ACTA ARITHMETICA
XCIII.2 (2000)

A note on a result of Bateman and Chowla

by

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1. Introduction. In 1961, answering a problem proposed by N. J. Fine, Besicovitch [2] constructed an example of a non-trivial real continuous function f on $[0,1]$ which is not odd with respect to the point $1/2$ and with the property that

$$\sum_{a=1}^{n} f\left(\frac{a}{n}\right) = 0 \quad \text{for each } n \in \mathbb{N}. \tag{1}$$

His proof consisted in the definition of the required function in inductive stages on small subintervals of $[0,1]$ and, in modern terminology, is rather akin to the construction of a complicated fractal function.

Bateman and Chowla [1], in 1963, pointed out that the more explicit functions

$$f_1(\theta) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n} \cos 2\pi n\theta \tag{2}$$

where λ denotes the Liouville function and

$$f_2(\theta) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cos 2\pi n\theta \tag{3}$$

where μ denotes the Möbius function also share the above properties of Besicovitch's function. The continuity of these two functions follows from the uniform convergence of the series involved, which is a classical result of Davenport [3]. The other properties including (1) are then comparatively trivial to demonstrate.

From a heuristic point of view, it is by no means clear from their paper why one might expect, a priori, functions such as (2) or (3) to be associated with Fine's problem.

In this paper, we show that a class of functions, which includes Davenport's function (3), arises naturally as formal infinite limits of a finite

2000 *Mathematics Subject Classification*: Primary 11N99.

minimizing problem involving sums of type (1). We then show that each member of this class provides in fact a solution to Fine's problem. To do this, we prove a Davenport-type uniform convergence result of the series involved using Vaughan's identity, and one interesting outcome of our work is that the function

$$f(\theta)=\sum_{n=1}^{\infty}\frac{\mu(n)}{\sigma(n)}\cos 2\pi n\theta, \tag{4}$$

where σ is the usual sum of divisors function, is in a sense a more natural solution to the original problem than is (3). Our main result is thus the following.

THEOREM 1. *Let $h(n)$ be any positive multiplicative function with*

$$h(p)=1+O\left(\frac{\log p}{p}\right)\quad \text{for primes } p.$$

Then $f(x)$ defined by

$$f(x)=\sum_{k=1}^{\infty}h(k)\frac{\mu(k)}{k}\cos 2\pi kx$$

is a non-trivial function, continuous on $[0,1]$, which satisfies $f(x)=f(1-x)$ and has the property that

$$\sum_{a=1}^{n}f(a/n)=0\quad \text{for each } n\in\mathbb{N}.$$

2. A finite minimizing problem and its solution. For any real function f, continuous on $[0,1]$, define its *deviation* $D(n)=D_f(n)$, of order n, by

$$D(n)=\frac{1}{n}\sum_{a=1}^{n}f\left(\frac{a}{n}\right)-\int_0^1 f(x)\,dx\quad \text{for any } n\in\mathbb{N}.$$

Clearly $D_f(n)=D_g(n)$ if f and g differ by a constant. For even trigonometric polynomials

$$f(x)=f_N(x)=\sum_{k\le N}\frac{c(k)}{k}\cos 2\pi kx \tag{5}$$

we see that, for any fixed $N\in\mathbb{N}$ and $c(k)\in\mathbb{R}$,

$$D(n)=\frac{1}{n}\sum_{a=1}^{n}f\left(\frac{a}{n}\right)=\sum_{\substack{k\le N\\ n|k}}\frac{c(k)}{k}. \tag{6}$$

We consider the problem of determining a function of the form (5) which minimizes the weighted l^2-norm $\|D_N\|$ of the deviations $D(n)$ defined by

$$\|D_N\|^2 = \sum_{n\le N} \alpha_n D^2(n) = \sum_{n\le N} \frac{\alpha_n}{n^2}\Big(\sum_{h\le N/n} \frac{c(hn)}{h}\Big)^2 \tag{7}$$

subject to the normalizing condition $c(1) = 1$ and where α_n are any given positive numbers.

