* $v$ is a function $F$ which is holomorphic in $\mathcal{H}$, meromorphic at $\infty$, and for which there exists a cusp form $G \in C^0(2-k,\overline{v})$ such that

$$(1)\qquad [F(\tau) - \overline{v}(V)(\gamma\tau+\delta)^{-k}F(V\tau)]^- = \int_{V^{-1}(\infty)}^{i\infty} G(z)(z-\overline{\tau})^{-k}\,dz,$$

for all $V \in \Gamma(1)$ and $\tau \in \mathcal{H}$. Here $[\cdot]^-$ denotes complex conjugation and the path of integration is a vertical line in $\mathcal{H}$. If $V$ is a translation, then the right side is defined to be identically zero. (Actually, an equivalent definition can be given [5, p. 5] in which relation (1) is assumed for all *non*-translations only.) It is clear that $F$ must possess a *Fourier expansion* of the type

$$F(\tau) = \sum_{n=-\mu}^{\infty} a_n e^{2\pi i(n+\kappa)\tau} \quad \forall \tau \in \mathcal{H}.$$

If $F$ is not identically zero, then we assume that $a_{-\mu} \neq 0$ and we say that the *order at* $\infty$ of $F$ is $-\mu+\kappa$. The vector space of Niebur modular integrals on $\Gamma(1)$ of weight $k$ and MS $v$ which have order at $\infty$ greater than or equal to $-\mu+\kappa$ is denoted by $I(\mu, k, v)$. If the right side of (1) is identically zero for all $V \in \Gamma(1)$, then of course $F$ is a *modular form on* $\Gamma(1)$ *of weight* $k$ *and MS* $v$. The subspace of all modular forms in $I(\mu, k, v)$ is denoted by $C(\mu, k, v)$ and the space of *cusp forms*, $C(-\lfloor 1-\kappa \rfloor, k, v)$, is denoted by $C^0(k, v)$. Here $\lfloor \cdot \rfloor$ is the greatest integer function. We note that each Niebur modular integral has a *unique* cusp form corresponding to it.

**3. Nonanalytic pseudo-Poincaré series.** We define the functions that will eventually give us Niebur modular integrals (in the desired weight range). Let $0 < k < 1$ and $\nu \in \mathbb{Z}$, $\nu < 0$. For $\tau \in \mathcal{H}$ and $\mathrm{Re}(s) > k$ put

$$(2) \quad H_\nu(\tau|s) = H_\nu(\tau|s; k, v)$$
$$= 2e^{2\pi i(\nu+\kappa)\tau} + \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1 \\ (c,d)\neq(0,\pm 1)}} \frac{e^{2\pi i(\nu+\kappa)M\tau}}{v(M)(c\tau+d)^k|c\tau+d|^s} g\left(\frac{-2\pi i(\nu+\kappa)}{c(c\tau+d)}; 1-k\right),$$

where $M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ and

$$(3) \qquad g(w) = g(w;\beta) = \frac{\int_0^w t^{\beta-1}e^{-t}\,dt}{\Gamma(\beta)}.$$

(Note that the numerator in (3) is the *incomplete gamma function*.) We call the functions $H_\nu$ *nonanalytic pseudo-Poincaré series*. They are motivated by certain conditionally convergent series discovered by Niebur [5, p. 43]. Although $H_\nu(\tau|s)$ is not holomorphic in $\tau$, it is holomorphic in $s$ for $\mathrm{Re}(s) > k$. (This latter fact follows from a basic bound on $g$ coupled with a modification of the usual reasoning which is used to study Poincaré series.) The modular relation for $H_\nu(\tau|s)$, which is derived below in Lemma 1, involves a multivariable Poincaré series. Specifically, with $z \in \mathcal{H}$ and all remaining notation as before, define

$$(4)\qquad G_{-\nu-1}(z|\tau|\overline{s}) = G_{-\nu-1}(z|\tau|\overline{s}; 2-k, \overline{v}) = \sum_{\substack{c,d\in\mathbb{Z}\\ (c,d)=1}} \frac{e^{2\pi i[(-\nu-1)+(1-\kappa)]Mz}}{\overline{v}(M)(cz+d)^{2-k}|c\tau+d|^{\overline{s}}}.$$

Observe that $G_{-\nu-1}(z|\tau|\overline{s})$ is analytic in $\overline{s}$ for $\mathrm{Re}(s) > k$.

We now outline the contents of this paper. Section 5 contains the construction of an analytic continuation of both $H_\nu(\tau|s)$ and $G_{-\nu-1}(z|\tau|\overline{s})$ into a region including the origin. These continuations, which rely heavily on the lemmas provided in Section 4, give us explicit formulas (see Section 5, Theorems 7(b) and 9(b)) for the Fourier expansions of $H_\nu(\tau|0)$ and $G_{-\nu-1}(z|\tau|0)$. In Section 6 we establish the

Main Theorem. *Assume* $0 < k < 1$ *and* $\nu \in \mathbb{Z}$, $\nu < 0$. *Let* $H_\nu(\tau) = H_\nu(\tau|0)$ *and* $G_{-\nu-1}(z) = G_{-\nu-1}(z|\tau|0)$, *where* $H_\nu(\tau|s)$ *and* $G_{-\nu-1}(z|\tau|\overline{s})$ *are defined initially by* (2) *and* (4), *respectively. The vector spaces* $I(\cdot,\cdot,\cdot)$ *and* $C^0(\cdot,\cdot)$ *are defined at the end of Section* 2.

