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Existence of a non-entire twist for a class of L-functions

by

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1. Settings and results. Given an integer $\mathbf{d} \geq 1$, we consider the class $\mathcal{C}_{\mathbf{d}}$ of functions with the following properties:

• (Arithmetical conditions) If $f \in \mathcal{C}_{\mathbf{d}}$, then

$$f(s) = \prod_p \prod_{j=1}^{\mathbf{d}} (1 - \alpha_j(p)p^{-s})^{-1}$$

where $|\alpha_j(p)| \leq 1$ for all j, p. As a consequence of this hypothesis f has a Dirichlet series representation $f(s) = \sum_n a_n n^{-s}$ that is absolutely convergent for $\sigma > 1$.

• (Analytical conditions) For all integers $q \geq 1$ and all primitive characters χ mod q, the twisted function $(f \otimes \chi)(s) := \sum_n \chi(n) a_n n^{-s}$ has continuation to $\mathbb{C}$ as a meromorphic function with at most a pole at $s = 1$; moreover, $(s-1)^m (f \otimes \chi)(s)$ is an entire function of finite order for some integer m, and $f \otimes \chi$ satisfies a functional equation of type

$$(f \otimes \chi)(1-s) = q^{\mathbf{d}(s-1/2)} \Phi_\chi^f(s) (\bar{f} \otimes \overline{\chi})(s)$$

where $\bar{f}(s) := \sum_n \overline{a}_n n^{-s}$, $\Phi_\chi^f(s)$ is an holomorphic function in $\sigma > 0$ and satisfies the estimate $|\Phi_\chi^f(s)| < c(\sigma, \chi)|t|^{B(\sigma,\chi)}$ for $|t| \geq 1$ on each vertical line $\sigma + it$, for some constants $c(\sigma, \chi)$, $B(\sigma, \chi) > 0$. Moreover, we assume that there exists $\widetilde{\sigma} > 0$ such that $c(\sigma, \chi) = c(\sigma)$ and $B(\sigma, \chi) = B(\sigma)$ for $\sigma > \widetilde{\sigma}$.

• In addition, for $f \in \mathcal{C}_1$ we assume that $\Phi_\chi^f(s) \ll |t|^\sigma$ uniformly for $|t| > 1$ and σ sufficiently large.

Remark 1. The above conditions are inspired by the work of Duke and Iwaniec [1].

Remark 2. With these hypotheses, $\mathcal{C}_{\mathbf{d}'} \subseteq \mathcal{C}_{\mathbf{d}}$ when $\mathbf{d}' \leq \mathbf{d}$, so the really interesting parameter associated with $f \in \mathcal{C}_{\mathbf{d}}$ is $\mathbf{d}(f) := \min\{\mathbf{d}' : f \in \mathcal{C}_{\mathbf{d}'}\}$; in the following we will assume that $\mathbf{d}(f) = \mathbf{d}$ whenever we write $f \in \mathcal{C}_{\mathbf{d}}$.

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Remark 3. The third condition is compatible with our knowledge of $\mathcal{C}_1$ and is necessary in a technical point of Section 2.

Remark 4. The set $\bigcup_{\mathbf{d}} \mathcal{C}_{\mathbf{d}}$ has a lot of algebraic structure provided by the product and the Rankin–Selberg convolution: in fact, let $f \in \mathcal{C}_{\mathbf{d}}$ and $g \in \mathcal{C}_{\mathbf{d}'}$; then the identity $(fg) \otimes \chi = (f \otimes \chi)(g \otimes \chi)$ shows that $fg \in \mathcal{C}_{\mathbf{d}+\mathbf{d}'}$. Moreover, if we assume that $f \otimes g$ satisfies the analytical conditions, then $f \otimes g \in \mathcal{C}_{\mathbf{d}\mathbf{d}'}$.

It is not completely trivial to show that the usual Dirichlet L-functions $L(s,\kappa)$ are in $\mathcal{C}_1$, the non-trivial part being the existence of a χ-uniform estimate for $f \otimes \chi = L(s,\kappa\chi)$; we prove this in the appendix.

Likewise, it can be proved that the normalized L-functions associated with holomorphic newforms for the Hecke group $\Gamma_0(N)$ with multiplier κ are in $\mathcal{C}_2$: in this case we know that the twisted function $L \otimes \chi$ is again a normalized L-function associated with a newform for a $\Gamma_0(\widetilde{N})$ and a new multiplier, so in this case $f \otimes \chi$ is always an entire function (see Theorem 4.3.12 in [4]).