THEOREM 2. *For $N \in \mathbb{N}$ define a class S_N of real trigonometric polynomials of order N by*

$$S_N = \Big\{f : f(x) = \sum_{k\le N} \frac{c(k)}{k}\cos 2\pi kx,\ c(k)\in\mathbb{R},\ c(1)=1\Big\}.$$

Then for any real positive α_n and any $f \in S_N$,

$$\|D_N\|^2 \ge \frac{1}{\sum_{n\le N}\mu^2(n)/\alpha_n}$$

with equality for the polynomial $f \in S_N$ with

$$\frac{c(k)}{k} = \frac{1}{\sum_{n\le N}\mu^2(n)/\alpha_n}\Big(\sum_{\substack{n\le N/k\\ (n,k)=1}} \frac{\mu^2(n)}{\alpha_{nk}}\Big)\mu(k) \quad \textit{for each } k, 1\le k\le N.$$

P r o o f. The condition $c(1) = 1$ can be expressed as

$$\sum_{\substack{h,n\\ hn\le N}} \frac{c(hn)\mu(n)}{hn} = \sum_{l\le N}\frac{c(l)}{l}\sum_{n\mid l}\mu(n) = 1$$

and hence

$$\sum_{n\le N}\frac{\mu(n)}{\alpha_n^{1/2}}\cdot\frac{\alpha_n^{1/2}}{n}\sum_{h\le N/n}\frac{c(hn)}{h} = 1.$$

We apply the Cauchy–Schwarz inequality to this condition in a manner reminiscent of Turán's proof of Selberg's Upper Bound Sieve (see Halberstam–Richert [4], p. 121) to obtain

$$\sum_{n\le N}\frac{\mu^2(n)}{\alpha_n}\sum_{n\le N}\frac{\alpha_n}{n^2}\Big(\sum_{h\le N/n}\frac{c(hn)}{h}\Big)^2 \ge 1,$$

i.e. that

$$\|D_N\|^2 \ge \frac{1}{\sum_{n\le N}\mu^2(n)/\alpha_n},$$

with equality when

$$\frac{\mu(n)}{\alpha_n^{1/2}} = C\frac{\alpha_n^{1/2}}{n}\sum_{h\le N/n}\frac{c(hn)}{h}$$

for some $C \neq 0$ and all $n \le N$. By Möbius inversion,

$$\frac{c(k)}{k} = \frac{1}{C}\sum_{h\le N/k}\frac{\mu(h)\mu(hk)}{\alpha_{hk}} = \frac{1}{C}\Bigg(\sum_{\substack{h\le N/k\\(h,k)=1}}\frac{\mu^2(h)}{\alpha_{hk}}\Bigg)\mu(k).$$

The condition $c(1) = 1$ forces the choice $C = \sum_{h\le N}\mu^2(h)/\alpha_h$, and this completes the proof of Theorem 2.

Now suppose that the positive weights α_n are multiplicative functions of n with

$$\alpha_p = 1 + O\bigg(\frac{\log p}{p}\bigg). \tag{8}$$

We shall determine the formal limit of the minimizing polynomial in Theorem 2 as $N \to \infty$ by calculating the limit of $c(k)/k$ as $N \to \infty$ for each fixed k. Clearly

$$\frac{c(k)}{k} = \bigg(\frac{\sum_{n\le N/k,\,(n,k)=1}\mu^2(n)/\alpha_n}{\sum_{n\le N}\mu^2(n)/\alpha_n}\bigg)\frac{\mu(k)}{\alpha_k}.$$

Writing $\beta(n) = 1/\alpha_n$, we have for $\operatorname{Re} s > 1$,

$$\begin{aligned}\sum_{\substack{n=1\\(n,k)=1}}^{\infty}\frac{\mu^2(n)\beta(n)}{n^s} &= \prod_p\bigg(1+\frac{\beta(p)}{p^s}\bigg)\prod_{p\mid k}\bigg(1+\frac{\beta(p)}{p^s}\bigg)^{-1}\\ &= F(s)G(s,k), \quad \text{say.}\end{aligned} \tag{9}$$

Writing $\beta(p) = 1 + R(p)$, where by hypothesis $R(p) = O((\log p)/p)$, we obtain

$$F(s) = \frac{\zeta(s)}{\zeta(2s)}\prod_p\bigg(1+\frac{R(p)}{p^s+1}\bigg)$$

and hence $F(s)$ is analytic in a region which includes $\operatorname{Re} s \ge 1$ except for a simple pole at $s = 1$ with residue

$$\frac{1}{\zeta(2)}\prod_p\bigg(1+\frac{R(p)}{p+1}\bigg).$$

Therefore by the Wiener–Ikehara Theorem, or indeed by more elementary means, it follows from (9) that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{\substack{n\le x\\(n,k)=1}}\mu^2(n)\beta(n)=\frac{1}{\zeta(2)}\prod_p\Big(1+\frac{R(p)}{p+1}\Big)\prod_{p\mid k}\Big(1+\frac{\beta(p)}{p}\Big)^{-1}.$$