(a) $H_\nu(\tau) \in I(-\nu, k, v)$. *Specifically, for all* $V = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$ *and* $\tau \in \mathcal{H}$ *we have*

$$[H_\nu(\tau) - \overline{v}(V)(\gamma\tau+\delta)^{-k} H_\nu(V\tau)]^- = \int_{V^{-1}(\infty)}^{i\infty} G^*_{-\nu-1}(z)(z-\overline{\tau})^{-k}\,dz,$$

*where*

$$G^*_{-\nu-1}(z) := \frac{[2\pi i(\nu+\kappa)]^{1-k}}{\Gamma(1-k)} G_{-\nu-1}(z) \in C^0(2-k, \overline{v}).$$

(b) *Let* $W(\mu, k, v)$, $\mu \in \mathbb{Z}^+$, *denote the space spanned by* $\{H_\nu\}_{\nu=-1}^{-\mu}$. *Then*

$$I(\mu, k, v) = W(\mu, k, v) \oplus C^0(k, v).$$

This result tells us that *every* Niebur modular integral on $\Gamma(1)$ of weight $k$ and MS $v$ can be uniquely written as the sum of a cusp form and a finite linear combination of the functions $H_\nu(\tau)$. In Section 7 we investigate when the Niebur modular integral $H_\nu(\tau)$ is in fact a modular form.

We present the behaviour of $H_\nu(\tau|s)$ under modular transformations.

Lemma 1. *Let* $H_\nu(\tau|s)$ *and* $G_{-\nu-1}(z|\tau|\overline{s})$ *be defined by* (2) *and* (4), *respectively. Also, let* $V = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$. *Then for all* $\tau \in \mathcal{H}$ *and* $\mathrm{Re}(s) > k$,

$$[H_\nu(\tau|s) - \overline{v}(V)(\gamma\tau+\delta)^{-k}|\gamma\tau+\delta|^{-s} H_\nu(V\tau|s)]^- = A \int_{V^{-1}(\infty)}^{i\infty} G_{-\nu-1}(z|\tau|\overline{s})(z-\overline{\tau})^{-k}\,dz,$$

*where*

$$A = \frac{[2\pi i(\nu+\kappa)]^{1-k}}{\Gamma(1-k)}.$$

Proof. If $V$ is a translation, then the statement is obvious and so we assume that $\gamma \neq 0$. Note that

$$g\left(\frac{-2\pi i(\nu+\kappa)}{c(c\tau+d)}\right) = 1 - \frac{1}{\Gamma(1-k)} \int_{\frac{-2\pi i(\nu+\kappa)}{c(c\tau+d)}}^{\infty} t^{-k} e^{-t}\, dt.$$

(Here the path of integration is a horizontal line in the right half-plane.) Putting this into the definition for $H_\nu(\tau|s)$ and rewriting the integral some more gives

$$\begin{aligned} H_\nu(\tau|s) &= 2e^{2\pi i(\nu+\kappa)\tau} \\ &\quad + \sum_{\substack{c,d\in\mathbb{Z}\\ (c,d)=1\\ (c,d)\neq(0,\pm1)}} \left\{ \frac{e^{2\pi i(\nu+\kappa)M\tau}}{v(M)(c\tau+d)^k|c\tau+d|^s} - \frac{\overline{A}}{v(M)(c\tau+d)^k|c\tau+d|^s} \right. \\ &\quad \left. \times \int_{M(\infty)}^{-i\infty} (u-M\tau)^{-k} e^{2\pi i(\nu+\kappa)u}\, du \right\}. \end{aligned}$$

From the above and traditional arguments we obtain

$$\begin{aligned} &H_\nu(\tau|s) - \overline{v}(V)(\gamma\tau+\delta)^{-k}|\gamma\tau+\delta|^{-s} H_\nu(V\tau|s) \\ &= -\frac{2\overline{A}}{v(V)(\gamma\tau+\delta)^k|\gamma\tau+\delta|^s} \int_{V(\infty)}^{-i\infty} (u-V\tau)^{-k} e^{2\pi i(\nu+\kappa)u}\, du \\ &\quad + 2\overline{A} \int_{V^{-1}(\infty)}^{-i\infty} (u-\tau)^{-k} e^{2\pi i(\nu+\kappa)u}\, du \\ &\quad + \sum_{\substack{c,d\in\mathbb{Z}\\ (c,d)=1\\ (c,d)\neq\pm(0,1)\\ (c,d)\neq\pm(\gamma,\delta)}} \frac{\overline{A}}{v(M)(c\tau+d)^k|c\tau+d|^s} \int_{MV^{-1}(\infty)}^{M(\infty)} (u-M\tau)^{-k} e^{2\pi i(\nu+\kappa)u}\, du. \end{aligned}$$

Note that the integral in the summand is over a (hyperbolic) geodesic in the lower half-plane. Transforming this integral as well as the first one and then combining everything we get

$$\begin{aligned} &H_\nu(\tau|s) - \overline{v}(V)(\gamma\tau+\delta)^{-k}|\gamma\tau+\delta|^{-s} H_\nu(V\tau|s) \\ &\qquad = \overline{A} \sum_{\substack{c,d\in\mathbb{Z}\\ (c,d)=1}} \int_{V^{-1}(\infty)}^{-i\infty} \frac{(z-\tau)^{-k} e^{2\pi i(\nu+\kappa)Mz}}{v(M)(cz+d)^{2-k}|c\tau+d|^s}\, dz. \end{aligned}$$