Moreover, let L be a normalized function associated with a holomorphic newform for $\mathrm{SL}_2(\mathbb{Z})$ and let $L(s, \mathrm{sym}^m)$ be the m-symmetric function generated by L, introduced by Serre in connection with the Sato–Tate conjecture. For $m \geq 1$ the Langlands program implies that $L(s,\mathrm{sym}^m) \in \mathcal{C}_{m+1}$ and that the twist $L(s,\mathrm{sym}^m) \otimes \chi$ is entire for all χ. For small values of m these conjectures are consequences of important results proved in the literature. In particular they are true for $m = 1$ (case already quoted) and for $m = 2$ (from Shimura [8]). They are "almost" true for $m = 3, 4, 5$ too, in the sense that for those values of m the functional equation and the meromorphic continuation to $\mathbb{C}$ have been established (Shahidi [6, 7]), but that the singularities are reduced at most to a pole at $s = 1$ is not yet proved.

Definition. We say that $f \in \mathcal{C}_{\mathbf{d}}$ has the $*$-*property* when $f \otimes \chi$ is an entire function for all primitive χ (hence f is entire as well, since $f = f \otimes \chi_0$ with $q = 1$).

The previous remarks show that there are elements with the $*$-property in $\mathcal{C}_{\mathbf{d}}$ for $\mathbf{d} = 2, 3$ (see Remark 2) and conjecturally for every $\mathbf{d} \geq 2$, but not every element of $\mathcal{C}_{\mathbf{d}}$ has the $*$-property, as the function $\zeta^2(s)$ shows. However, there is strong evidence, but no proof, that the elements of $\mathcal{C}_{\mathbf{d}}$ with $\mathbf{d} \geq 2$ have the $*$-property if they are not a product or Rankin–Selberg convolution of functions in some $\mathcal{C}_{\mathbf{d}'}$ (see Remark 4). The main result of this paper is that the restriction to $\mathbf{d} \geq 2$ is in fact a necessary condition for the $*$-property.

Theorem. *Let $f \in \mathcal{C}_1$ have the $*$-property. Then f is the constant function $f(s) = 1$.*

The class $\mathcal{C}_{\mathbf{d}}$ appears to be related to the Selberg class $\mathcal{S}_{\mathbf{d}}$ (see [5] and [3]) but there are some important differences. Firstly, in $\mathcal{C}_{\mathbf{d}}$ the kernel Φ_χ^f of the functional equation is not necessarily a product of Γ-factors; secondly, in $\mathcal{C}_{\mathbf{d}}$ we assume a "well-behaviour" of $f\otimes\chi$ that probably holds in $\mathcal{S}_{\mathbf{d}}$ as well, but $f\otimes\chi$ does not necessarily belong to $\mathcal{S}_{\mathbf{d}}$. Finally, in our arithmetical definition $\mathbf{d}$ is always an integer, while in the Selberg setting every positive real value is in principle possible for $\mathbf{d}$, as a consequence of a different (analytical) definition. In all the known cases the two definitions provide the same result: this reveals that there are deep aspects of the theory that are not yet well understood. Kaczorowski and Perelli [3] have proved that the Dirichlet L-functions $L(s,\kappa)$ and their shifts are the only elements of $\mathcal{S}_1$, so it is natural to conjecture that these functions exhaust $\mathcal{C}_1$ as well. We are not able to prove this conjecture at present; however, our Theorem agrees with this conjecture.

The Theorem is a consequence of the following two lemmas.

LEMMA 1. *Let* $f(s)=\sum_n a_n n^{-s}\in\mathcal{C}_1$ *and* $g(s)=\sum_n b_n n^{-s}\in\mathcal{C}_{\mathbf{d}}$ *for some* $\mathbf{d}\ge 2$, *and assume that* f *and* g *have the* $*$*-property. Then*

$$\sum_{x/2<n<x} a_n b_n \eta^2(n/x) \ll_A x^{-A} \qquad \forall A>0$$

with an arbitrary positive function $\eta\in C_0^\infty([1/2,1])$.

LEMMA 2. *Let* $\sum_k h_k x^k=\prod_{j=1}^u(1-\beta_j x)^{-1}$ *with* $0<|\beta_j|\le 1$ *for any* j. *Assume that* $|\beta_j|=1$ *for some* j *and let* $m_i=\#\{j:\beta_j=\beta_i \text{ with } |\beta_i|=1\}$, $M=\max\{m_i\}$. *Then* $h_k=\Omega(k^{M-1})$; *in particular* $h_k=\Omega(1)$.