A simple calculation then yields that, for fixed $k\in\mathbb{N}$,

$$\lim_{N\to\infty}c(k)=\prod_{p\mid k}\Big(\alpha_p+\frac{1}{p}\Big)^{-1}\mu(k).$$

Hence the formal limit of the minimizing polynomial is given by

$$f(x)=\sum_{k=1}^{\infty}\prod_{p\mid k}\Big(\alpha_p+\frac{1}{p}\Big)^{-1}\frac{\mu(k)}{k}\cos 2\pi kx. \tag{10}$$

Note that the choice $\alpha_p=1-1/p$, i.e. $\alpha_k=\phi(k)/k$, yields Davenport's function (3) whilst the equal weights $\alpha_k=1$ give the function (4) mentioned in the introduction.

Remark. Although the condition (8) on α_p is principally chosen here to facilitate calculations in the application of Vaughan's identity, in particular it ensures that α_p are not too small and hence the function

$$h(n)=\prod_{p\mid n}\Big(\alpha_p+\frac{1}{p}\Big)^{-1}$$

satisfies $h(n)\ll(\log n)^c$ for some $c>0$; it is equally true that α_p cannot be too large since we can show that $\sum_{n=1}^{\infty}h(n)/n$ needs to be necessarily divergent for the overall function $f(x)$ to have all the desired properties.

3. Proof of Theorem 1. Our Theorem 3 proved below implies that

$$\sum_{k\le y}\mu(k)h(k)\cos 2\pi kx\ll y/\log^{\lambda} y$$

uniformly in x, for any $\lambda>0$. Writing

$$S_N(x)=\sum_{k\le N}\frac{\mu(k)h(k)}{k}\cos 2\pi kx,$$

we deduce, by partial summation, that

$$\begin{aligned}S_{N+M}(x)-S_N(x)\\ &=\Big(\sum_{k\le N+M}\mu(k)h(k)\cos 2\pi kx\Big)\frac{1}{N+M}\\ &\quad-\Big(\sum_{k\le N}\mu(k)h(k)\cos 2\pi kx\Big)\frac{1}{N}+\int_N^{N+M}\Big(\sum_{k\le t}\mu(k)h(k)\cos 2\pi kx\Big)\frac{dt}{t^2}.\end{aligned}$$

This implies, using Theorem 3 with $\lambda > 1$, that $S_N(x)$ converges uniformly in x and hence that $f(x)$ given by (10) is continuous. Integrating the series term by term, we deduce that

$$\int_0^1 f(x)\,dx = 0$$

and, by Parseval's identity,

$$\int_0^1 f^2(x)\,dx = \frac{1}{2}\sum_{k=1}^{\infty}\frac{\mu^2(k)h^2(k)}{k^2} \ge \frac{1}{2}$$

so that $f(x)$ is non-trivial. In addition, setting $g(k) = \mu(k)h(k)$, we find that for any $n \in \mathbb{N}$,

$$\sum_{a=1}^{n} f\left(\frac{a}{n}\right) = \sum_{a=1}^{n}\left(\sum_{k=1}^{\infty}\frac{g(k)}{k}\cos\frac{2\pi ka}{n}\right)$$
$$= \sum_{k=1}^{\infty}\frac{g(k)}{k}\sum_{a=1}^{n}\cos\frac{2\pi ka}{n} = n\sum_{\substack{k=1\\ n\mid k}}^{\infty}\frac{g(k)}{k} = \Bigg(\sum_{\substack{h=1\\ (h,n)=1}}^{\infty}\frac{g(h)}{h}\Bigg)g(n).$$

Now for $\operatorname{Re} s > 1$, observe that

$$\sum_{\substack{h=1\\ (h,n)=1}}^{\infty}\frac{g(h)}{h^s} = \prod_{p\nmid n}\left(1+\frac{g(p)}{p^s}\right) = \frac{1}{\zeta(s)}G(s)$$

where $G(s)$ is analytic in a region which contains the point $s = 1$. Hence by the continuity theorem for Dirichlet series, we see that for all $n \in \mathbb{N}$,

$$\sum_{\substack{h=1\\ (h,n)=1}}^{\infty}\frac{g(h)}{h} = \lim_{s\to 1}\frac{G(s)}{\zeta(s)} = 0,$$

which implies that $\sum_{a=1}^{n} f(a/n) = 0$ for all $n \in \mathbb{N}$, as required.