The proof follows readily from the above equality. ∎

**4. Expansions and auxiliary lemmas.** We next record the Fourier expansion of $H_\nu(\tau|s)$.

LEMMA 2. *Let $H_\nu(\tau|s)$ be defined by* (2). *Then for* $\mathrm{Re}(s) > k$ *and* $y = \mathrm{Im}(\tau) > 0$ *we have*

$$\begin{aligned} H_\nu(\tau|s) = 2e^{2\pi i(\nu+\kappa)\tau} &+ 2i^{-k}\frac{(2\pi)^{s+2-k}(-\nu-\kappa)^{1-k}}{\Gamma(s/2)} \\ &\times \bigg\{ \sum_{n=0}^{\infty} (n+\kappa)^s e^{2\pi i(n+\kappa)\tau} \sum_{p=0}^{\infty} \frac{[-4\pi^2(n+\kappa)(\nu+\kappa)]^p}{\Gamma(p+2-k)\Gamma(p+s/2+1)} \\ &\times \sigma(4\pi(n+\kappa)y, p+s/2+1, s/2) Z_{\nu,n}(s/2+p+1-k/2) \\ &+ \sum_{n=1}^{\infty} (n-\kappa)^s e^{-2\pi i(n-\kappa)\overline{\tau}} \sum_{p=0}^{\infty} \frac{[-4\pi^2(n-\kappa)(\nu+\kappa)]^p}{\Gamma(p+2-k)\Gamma(p+s/2+1)} \\ &\times \sigma(4\pi(n-\kappa)y, s/2, p+s/2+1) Z_{\nu,-n}(s/2+p+1-k/2) \bigg\}. \end{aligned}$$

*Here*,

$$\sigma(\eta,\alpha,\beta) = \int_0^\infty (u+1)^{\alpha-1} u^{\beta-1} e^{-\eta u}\, du \tag{5}$$

*and*

$$Z_{m,n}(w) = Z(w; m, n, k, v) = \sum_{c=1}^{\infty} \frac{A_c(m,n)}{c^{2w}} \tag{6}$$

*is Selberg's Kloosterman zeta-function*, *where*

$$A_c(m,n) = A(c; m, n, k, v) = \sum_{\substack{-d=0 \\ (c,d)=1}}^{c-1} \overline{v}(M) e^{\frac{2\pi i}{c}[(m+\kappa)a+(n+\kappa)d]} \tag{7}$$

*is the generalized Kloosterman sum and* $M = \left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$.

The above lemma is derived in very much the same way as one finds the Fourier expansion for nonanalytic Poincaré series. The main difference is that along the way we employed the power series

$$e^w g(w;\beta) = \sum_{p=0}^{\infty} \frac{w^{p+\beta}}{\Gamma(p+1+\beta)},$$

which itself follows from the fact that $e^w g(w;\beta)$ satisfies the first order linear ODE $u' - u = w^{\beta-1}/\Gamma(\beta)$, $u(0) = 0$.

We now concoct an expansion for $G_{-\nu-1}(z|\tau|s)$, the series (with $s$ replaced by its complex conjugate) which appears in the modular relation for $H_\nu(\tau|s)$. Although $G_{-\nu-1}(z|\tau|s)$ cannot have Fourier expansions in $z$ or $\tau$

(since it is neither periodic in $z$ nor in $\tau$ individually), it does possess a hybrid expansion, one which exploits the periodicity of $G_{-\nu-1}(z|\tau|s)$ in $z$ and $\tau$ simultaneously.

LEMMA 3. *Let $G_{-\nu-1}(z|\tau|s)$ be defined by* (4). *Then for* $\mathrm{Re}(s) > k$, $y = \mathrm{Im}(\tau) > 0$ *and* $z \in \mathcal{H}$ *we have*

$$G_{-\nu-1}(z|\tau|s) = 2e^{2\pi i[(-\nu-1)+(1-\kappa)]z} - \frac{2(2\pi)^{s+2-k}i^k}{\Gamma^2(s/2)}$$
$$\times \Big\{ \sum_{l=0}^{\infty} e^{2\pi i(l+1-\kappa)z} \sum_{n=0}^{\infty} e^{2\pi i n\tau} \sum_{p=0}^{\infty} \frac{\{-4\pi^2[(-\nu-1)+(1-\kappa)]\}^p}{p!\Gamma(p+2-k)}$$
$$\times \mathcal{D}_1(s;l,n,p) Z(s/2+p+1-k/2; -\nu-1, l+n, 2-k, \overline{v})$$
$$+ \sum_{l=0}^{\infty} e^{2\pi i(l+1-\kappa)z} \sum_{n=1}^{\infty} e^{-2\pi i n\overline{\tau}} \sum_{p=0}^{\infty} \frac{\{-4\pi^2[(-\nu-1)+(1-\kappa)]\}^p}{p!\Gamma(p+2-k)}$$
$$\times \mathcal{D}_2(s;l,n,p) Z(s/2+p+1-k/2; -\nu-1, l-n, 2-k, \overline{v})\Big\},$$

*where the Dirichlet series $Z$ is defined by* (6),

$$\mathcal{D}_1(s;l,n,p) = \int_0^b (n+t)^{s-1}(l+1-\kappa-t)^{p+1-k} e^{2\pi i(\tau-z)t} \sigma(4\pi(n+t)y, s/2, s/2)\, dt$$

*and*

$$\mathcal{D}_2(s;l,n,p) = \int_0^b (n-t)^{s-1}(l+1-\kappa-t)^{p+1-k} e^{2\pi i(\overline{\tau}-z)t} \sigma(4\pi(n-t)y, s/2, s/2)\, dt.$$