For the proof of Lemma 1 we follow, with some non-trivial simplifications, the approach used by Duke and Iwaniec [1] to treat a similar problem. Section 2 is devoted to the proof of this lemma.

Lemma 2 is an easy consequence of explicit computations of linear algebra (see Section 3).

Proof of the Theorem. If we assume the lemmas, the proof of the Theorem is simple; in fact Lemma 1 implies

$$|a_n b_n| < c(A) n^{-A} \qquad \forall A>0. \tag{1}$$

We write $f(s)=\prod_p(1-\alpha(p)p^{-s})^{-1}$, $g(s)=\prod_p\prod_{j=1}^{\mathbf{d}}(1-\beta_j(p)p^{-s})^{-1}$. Given any prime p, we select a function g such that $|\beta_j(p)|=1$ for some j (this is always possible, for example in $\mathcal{C}_2$ with g a normalized L-function associated with a holomorphic newform for $\mathrm{SL}_2(\mathbb{Z})$). Then the sequence b_{p^k} satisfies the hypothesis of Lemma 2, so there is a subsequence $\{b_{p^{k_n}}\}$ such that $|b_{p^{k_n}}|>c$ for some positive constant c and every n. The complete multiplicativity of a_n and (1) give

$$|\alpha(p)|^{k_n}c = |a_{p^{k_n}}|c \le |a_{p^{k_n}}b_{p^{k_n}}| \le c(A)p^{-k_nA},$$

so $|\alpha(p)| \le (c(A)/c)^{1/k_n}p^{-A}$, and hence taking $n \to \infty$, for any p and A we have $|\alpha(p)| \le p^{-A}$. Therefore $\alpha(p) = 0$ for every p, and the result follows. ∎

2. Proof of Lemma 1

2.1. *Preliminary identities*

REMARK 5. Here and in the following section $\int_{\sigma>a}$ is the integral on the vertical line with abscissa $\sigma > a$.

Let η be as in Lemma 1, $Y(x) := \sum_q \eta(q/\sqrt{x}) \sim \sqrt{x}\int_{\mathbb{R}} \eta(u)\,du$, and define

$$\mathcal{D}(x) := \sum_n a_n b_n \eta^2(n/x).$$

In order to analyze the asymptotic behaviour of $\mathcal{D}(x)$ and prove the lemma, we begin by performing the same transformations as in Section 3 of [1], with some little changes. In particular, the decomposition of a_{rm} is now obvious by complete multiplicativity, and the other arithmetical functions $b_r(b)$, $c_t(c)$, $d_t(d)$, which are necessary for the decomposition of b_{rn} and to relax the constraints $(m,t) = 1$ and $(n,t) = 1$ respectively, are now defined by

$$\text{(2a)} \qquad b_{rn} = \sum_{bn'=n,\, b|r^{\mathbf{d}-1}} b_r(b)b_{n'}, \qquad b_r(b) \ll r^{\varepsilon},$$

$$\text{(2b)} \qquad \sum_{dn'=n,\, d|t^{\mathbf{d}}} d_t(d)b_{n'} = \begin{cases} b_n & \text{if } (n,t)=1, \\ 0 & \text{otherwise,} \end{cases} \qquad d_t(d) \ll t^{\varepsilon},$$

$$\text{(2c)} \qquad \sum_{cm'=m,\, c|t} c_t(c)a_{m'} = \begin{cases} a_m & \text{if } (m,t)=1, \\ 0 & \text{otherwise,} \end{cases} \qquad c_t(c) \ll t^{\varepsilon}.$$

The existence of $b_r(b)$ for $\mathbf{d} = 2$ is proved in [2], and the general case is similar; the existence of $c_t(c)$ and $d_t(d)$ is granted by the Euler product (in particular $c_t(c) = \mu(c)a_c$, with μ the Möbius function).