This completes the proof of Theorem 1. We now prove, as required, Theorem 3.

THEOREM 3. *Let $h(n)$ be any positive multiplicative function with*

$$h(p) = 1 + O\left(\frac{\log p}{p}\right) \qquad \text{for primes } p.$$

Then, for any $\lambda > 0$,

$$\max_{\alpha\in[0,1]}\Big|\sum_{n\le x}\mu(n)h(n)e(n\alpha)\Big| \ll_\lambda x/\log^\lambda x$$

where, as usual, $e(n\alpha) = \exp(2\pi i n\alpha)$ and $\ll_\lambda$ indicates the Vinogradov symbol with the implicit constant depending at most on λ.

REMARK. With a more judicious choice of the parameters involved, it is easily seen that the hypothesis on h can be relaxed to

$$h(p) = 1 + O(1/p^{1/2})$$

and the bound obtained can be sharpened to

$$\ll x\exp(-c_0(\log x)^{1/2}).$$

We have refrained from doing this since we only need Theorem 3 as stated and even so in fact only for some $\lambda > 1$.

Proof (of Theorem 3). Set $g(n) = \mu(n)h(n)$ and note that $g(n) \ll \log^c n$ for some fixed $c \geq 1$. We need the following Siegel–Walfisz type result due to Siebert [5], Satz 4.

LEMMA 1. *Let $f(n)$ be a multiplicative function with*

$$\sum_{p\leq x} |f(p) + \tau| \ll x^{1-\varepsilon}$$

where $\varepsilon > 0, \tau \in \mathbb{N}$ and $|f(p^a)| \leq c_1 a^{c_2}$ with $a \in \mathbb{N}$ and $c_1, c_2 > 0$. Then for any $h > 0$ and $\theta = \theta(h) > 0$,

$$\sum_{\substack{n\leq x\\ n\equiv l \pmod k}} f(n) \ll x\exp(-\theta(\log x)^{1/2})$$

uniformly for $k \leq \log^h x$.

Observe that $g(n)$ satisfies the hypotheses of Lemma 1 with $\tau = 1$. Note also that the upper bound in Theorem 3 for $\alpha = 0$ and $\alpha = 1$ follows immediately from this lemma so that we may assume henceforth that $\alpha \in (0,1)$.

For any $Q \in \mathbb{N}$, Dirichlet's theorem implies that there exist $a, q \in \mathbb{N}$ with $(a,q) = 1$ and $q \leq Q$ such that

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{qQ}.$$

Put $Q = x(\log x)^{-\lambda_1}, \delta = (\log x)^{\lambda_1}$ where λ_1 satisfies

$$\lambda_1 \geq 2\lambda + 2c + 5,$$

c as in the upper bound for $g(n)$. We define the major arcs to consist of those α with corresponding $q \leq \delta$ and the minor arcs those α with $\delta < q \leq Q$.

Write

$$M_n = \sum_{m\leq n} g(m)e(am/q)$$

for each $\alpha \in (0,1)$. A simple calculation involving partial summation yields

$$\Big|\sum_{n\le x} g(n)e(n\alpha)\Big| \le \left(1+\frac{2\pi x}{qQ}\right)\max_{n\le x}|M_n|. \tag{11}$$

On the major arcs, writing

$$M_n = \sum_{r=0}^{q-1} e(ar/q) \sum_{\substack{m\le n\\ m\equiv r \pmod q}} g(m)$$

and using (11) and Lemma 1, one easily obtains

$$\Big|\sum_{n\le x} g(n)e(n\alpha)\Big| \le \max_{n\le x}\max_{0\le r\le q-1}(q+2\pi x/Q)\Big|\sum_{\substack{m\le n\\ m\equiv r \pmod q}} g(m)\Big|$$
$$\ll x\exp(-\theta(\lambda_1)(\log x)^{1/2})(\log x)^{\lambda_1} \ll x/\log^{\lambda} x.$$

On the minor arcs we have $qQ > x$ and hence from (11), it suffices to show that

$$\max_{n\le x}|M_n| \ll_\lambda x/\log^\lambda x.$$

Since, trivially, $M_n \ll n(\log n)^c$, it suffices to prove that

$$M_N \ll x/\log^\lambda x$$

for any N with $x(\log x)^{-\lambda_1} \le N \le x$.