*Here the function $\sigma$ is defined by* (5) *and*

$$b = b(l) = \begin{cases} l-\kappa & \text{if } l = 0, \\ 1 & \text{if } l = 1, 2, \ldots \end{cases}$$

(*Note that we have suppressed the dependence of both $\mathcal{D}_1$ and $\mathcal{D}_2$ on $z, \tau, k$ and $v$.*)

Proof. Rewriting $G_{-\nu-1}(z|\tau|s)$ in the usual way we get

$$G_{-\nu-1}(z|\tau|s) = 2e^{2\pi i(-\nu-\kappa)z} + 2\sum_{c=1}^{\infty} \frac{1}{c^{2-k+s}} \sum_{\substack{-h=0 \\ (c,h)=1}}^{c-1} v(M_{c,h}) e^{2\pi i(-\nu-\kappa)\frac{a}{c}}$$

$$\times \sum_{m=-\infty}^{\infty} \frac{e^{2\pi i\kappa m}}{(z+h/c+m)^{2-k}|\tau+h/c+m|^s} \times \sum_{p=0}^{\infty} \frac{\left(\frac{-2\pi i(-\nu-\kappa)}{c^2(z+h/c+m)}\right)^p}{p!},$$

where $M_{c,h} = \left(\begin{smallmatrix} a & * \\ c & h \end{smallmatrix}\right) \in \Gamma(1)$ and we let $d = h + cm$ with $0 \le -h \le c-1$, $(c,h) = 1$, and $m \in \mathbb{Z}$. Interchanging summations on $m$ and $p$ we are left staring at the following innermost sum:

$$\sum_{m=-\infty}^{\infty} \frac{e^{-2\pi i(1-\kappa)m}}{(\tau+h/c+m)^{s/2}(\overline{\tau}+h/c+m)^{s/2}(z+h/c+m)^{2-k+p}}.$$

We cannot use Poisson summation here! Nonetheless, we have developed a two-variable summation formula specifically designed to handle this sum. An application of this (see [7] or [8, pp. 63–68]; we do not display this gargantuan identity here) gives

$$\begin{aligned} G_{-\nu-1}(z|\tau|s) = 2e^{2\pi i(-\nu-\kappa)z} & \\ - \frac{2(2\pi)^{s+2-k} i^k}{\Gamma^2(s/2)} & \sum_{c=1}^{\infty} \frac{1}{c^{s+2-k}} \sum_{\substack{-h=0 \\ (c,h)=1}}^{c-1} v(M_{c,h}) e^{2\pi i(-\nu-\kappa)\frac{a}{c}} \\ & \times \sum_{p=0}^{\infty} \frac{\left(\frac{-4\pi^2(-\nu-\kappa)}{c^2}\right)^p}{p!\,\Gamma(p+2-k)} \sum_{l=0}^{\infty} e^{2\pi i(l+1-\kappa)(z+\frac{h}{c})} \\ & \times \Big\{ \sum_{n=0}^{\infty} e^{2\pi i n(\tau+\frac{h}{c})} \mathcal{D}_1(s;l,n,p) \\ & + \sum_{n=1}^{\infty} e^{-2\pi i n(\overline{\tau}+\frac{h}{c})} \mathcal{D}_2(s;l,n,p) \Big\}. \end{aligned}$$

Lastly, we change the order of summation. This is permitted by absolute convergence, for $\mathrm{Re}(s) > k$, of both quadruple sums on $c, p, l$, and $n$. ∎

In order to further examine $H_\nu(\tau|s)$ (as well as $G_{-\nu-1}(z|\tau|\overline{s})$) we require some properties of $Z_{m,n}(w)$, which is defined by (6) and (7) for all integers $m$ and $n$. It is easy to see that $Z_{m,n}(w)$ is holomorphic in $\mathrm{Re}(w) > 1$. In his groundbreaking work Selberg [12] proved that $Z_{m,n}(w)$ has an analytic continuation to a function meromorphic in the whole $w$-plane. Furthermore, he showed that $Z_{m,n}(w)$ is holomorphic in $\mathrm{Re}(w) > 1/2$ with the possible exception of a finite number of simple poles on the real segment $1/2 < w \le 1$ (the "exceptional poles"). The following derives from Roelcke's study [11] of the non-Euclidean Laplacian (in weight $k$).

Lemma 4. *Let* $0 < k < 2$ *and* $m$ *and* $n$ *be any integers. Then* $Z(w; m, n, k, v)$ *is holomorphic in* $\mathrm{Re}(w) > 1/2 + |k-1|/2$. *Moreover*, *we have the following*:

(a) *If* $0 < k < 1$, *then* $Z(w; m, n, k, v)$ *is holomorphic in* $\mathrm{Re}(w) \geq 1 - k/2$ *with the possible exception of a simple pole at* $1 - k/2$. *This pole occurs if and only if* $m$ *and* $n$ *are both nonnegative and there exists an* $f \in C^0(k, v)$ *such that the terms* $e^{2\pi i(m+\kappa)\tau}$ *and* $e^{2\pi i(n+\kappa)\tau}$ *both occur in its Fourier expansion.*