The result of these transformations is the following identity, which is analogous to (9) of [1]:

$$\text{(3)} \qquad Y\mathcal{D}(x) = \sum_{q,r,t} \phi(qt)^{-1} \sum_{\substack{(b,qt)=1 \\ b|r^{\mathbf{d}-1}}} a_r b_r(b) \sum_{\substack{(cd,q)=1 \\ c|t,\, d|t^{\mathbf{d}}}} c_t(c)d_t(d)$$
$$\times \sideset{}{^*}\sum_{\chi \bmod q} \sum_{m,n} \chi(cm)\overline{\chi}(bdn)a_m b_n h\Big(\frac{crm}{x}, \frac{bdrn}{x}, \frac{qrt}{\sqrt{x}}\Big),$$

where $h(x,y,z) := \eta(x)\eta(y)(\eta(z)-\eta(|x-y|/z))$ has support in $[1/2,1]\times[1/2,1]\times(0,1]$ and $\sum^*$ is a sum over the primitive characters only.

Now we adapt to our case the argument in Section 4 of [1], but we avoid using the Kloosterman sums.

Let

$$\varrho_1 := cr/x,\quad \varrho_2 := bdr/x,\quad z := qrt/\sqrt{x},\quad \mathfrak{h}(u,v) := h(\varrho_1 u, \varrho_2 v, z)$$

and

$$\Delta(\chi) := \sum_{m,n} \chi(m)\overline{\chi}(n) a_m b_n \mathfrak{h}(m,n).$$

Then $\mathfrak{h}(u,v)$ is a smooth function with compact support that is zero in $\{|u|<1/(2\varrho_1)\}\times\{|v|<1/(2\varrho_2)\}$, hence

$$\check{\mathfrak{h}}(s_1,s_2) := \int_0^\infty\int_0^\infty \mathfrak{h}(u,v)u^{-s_1}v^{-s_2}\,du\,dv$$

is entire in $\mathbb{C}\times\mathbb{C}$.

Moreover, the equality $\check{\mathfrak{h}}(s_1,s_2) = \varrho_1^{s_1-1}\varrho_2^{s_2-1}\check{h}(s_1,s_2,z)$ holds with

$$\check{h}(s_1,s_2,z) := \int_0^\infty\int_0^\infty h(u,v,z)u^{-s_1}v^{-s_2}\,du\,dv, \tag{4}$$

therefore

$$\varrho_1^{-s_1}\varrho_2^{-s_2}\check{h}(1-s_1,1-s_2,z) = \int_0^\infty\int_0^\infty \mathfrak{h}(u,v)u^{s_1-1}v^{s_2-1}\,du\,dv.$$

The inverse of this Mellin integral gives

$$\mathfrak{h}(u,v) = \frac{-1}{4\pi^2}\iint_{\sigma_1,\sigma_2>1} \check{h}(1-s_1,1-s_2,z)(\varrho_1 u)^{-s_1}(\varrho_2 v)^{-s_2}\,ds_1\,ds_2,$$

therefore

$$\Delta(\chi) = \frac{-1}{4\pi^2}\iint_{\sigma_1,\sigma_2>1} \check{h}(1-s_1,1-s_2,z)(f\otimes\chi)(s_1)(g\otimes\overline{\chi})(s_2)\varrho_1^{-s_1}\varrho_2^{-s_2}\,ds_1\,ds_2$$

for the uniform convergence of $\sum a_n n^{-s}$ and $\sum b_n n^{-s}$ in $\sigma>1+\varepsilon$.

The functions $f\otimes\chi$ and $g\otimes\overline{\chi}$ are entire by the $*$-property and have a polynomial behaviour on the vertical strips by the hypothesis on the functional equations. In the next subsection we prove that $\check{h}$ tends to zero on the vertical lines more quickly than any power, so the changes $s_1\mapsto 1-s_1$, $s_2\mapsto 1-s_2$ and the subsequent applications of the Fubini and Cauchy theorems give

$$\Delta(\chi) = \frac{-1}{4\pi^2}\iint_{\sigma_1,\sigma_2>1} \check{h}(s_1,s_2,z)(f\otimes\chi)(1-s_1)(g\otimes\overline{\chi})(1-s_2)\varrho_1^{s_1-1}\varrho_2^{s_2-1}\,ds_1\,ds_2.$$

Now we introduce the functional equations and the Dirichlet series again, thus getting

$$\varDelta(\chi)=\frac{q^{-(1+\mathbf{d})/2}}{\varrho_1\varrho_2}\sum_{m,n}\overline{\chi}(m)\chi(n)\overline{a}_m\overline{b}_n\mathcal{H}_\chi\Big(\frac{m}{q\varrho_1},\frac{n}{q^{\mathbf{d}}\varrho_2},\frac{qrt}{\sqrt{x}}\Big)$$

where

$$\mathcal{H}_\chi(u,v,z):=\frac{-1}{4\pi^2}\iint\limits_{\sigma_1,\sigma_2>0}\check{h}(s_1,s_2,z)\varPhi^f_\chi(s_1)\varPhi^g_{\overline{\chi}}(s_2)u^{-s_1}v^{-s_2}\,ds_1\,ds_2. \tag{5}$$