Put $u = N^{2/5}$. Vaughan's identity [6] yields the decomposition

$$M_N = S_0 + S_1 - S_2 - S_3$$

where

$$S_0 = \sum_{n\le u} g(n)e(na/q),$$
$$S_1 = \sum_{d\le u}\mu(d)\sum_{r\le N/d}\ \sum_{n\le N/(dr)} g(n)e(drna/q),$$
$$S_2 = \sum_{d\le u}\mu(d)\sum_{n\le u}\ \sum_{r\le N/(dn)} g(n)e(drna/q),$$
$$S_3 = \sum_{u\le m\le N/u}\tau(m)\sum_{u<n\le N/m} g(n)e(mna/q).$$

Trivially, we have, for any $\varepsilon > 0$,

$$S_0 \ll u(\log x)^c \ll x^{2/5+\varepsilon}.$$

To estimate S_1, writing $rn = k$, we see that

$$S_1 = \sum_{d\le u}\mu(d)\sum_{k\le N/d} e(dka/q)\sum_{n\mid k} g(n)$$

and hence

$$S_1 \ll \sum_{d\le u} \sum_{k\le N/d} \Big|\sum_{n|k} g(n)\Big|.$$

Using $|\sum_{n|k} g(n)| = \prod_{p|k} |1-h(p)|$, we deduce that

$$\sum_{k\le N/d} \Big|\sum_{n|k} g(n)\Big| \le \sum_{k\le N/d} \Big|\sum_{n|k} g(n)\Big| \left(\frac{N}{dk}\right)^{1/2} \ll \left(\frac{N}{d}\right)^{1/2}$$

and hence

$$S_1 \ll N^{1/2} u^{1/2} \ll x^{7/10}.$$

For the estimation of S_2 and S_3, we need Lemma 2.2 of Vaughan [6] which we state here in two parts.

Lemma 2. (i) *For $N_1, N_2 \in \mathbb{Z}$ and $N_2 \ge N_1$,*

$$\Big|\sum_{n=N_1}^{N_2} e\left(\frac{na}{q}\right)\Big| \le \min\left(N_2 - N_1 + 1, \frac{1}{|\sin(\pi a/q)|}\right).$$

(ii) *If $S \ge 1$ and $(a,q)=1$ then*

$$\sum_{n\le S} \min\left(\frac{N}{n}, \frac{1}{|\sin(\pi na/q)|}\right) \ll \left(\frac{N}{q} + S + q\right)\log(2qS).$$

Put $dn = k$ in the expression for S_2 to obtain

$$\begin{aligned} S_2 &= \sum_{k\le u^2} \sum_{r\le N/k} \Big(\sum_{d\le u} \sum_{\substack{n\le u\\ dn=k}} \mu(d) g(n)\Big) e(kra/q) \\ &\ll (\log x)^c \sum_{k\le u^2} \tau(k) \Big| \sum_{r\le N/k} e(kra/q)\Big|. \end{aligned}$$

Splitting the k-sum according to $\tau(k) > T$ and $\tau(k) \le T$ and applying Lemma 2 with the choice of $T = (\log x)^{\lambda+4+c}$ yields $S_2 \ll x/\log^\lambda x$. We write S_3 as

$$S_3 = \sum_{j=0}^{K} \sum_{m\in I_j} \tau(m) \sum_{u<n\le N/m} g(n) e(mna/q)$$

where K is defined by $2^K u \le N/u < 2^{K+1} u$, $I_j = (2^j u, 2^{j+1} u]$ for each $0 \le j \le K-1$ and $I_K = (2^K u, N/u]$. Hence

$$S_3 = \sum_{j=0}^{K} U_j$$

where, putting $Y_j = 2^j u$ and using the Cauchy–Schwarz inequality, we obtain

$$|U_j|^2 \le \sum_{m\in I_j} \tau^2(m) \sum_{m\in I_j} \Big| \sum_{u<n\le N/m} g(n)e(mna/q)\Big|^2$$
$$\ll Y_j(\log x)^{2c+3} \sum_{\substack{n_1,n_2\\ u<n_i\le N/Y_j}} \Big| \sum_{\substack{Y_j<m\le 2Y_j\\ m\le \min(N/n_1,N/n_2)}} e(m(n_1-n_2)a/q)\Big|$$

which by Lemma 2 yields

$$|U_j| \ll Y_j^{1/2} x^{1/2} (\log x)^{c+3/2} + x/(\log x)^{\lambda+1}.$$

So finally,

$$S_3 = \sum_{j=0}^{K} U_j \ll x/\log^\lambda x.$$

This completes the proof of Theorem 3.

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Received on 18.6.1999
and in revised form on 4.10.1999 (3629)