(b) *If* $1 < k < 2$, *then* $Z(w; m, n, k, v)$ *is holomorphic in* $\mathrm{Re}(w) \geq k/2$ *with the possible exception of a simple pole at* $k/2$. *This pole occurs if and only if* $m$ *and* $n$ *are both negative and there exists an* $f \in C^0(2-k, \overline{v})$ *such that the terms* $e^{2\pi i[(-m-1)+(1-\kappa)]\tau}$ *and* $e^{2\pi i[(-n-1)+(1-\kappa)]\tau}$ *both occur in its Fourier expansion.*

It is easy to bound $Z_{m,n}(w)$ if $\mathrm{Re}(w) \geq 1 + \delta$, $\delta > 0$, but much more difficult otherwise. In the early 80's, however, Goldfeld and Sarnak [2] established

Lemma 5. *Let* $k \in \mathbb{R}$ *and* $m, n \in \mathbb{Z}$. *Also assume that* $(m+\kappa)(n+\kappa) \neq 0$. *If* $1/2 < \mathrm{Re}(w) < 3/2$ *and* $|\mathrm{Im}(w)| \geq 1$, *then*

$$|Z_{m,n}(w)| \leq C \frac{|m+\kappa| \cdot |n+\kappa| \cdot |\mathrm{Im}(w)|^{1/2}}{\mathrm{Re}(w) - 1/2},$$

*where* $C$ *is a positive constant depending at most upon* $k$ *and* $\kappa$.

From the above bound on $|Z_{m,n}(w)|$ Knopp [3, 4] proved

Lemma 6. *Let* $0 < k < 2$ *and* $m, n \in \mathbb{Z}$.

(a) *If* $0 < k < 2/3$ *and* $Z_{m,n}(w)$ *is holomorphic at* $w = 1 - k/2$, *then*

$$Z_{m,n}(1 - k/2) = \sum_{c=1}^{\infty} \frac{A_c(m,n)}{c^{2-k}}.$$

(b) *If* $4/3 < k < 2$ *and* $Z_{m,n}(w)$ *is holomorphic at* $w = k/2$, *then*

$$Z_{m,n}(k/2) = \sum_{c=1}^{\infty} \frac{A_c(m,n)}{c^{k}}.$$

## 5. Analytic continuation and Fourier expansion

Theorem 7. *Let* $H_\nu(\tau|s)$ *be defined by* (2), *where* $0 < k < 1$ *and* $\nu \in \mathbb{Z}$, $\nu < 0$. *Then*

(a) $H_\nu(\tau|s)$ *has an analytic continuation in* $s$ *into the closed half-plane* $\mathrm{Re}(s) \geq 0$.

(b) $H_\nu(\tau) = H_\nu(\tau|0)$ *is analytic in* $\tau$ *and has the following Fourier expansion*:

$$H_\nu(\tau) = 2e^{-2\pi i|\nu+\kappa|\tau} + \sum_{n=0}^{\infty} a_n(\nu,k,v)e^{2\pi i(n+\kappa)\tau}, \qquad \tau \in \mathcal{H},$$

*where*

$$\begin{aligned} a_n(\nu,k,v) &\\ = 4\pi i^{-k}\Bigg\{ \sum_{c=1}^{\infty} \Bigg[ &\frac{A_c(\nu,n)}{c}\left(\frac{|\nu+\kappa|}{n+\kappa}\right)^{(1-k)/2} I_{1-k}\left(\frac{4\pi}{c}\sqrt{(n+\kappa)|\nu+\kappa|}\right) \\ &- \frac{[2\pi|\nu+\kappa|]^{1-k}}{\Gamma(2-k)}\cdot\frac{A_c(\nu,n)}{c^{2-k}}\Bigg] + \frac{[2\pi|\nu+\kappa|]^{1-k}}{\Gamma(2-k)} Z_{\nu,n}(1-k/2)\Bigg\}. \end{aligned}$$

*Here*

$$I_{1-k}(u) = \sum_{p=0}^{\infty} \frac{(u/2)^{2p+1-k}}{p!\Gamma(p+2-k)}, \qquad u > 0,$$

*is the modified Bessel function of the first kind of order* $1-k$.

Proof. We shall perform an analytic continuation of the Fourier expansion (Lemma 2) of $H_\nu(\tau|s)$ into an open half-plane containing the origin $s = 0$. The key is to examine the "$\sigma$-functions" and the zeta-functions. From (5) we see that the function $\sigma(4\pi(n-\kappa)y, s/2, p+s/2+1)$ is itself analytic in $\mathrm{Re}(s) > -2$ (because $p \ge 0$), whereas $[\Gamma(s/2)]^{-1}\sigma(4\pi(n+\kappa)y, p+s/2+1, s/2)$ has an analytic continuation to $\mathrm{Re}(s) > -2$ which can be found by using integration by parts once. The zeta-functions require more delicate analysis. For $p \ge 1$, however, both $Z_{\nu,n}(s/2+p+1-k/2)$ and $Z_{\nu,-n}(s/2+p+1-k/2)$ are easily seen to be holomorphic in $\mathrm{Re}(s) > k-2$. This leaves us with the case $p = 0$. But from Lemma 4(a), we know that both $Z_{\nu,n}(s/2+1-k/2)$ and $Z_{\nu,-n}(s/2+1-k/2)$ are holomorphic in $\mathrm{Re}(s) \ge 0$. So there exists a $\delta > 0$ (and independent of $n$) such that, once we pull in the function $1/\Gamma(s/2)$, both major summands on the right-hand side of the Fourier expansion of $H_\nu(\tau|s)$ are analytic in $\mathrm{Re}(s) > -\delta$.