In the definition of $\mathcal{H}_\chi$ we can allow every positive value for σ_1 and σ_2 by the hypothesis about $\varPhi^f_\chi$ and $\varPhi^g_\chi$ and the behaviour of $\check{h}$ on the vertical lines. Substituting this expression in (3) we obtain the final equality

$$Y\mathcal{D}(x)=x^2\sum_{rt<\sqrt{x}}a_r\sum_{\substack{b\mid r^{\mathbf{d}-1}\\(b,t)=1}}\sum_{\substack{c\mid t\\d\mid t^{\mathbf{d}}}}b_r(b)c_t(c)d_t(d)\frac{\mathcal{E}}{bcdr^2}, \tag{6}$$

where

$$\mathcal{E}:=\sum_{\substack{m,n,q\\(bcdmn,q)=1}}\frac{q^{-(1+\mathbf{d})/2}}{\varphi(qt)}\overline{a}_m\overline{b}_n \times\sideset{}{^*}\sum_{\chi \bmod q}\chi(cn\overline{bdm})\mathcal{H}_\chi\Big(\frac{mx}{crq},\frac{nx}{bdrq^{\mathbf{d}}},\frac{qrt}{\sqrt{x}}\Big), \tag{7}$$

which is analogous to (10) of [1].

2.2. *Estimate of* $\mathcal{H}_\chi$

Remark 6. In this and the following sections ε is an arbitrary (small) positive parameter not always with the same value.

We recall that $h(u,v,z)=\eta(u)\eta(v)(\eta(z)-\eta(|u-v|/z))$ has support in $[1/2,1]\times[1/2,1]\times(0,1]$ and the definitions of $\check{h}(s_1,s_2,z)$ and $\mathcal{H}_\chi(u,v,z)$ in (4) and (5).

By partial integration we have, for all $A,B\ge 0$,

$$\check{h}(s_1,s_2,z)=\int_0^\infty\int_0^\infty\frac{\partial h(u,v,z)}{\partial^A u\partial^B v}\times\frac{u^{A-s_1}}{(s_1-A)\dots(s_1-1)}\cdot\frac{v^{B-s_2}}{(s_2-B)\dots(s_2-1)}\,du\,dv;$$

moreover, $z^{A+B}\frac{\partial h(u,v,z)}{\partial^A u\partial^B v}$ is uniformly bounded on its support, since it is a polynomial expression in z, $\eta^{(i)}(u)$, $\eta^{(j)}(v)$, $\eta^{(k)}(|u-v|/z)$, so the former relation gives the estimate

$$\check{h}(s_1,s_2,z)\ll z^{-A-B}(1+|s_1|)^{-A}(1+|s_2|)^{-B}\qquad\forall A,B\ge 0 \tag{8}$$

where the implied constant depends only on A, B, σ_1, σ_2. Hence (8) is uniform on the vertical lines. Therefore

$$\mathcal{H}_\chi \ll u^{-\sigma_1} v^{-\sigma_2} z^{-A-B} \iint\limits_{\sigma_1,\sigma_2>0} \frac{|\Phi_\chi^f(s_1)|}{(1+|s_1|)^A} \cdot \frac{|\Phi_{\overline{\chi}}^g(s_2)|}{(1+|s_2|)^B}\, dt_1\, dt_2,$$

the estimate being independent of the character χ if σ_1 and σ_2 are sufficiently large. Moreover, we have supposed that $\Phi_\chi^f(s_1) \ll |t|^{\sigma_1}$ and $\Phi_\chi^f(s_2) \ll |t|^{B(\sigma_2)}$ for $|t| > 1$ and σ_i large, so

$$\mathcal{H}_\chi \ll u^{-\sigma_1} v^{-\sigma_2} z^{-A-B} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} (1+|t_1|)^{\sigma_1 - A} (1+|t_2|)^{B(\sigma_2)-B}\, dt_1\, dt_2,$$

where by (8) we have supposed A and B sufficiently large to assure the convergence of the integral. Choosing $A = \sigma_1 + 1 + \varepsilon$ and $B = B(\sigma_2) + 1 + \varepsilon$, we have