In order to finish the proof of part (a) it suffices to show that (once $1/\Gamma(s/2)$ is pulled in) both infinite sums on $n$ converge uniformly in compact subsets of $\mathrm{Re}(s) > -\delta$. But this follows from Lemma 5 and known bounds on the $\sigma$-function (see, for example, Lemma 6 in [9]).

We now show part (b) by evaluating $H_\nu(\tau|s)$ at $s = 0$. Using the fact that

$$\frac{\sigma(4\pi(n+\kappa)y, k+p+s/2, s/2)}{\Gamma(s/2)}\bigg|_{s=0} = 1$$

and noting that the second infinite sum vanishes at $s=0$ we find that

$$a_n(\nu,k,v) = 2i^{-k}(2\pi)^{2-k}|\nu+\kappa|^{1-k}\sum_{p=0}^{\infty}\frac{[-4\pi^2(n+\kappa)(\nu+\kappa)]^p}{p!\Gamma(p+2-k)}Z_{\nu,n}(p+1-k/2).$$

By definition (6),

$$Z_{\nu,n}(p+1-k/2)=\sum_{c=1}^{\infty}\frac{A_c(\nu,n)}{c^{2p+2-k}}\qquad \text{for } p\ge 1.$$

The desired expansion results from plugging this in, interchanging sums (the double sum on $p\ge 1$ and $c$ is absolutely convergent) and recalling the definition of $I_{1-k}$. ∎

COROLLARY 8. *Let $H_\nu(\tau)$ be defined as before but now assume that $0<k<2/3$. Then $H_\nu(\tau)$ is analytic in $\tau$ and has the following Fourier expansion*:

$$H_\nu(\tau)=2e^{-2\pi i|\nu+\kappa|\tau}+\sum_{n=0}^{\infty}c_n(\nu,k,v)e^{2\pi i(n+\kappa)\tau},\qquad \tau\in\mathcal{H},$$

*where*

$$c_n(\nu,k,v) = 4\pi i^{-k}\left(\frac{|\nu+\kappa|}{n+\kappa}\right)^{(1-k)/2}\sum_{c=1}^{\infty}\frac{A_c(\nu,n)}{c}I_{1-k}\left(\frac{4\pi}{c}\sqrt{(n+\kappa)|\nu+\kappa|}\right).$$

P r o o f. This follows right away from Lemma 6(a) and Theorem 7. ∎

THEOREM 9. *Let $G_{-\nu-1}(z|\tau|\overline{s})$ be defined by* (4), *where* $0<k<1$ *and* $\nu\in\mathbb{Z}$, $\nu<0$.

(a) *For both $z$ and $\tau$ fixed in $\mathcal{H}$, $G_{-\nu-1}(z|\tau|\overline{s})$ has an analytic continuation in $\overline{s}$ into the closed half-plane* $\mathrm{Re}(s)\ge 0$.

(b) *$G_{-\nu-1}(z)=G_{-\nu-1}(z|\tau|0)$ is analytic in $z$ and has the following Fourier expansion*:

$$G_{-\nu-1}(z)=2e^{2\pi i[(-\nu-1)+(1-\kappa)]z} + \sum_{l=0}^{\infty}b_l(-\nu-1,2-k,\overline{v})e^{2\pi i(l+1-\kappa)z},\qquad z\in\mathcal{H},$$

*where*

$$\begin{aligned}
&b_l(-\nu-1,2-k,\overline{v})\\
&\quad= -4\pi i^k\Bigg\{\sum_{c=1}^{\infty}\Bigg[\frac{A(c;-\nu-1,l,2-k,\overline{v})}{c}\bigg(\frac{l+1-\kappa}{[(-\nu-1)+(1-\kappa)]}\bigg)^{(1-k)/2}\\
&\qquad\times J_{1-k}\bigg(\frac{4\pi}{c}\sqrt{[(-\nu-1)+(1-\kappa)](l+1-\kappa)}\bigg)\\
&\qquad-\frac{[2\pi(l+1-\kappa)]^{1-k}}{\Gamma(2-k)}\cdot\frac{A(c;-\nu-1,l,2-k,\overline{v})}{c^{2-k}}\Bigg]\\
&\qquad+\frac{[2\pi(l+1-\kappa)]^{1-k}}{\Gamma(2-k)}Z(1-k/2;-\nu-1,l,2-k,\overline{v})\Bigg\}.
\end{aligned}$$

*Here*

$$J_{1-k}(u)=\sum_{p=0}^{\infty}\frac{(-1)^p(u/2)^{2p+1-k}}{p!\Gamma(p+2-k)},\qquad u>0,$$

*is the Bessel function of the first kind of order* $1-k$.

Proof. Obviously, proving part (a) is equivalent to showing that $G_{-\nu-1}(z|\tau|s)$ has an analytic continuation in $s$ into $\mathrm{Re}(s)\ge 0$. We shall work with the expansion given in Lemma 3. First note that the factor preceding both infinite sums on $l$ is an entire function in $s$ with a double zero at $s=0$ (and also at negative even integers). Next observe that all of the factors in both of the summands are independent of $s$ except for $\mathcal{D}_1$, $\mathcal{D}_2$, and the zeta-functions. Now, Lemma 4(b) tells us that both $Z(s/2+p+1-k/2;-\nu-1,l+n,2-k,\overline{v})$ and $Z(s/2+p+1-k/2;-\nu-1,l-n,2-k,\overline{v})$ are holomorphic in $\mathrm{Re}(s)\ge 0$. Here the fact that $-\nu-1\ge 0$ is crucial. Furthermore, we know that there exists a $\delta$, $0<\delta\le 1-k$, which is independent of $l$ and $n$, such that both zeta-functions are holomorphic in $\mathrm{Re}(s)>-\delta$.

We next examine the analyticity, in $s$, of the integrals $\mathcal{D}_1(s;l,n,p)$ and $\mathcal{D}_2(s;l,n,p)$. The function $\sigma(4\pi(n\pm t)y,s/2,s/2)$ is clearly holomorphic in $\mathrm{Re}(s)>0$ and has a simple pole at $s=0$. This implies that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are holomorphic in $\mathrm{Re}(s)>0$ and have a (not necessarily simple) pole at the origin. In fact the poles at the origin are at most double poles and their order depends on $l$ and $n$. It is not hard to see that the following functions are analytic in $\mathrm{Re}(s)>-2$ except for a simple pole at $s=0$: (i) $\mathcal{D}_1(s;l,n,p)$ for $n\ge 1$, (ii) $\mathcal{D}_2(s;l,n,p)$ for $n\ge 2$ and (iii) $\mathcal{D}_2(s;0,1,p)$. It remains to consider $\mathcal{D}_1(s;l,0,p)$, $l\ge 0$ and $\mathcal{D}_2(s;l,1,p)$, $l\ge 1$. We claim that both of these functions have a double pole at $s=0$ and are otherwise analytic in $\mathrm{Re}(s)>-1$. The proof is tedious and we omit it.

We now finish showing that $G_{-\nu-1}(z|\tau|s)$ is analytic in $\mathrm{Re}(s)>-\delta$. From the above analysis we see that, once we pull in $1/\Gamma^2(s/2)$, both of the major summands in our expansion for $G_{-\nu-1}(z|\tau|s)$ are holomorphic in

$\mathrm{Re}(s) > -\delta$. We also know that $G_{-\nu-1}(z|\tau|s)$ is analytic in $\mathrm{Re}(s) > k$ and so it suffices to prove that both infinite sums on $l$ (with the gamma-factors included) converge uniformly on compact subsets of $-\delta < \mathrm{Re}(s) < 1$. It can be shown that

$$|s^2\mathcal{D}_j(s;l,n,p)| \le K_j(n+1)(p+1)(l+1)^{p+1},$$

for $j = 1, 2$, where $K_j$ is a constant which depends at most upon the compact set, $z, \tau, k$ and $\kappa$. The use of these bounds in conjunction with those from Lemma 5 secures part (a).

In order to demonstrate part (b) we must evaluate $G_{-\nu-1}(z|\tau|s)$ at $s = 0$. Substantial vanishing gives

$$\begin{aligned} &G_{-\nu-1}(z|\tau|0) \\ &\quad = 2e^{2\pi i[(-\nu-1)+(1-\kappa)]z} - \frac{1}{2}(2\pi)^{2-k} i^k \\ &\qquad \times \Big\{ \sum_{l=0}^{\infty} e^{2\pi i(l+1-\kappa)z} \sum_{p=0}^{\infty} \frac{\{-4\pi^2[(-\nu-1)+(1-\kappa)]\}^p}{p!\Gamma(p+2-k)} \\ &\qquad \times [s^2\mathcal{D}_1(s;l,0,p)]\big|_{s=0} \cdot Z(p+1-k/2; -\nu-1, l, 2-k, \overline{v}) \\ &\qquad + \sum_{l=1}^{\infty} e^{2\pi i(l+1-\kappa)z} e^{-2\pi i\overline{\tau}} \sum_{p=0}^{\infty} \frac{\{-4\pi^2[(-\nu-1)+(1-\kappa)]\}^p}{p!\Gamma(p+2-k)} \\ &\qquad \times [s^2\mathcal{D}_2(s;l,1,p)]\big|_{s=0} \cdot Z(p+1-k/2; -\nu-1, l-1, 2-k, \overline{v}) \Big\}. \end{aligned}$$

From a calculation we get

$$[s^2\mathcal{D}_1(s;l,0,p)]\big|_{s=0} = 2(l+1-\kappa)^{p+1-k}$$

and

$$[s^2\mathcal{D}_2(s;l,1,p)]\big|_{s=0} = 2e^{2\pi i(\overline{\tau}-z)}(l-\kappa)^{p+1-k}.$$

Plugging in these expressions and rearranging completes our derivation. Note that the expansion is indeed independent of $\tau$ and thus we are justified in letting $G_{-\nu-1}(z) = G_{-\nu-1}(z|\tau|0)$. ∎

Corollary 10. *Let $G_{-\nu-1}(z)$ be defined as before but assume that $0 < k < 2/3$. Then $G_{-\nu-1}(z)$ is analytic in $z$ and has the following Fourier expansion*:

$$\begin{aligned} G_{-\nu-1}(z) &= 2e^{2\pi i[(-\nu-1)+(1-\kappa)]z} \\ &\qquad + \sum_{l=0}^{\infty} d_l(-\nu-1, 2-k, \overline{v}) e^{2\pi i(l+1-\kappa)z}, \qquad z \in \mathcal{H}, \end{aligned}$$

*where*

$$d_l(-\nu-1,2-k,\overline{v}) = -4\pi i^k\left(\frac{l+1-\kappa}{[(-\nu-1)+(1-\kappa)]}\right)^{(1-k)/2} \times \sum_{c=1}^{\infty}\frac{A(c;-\nu-1,l,2-k,\overline{v})}{c} \times J_{1-k}\left(\frac{4\pi}{c}\sqrt{[(-\nu-1)+(1-\kappa)](l+1-\kappa)}\right).$$

Proof. This is an instant consequence of Knopp's result (Lemma 6(b)) and the previous theorem. ∎

We conclude this section by observing that the space spanned by the functions $\{G_{-\nu-1}\}_{\nu=-1}^{-\infty}$ is in fact the space of cusp forms $C^0(2-k,\overline{v})$, where $0<k<1$. This follows from our knowledge that the above Fourier expansions for $G_{-\nu-1}$ were already shown to represent cusp forms (see [4] for forms of weight between 4/3 and 2 and [9] for the remainder of the weight range). We shall not prove this anew.