$$\begin{aligned}\mathcal{H}_\chi &\ll_{\sigma_1,\sigma_2} u^{-\sigma_1} v^{-\sigma_2} z^{-\sigma_1 - B(\sigma_2) - 2 - \varepsilon} \\ &= u^{-(\sigma_1 - B(\sigma_2) - 2 - \varepsilon)/2} v^{-\sigma_2} (uz^2)^{-(\sigma_1 + B(\sigma_2) + 2 + \varepsilon)/2}\end{aligned}$$

for all σ_1, σ_2 large, therefore

$$\mathcal{H}_\chi \ll_{A,D} u^{-A} v^{-D} (uz^2)^{-\widetilde{B}}$$

for all $A, D > 0$ large, for some $\widetilde{B} = \widetilde{B}(A, D) > 0$. Hence

$$(9)\quad \mathcal{H}_\chi\Big(\frac{mx}{crq}, \frac{nx}{bdrq^{\mathbf{d}}}, \frac{qrt}{\sqrt{x}}\Big) \ll_{A,D} \Big(\frac{crq}{mx}\Big)^A \Big(\frac{bdrq^{\mathbf{d}}}{nx}\Big)^D \Big(\frac{mx}{crq} \cdot \frac{q^2r^2t^2}{x}\Big)^{-\widetilde{B}}.$$

In view of the support of h, $\mathcal{H}_\chi(u, v, z)$ is zero when $z > 1$, so we can greatly simplify the estimate (9) by assuming $0 < z \le 1$, i.e., $q \le Q := \sqrt{x}/(rt)$. In fact

$$\frac{crq}{mx} \le \frac{cr}{mx} \cdot \frac{x^{1/2}}{rt} \le \frac{x^{-1/2}}{m}$$

by (2c),

$$\frac{bdrq^{\mathbf{d}}}{nx} \le \frac{b}{r^{\mathbf{d}-1}} \cdot \frac{d}{t^{\mathbf{d}}} \cdot \frac{x^{(\mathbf{d}-2)/2}}{n} \le \frac{x^{(\mathbf{d}-2)/2}}{n}$$

by (2a) and (2b), and

$$\frac{mx}{crq} \cdot \frac{q^2r^2t^2}{x} \ge 1$$

by (2c). Thus (9) becomes

$$\mathcal{H}_\chi\Big(\frac{mx}{crq}, \frac{nx}{bdrq^{\mathbf{d}}}, \frac{qrt}{\sqrt{x}}\Big) \ll_{A,D} \frac{x^{-A/2+(\mathbf{d}-2)D/2}}{m^A n^D} \qquad \forall A, D > 0.$$

Finally, with a suitable choice of $D = D(A)$ we have

$$\mathcal{H}_\chi\Big(\frac{mx}{crq}, \frac{nx}{bdrq^{\mathbf{d}}}, \frac{qrt}{\sqrt{x}}\Big) \ll_A \frac{x^{-A}}{m^A n^A} \qquad \forall A > 0, \tag{10}$$

uniformly in χ.

2.3. *Estimate of $\mathcal{E}$.* Estimate (10) is so strong that we can bound $\mathcal{E}$ trivially, using the uniformity in χ and taking the absolute values in (7), thus getting

$$\mathcal{E} \ll_A \sum_{q\le Q} \frac{q^{(1-\mathbf{d})/2}}{\varphi(qt)} \sum_m \frac{|a_m|}{m^A} \sum_n \frac{|b_n|}{n^A} x^{-A} \ll_A \frac{x^{-A}}{t^{1-\varepsilon}} \qquad \forall A > 1, \tag{11}$$

where the q-series is convergent since we have assumed $\mathbf{d} \ge 2$, and the same holds for the m and n-series when $A > 1$.

2.4. *Proof of Lemma 1.* The bound in (11), the trivial estimates $a_r, b_r(b) \ll r^\varepsilon$, $c_t(c), d_t(d) \ll t^\varepsilon$ and $b, c, d \ge 1$ give, when introduced in (6),

$$\begin{aligned} Y\mathcal{D}(x) &\ll_A x^{2-A} \sum_{rt\le\sqrt{x}} \frac{r^\varepsilon t^\varepsilon}{r^2 t} \sum_{\substack{b|r^{\mathbf{d}-1}\\ c|t,\, d|t^{\mathbf{d}}}} 1 \ll_A x^{2-A} \sum_{rt\le\sqrt{x}} \frac{r^\varepsilon t^\varepsilon}{r^2 t} \\ &\ll_A x^{2+\varepsilon-A} \qquad \forall A > 1. \end{aligned}$$

This completes the proof of Lemma 1, since $Y \asymp \sqrt{x}$.