**6. Niebur modular integrals.** Our work now culminates with a proof of the Main Theorem, which was stated in Section 3. We first consider part (a). From Lemma 1 we have

$$(8)\quad [H_\nu(\tau|s)-\overline{v}(V)(\gamma\tau+\delta)^{-k}|\gamma\tau+\delta|^{-s}H_\nu(V\tau|s)]^- = \frac{[2\pi i(\nu+\kappa)]^{1-k}}{\Gamma(1-k)}\int_{V^{-1}(\infty)}^{i\infty} G_{-\nu-1}(z|\tau|\overline{s})(z-\overline{\tau})^{-k}\,dz$$

for $\mathrm{Re}(s)>k$. Now by Theorem 7(a), for fixed $\tau\in\mathcal{H}$ the left-hand side above is analytic in $\overline{s}$ for $\mathrm{Re}(s)\ge 0$. Next we use Theorem 9(a), which tells us that for fixed $z$ and $\tau$ in $\mathcal{H}$, $G_{-\nu-1}(z|\tau|\overline{s})$ is analytic in $\overline{s}$ for $\mathrm{Re}(s)\ge 0$. Also, for fixed $\tau$ and $\overline{s}$ we know that $G_{-\nu-1}(z|\tau|\overline{s})$ vanishes exponentially in $z$ as $\mathrm{Im}(z)\to\infty$. These facts imply that the right-hand side of (8) is analytic in $\overline{s}$ for $\mathrm{Re}(s)\ge 0$. Invoking the identity principle and setting $s=0$ in (8) establishes the modular relation for $H_\nu$. Combining this with Theorem 7(b) and the ultimate paragraph of the previous section (which tells us that $G_{-\nu-1}$ is a cusp form) finishes the proof of part (a).

The proof of part (b) relies on Petersson's Riemann–Roch Theorem for automorphic forms. It is the same as that given in the prequel to this paper (see the proof of Theorem 17 in [9]) and we do not reproduce it here. ∎

**7. Modular forms.** In the previous section we proved that $H_\nu(\tau)$ is a Niebur modular integral. But is it a modular form? Clearly, it is one if and only if $G^*_{-\nu-1}(z)$, the cusp form of complementary weight and conjugate multiplier system corresponding to it, is identically zero. Recalling the paucity of nontrivial cusp forms on $\Gamma(1)$ of small weight we have the following

THEOREM 11. *Let the Niebur modular integral $H_\nu(\tau) = H_\nu(\tau; k, v)$ be defined as before, with $\nu < 0$ and $0 < k < 1$. Also, let $v_\eta$ be the MS associated with the Dedekind eta-function, defined by*

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{l=1}^{\infty}(1 - e^{2\pi il\tau}), \qquad \tau \in \mathcal{H}.$$

(a) *If $v \equiv v_\eta^{2k+4j}$, $j \in \{0, 1, 2, 3, 4\}$, then $H_\nu(\tau)$ is a modular form for all $\nu$.*

(b) *If $v \equiv v_\eta^{2k+20}$, however, then $H_\nu(\tau)$ is a modular form if and only if the $(-\nu-1)$th Fourier coefficient of $\eta^{4-2k}(\tau)$ vanishes. In particular, we know that $H_{-1}(\tau)$ is* **not** *a modular form.*

Proof. Part (a) is a consequence of the fact that $C^0(2-k, \overline{v})$ is either 0-dimensional or is spanned by just one function, $\eta^{4-2k}(\tau)$. The latter happens for only one out of the six possible MS for $\Gamma(1)$ in weight $k$, namely $\overline{v} \equiv v_\eta^{4-2k}$. Now, if $v \equiv v_\eta^{2k-4} \equiv v_\eta^{2k+20}$, then $H_\nu(\tau)$ is a modular form if and only if $G_{-\nu-1}(z) \equiv 0$. But this happens only when the $(-\nu-1)$th Fourier coefficient of $\eta^{4-2k}(\tau)$ equals zero. (This follows from consideration of the Petersson inner product of $G_{-\nu-1}$ and $\eta^{4-2k}$.) Since the 0th Fourier coefficient of $\eta^{4-2k}(\tau)$ is clearly nonzero, we see that $H_{-1}(\tau)$ is not a modular form. ■

COROLLARY 12. *Let the Niebur modular integral $H_\nu(\tau; k, v)$ be defined as before. Then $H_\nu(\tau; 1/2, v_\eta^{21})$ is a modular form if and only if $\nu \neq -m(m+1)/2 - 1$, $m \in \mathbb{Z}$.*

Proof. This is a consequence of part (b) of the previous theorem and Jacobi's famous triple product identity (see, for example, [1, pp. 21–22]), which implies that the Fourier coefficients of $e^{-\pi i\tau/4}\eta^3(\tau)$ are supported on the triangular numbers: $m(m+1)/2$, $m \in \mathbb{Z}$. ■

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