3. Some explicit formulas

3.1. *Proof of Lemma 2.* Writing

$$\sum_k h_k x^k = \prod_{j=1}^u (1 - \beta_j x)^{-1},$$

we have

$$h_k = \sum_{\substack{a_1+\ldots+a_u=k\\ a_i\ge 0}} \beta_1^{a_1} \ldots \beta_u^{a_u}. \tag{12}$$

Let $s_1, \ldots, s_u$ be the elementary symmetric polynomials in the β_j. Then the identity $(1 - s_1 x + \ldots + (-1)^u s_u x^u) \sum_k h_k x^k = 1$ gives the recursive relations

$$\begin{cases} h_k - s_1 h_{k-1} + s_2 h_{k-2} + \ldots + (-1)^u s_u h_{k-u} = 0 & \text{if } k > 0,\\ h_0 = 1, & \\ h_k = 0 & \text{if } k < 0. \end{cases} \tag{13}$$

The recursion can be solved in this way: denoting by v_n the column vector

$(h_n, h_{n-1}, \dots, h_{n-u+1})^t$, (13) is equivalent to $v_0 = (1, 0, \dots, 0)^t$ and $v_n = \mathcal{A}v_{n-1}$, i.e., $v_n = \mathcal{A}^n v_0$ with

$$\mathcal{A} := \begin{pmatrix} s_1 & -s_2 & s_3 & \dots & (-1)^u s_u \\ & & I_{u-1} & & 0 \end{pmatrix},$$

where I_{u-1} is the identity matrix of order $u-1$.

It is known that $\beta_1, \dots, \beta_u$ are the eigenvalues of $\mathcal{A}$ having $w_j := (\beta_j^{u-1}, \beta_j^{u-2}, \dots, 1)^t$ as eigenvectors, so $\mathcal{A}$ is diagonalizable if we suppose $\beta_i \neq \beta_j$ for all $i \neq j$; in this case we set $\mathcal{M} := (w_1, \dots, w_u)$ so that $\mathcal{G} := \mathcal{M}^{-1}\mathcal{A}\mathcal{M}$ is diagonal, $\mathcal{G} = \operatorname{diag}(\beta_1, \dots, \beta_u)$. Hence $v_n = \mathcal{M}\mathcal{G}^n\mathcal{M}^{-1}v_0$ and if $V(c_1, \dots, c_u)$ denotes the Vandermonde determinant $\prod_{1 \le i < j \le u}(c_i - c_j)$, it follows that

$$h_k = \sum_{j=1}^{u} \beta_j^{k+u-1}(-1)^{j+1} \frac{V(\beta_1, \overset{j}{\overset{\vee}{\dots}}, \beta_u)}{V(\beta_1, \dots, \beta_u)} = \sum_{j=1}^{u} \frac{\beta_j^{k+u-1}}{\prod_{i \neq j}(\beta_i - \beta_j)}. \tag{14}$$

In the general case suppose $\beta_1, \dots, \beta_l$ distinct and let $m_i = \#\{j : \beta_j = \beta_i\}$ for $i = 1, \dots, l$. Then (12) can be written as

$$h_k = \sum_{\substack{a_1+\dots+a_l=k \\ a_i \ge 0}} \beta_1^{a_1} \dots \beta_l^{a_l} \Big(\sum_{\substack{c_1+\dots+c_{m_1}=a_1 \\ c_i \ge 0}} 1 \Big) \dots \Big(\sum_{\substack{c_1+\dots+c_{m_l}=a_l \\ c_i \ge 0}} 1 \Big).$$

But $\sum_{c_1+\dots+c_m=a,\, c_i \ge 0} 1 = \binom{a+m-1}{m-1} =: P_m(a)$ is a polynomial in a of degree $m-1$ and $a^k\beta^a = \big(\beta\frac{d}{d\beta}\big)^k\beta^a$, so that the former equality becomes

$$h_k = P_{m_1}\Big(\beta_1\frac{\partial}{\partial\beta_1}\Big) \dots P_{m_l}\Big(\beta_l\frac{\partial}{\partial\beta_l}\Big) \sum_{\substack{a_1+\dots+a_l=k \\ a_i \ge 0}} \beta_1^{a_1} \dots \beta_l^{a_l}. \tag{15}$$

We substitute (14) in (15) obtaining

$$h_k = P_{m_1}\Big(\beta_1\frac{\partial}{\partial\beta_1}\Big) \dots P_{m_l}\Big(\beta_l\frac{\partial}{\partial\beta_l}\Big) \sum_{j=1}^{l} \frac{\beta_j^{k+l-1}}{\prod_{i \neq j}(\beta_i - \beta_j)}, \tag{16}$$

which finally gives the relation

$$h_k = \sum_{j=1}^{l} p_j(k)\beta_j^k, \tag{17}$$

where each $p_j(k)$ is a polynomial of degree $\le m_j - 1$ in the k variable.

We prove that $\partial_k p_j = m_j - 1$; it is sufficient to prove that the coefficient of $k^{m_1-1}\beta_1^k$ in (16) is not zero. But this coefficient is

$$\begin{aligned}\beta_1^{l-1}P_{m_2}\Big(\beta_2\frac{\partial}{\partial\beta_2}\Big)\dots P_{m_l}\Big(\beta_l\frac{\partial}{\partial\beta_l}\Big)\frac{1}{\prod_{i=2}^l(\beta_i-\beta_1)}\\
&=\beta_1^{l-1}\prod_{i=2}^l P_{m_i}\Big(\beta_i\frac{\partial}{\partial\beta_i}\Big)\frac{1}{\beta_i-\beta_1}=\prod_{i=2}^l P_{m_i}\Big(x_i\frac{\partial}{\partial x_i}\Big)\frac{1}{x_i-1}\\
&=\prod_{i=2}^l\frac{-1}{(1-x_i)^{m_i}},\end{aligned}$$

where $x_i:=\beta_i/\beta_1\neq 1$ by hypothesis, and hence this expression is obviously non-zero.

Now we can prove Lemma 2. The terms with $|\beta_j|<1$ in (17) are $o(1)$, the others β_j are of absolute value 1 by the hypothesis of Lemma 2. Let M be the maximum multiplicity of the terms with absolute value 1; then we know that in (17) there are terms of order k^{M-1}. Collecting these terms we have

$$h_k=k^{M-1}\Big(\sum_{j=1}^l r_j e^{ik\theta_j}+O(1/k)\Big),$$

for some real θ_j with $\theta_i\neq\theta_j$ for $i\neq j$, and $r_j\neq 0$. Lemma 2 follows if we prove that $R_k:=\sum_{j=1}^l r_je^{ik\theta_j}\not\to 0$ as $k\to\infty$. By contradiction let us assume that $R_k\to 0$. Then $R_ke^{-ik\theta_1}\to 0$ as well, and by the Cesàro mean value we have

$$o(1)=\frac{1}{N}\sum_{k=1}^N R_ke^{-ik\theta_1}=\sum_{j=1}^l r_j\frac{1}{N}\sum_{k=1}^N e^{ik(\theta_j-\theta_1)}=r_1+O(1/N),$$

a contradiction.

3.2. *A remarkable relation.* We show here the deduction of an interesting formula, identity (18) below, for the p-component of the coefficients of $L_f(s,\mathrm{sym}^m)$, where f is a holomorphic newform for $\mathrm{SL}_2(\mathbb{Z})$. This formula is not necessary for the proof of our Theorem, but in some sense it completes the topics presented in the previous section. If we introduce the polynomials

$$D_u(N):=\begin{vmatrix}\beta_1^N & \beta_2^N & \dots & \beta_u^N\\ \beta_1^{u-2} & \beta_2^{u-2} & \dots & \beta_u^{u-2}\\ \beta_1^{u-3} & \beta_2^{u-3} & \dots & \beta_u^{u-3}\\ \vdots & \vdots & & \vdots\\ \beta_1 & \beta_2 & \dots & \beta_u\\ 1 & 1 & \dots & 1\end{vmatrix},$$

identity (14) can be formulated as $h_k=D_u(k+u-1)/D_u(u-1)$.

Now we suppose that $u = m+1$ and $\{\beta_j\}_{j=1}^{u} \equiv \{z^{m-2j}\}_{j=0}^{m}$ with $|z| = 1$: this happens when we consider the m-symmetric power of an L-function associated with a normalized newform for $\mathrm{SL}_2(\mathbb{Z})$, with $(1-zp^{-s})(1-\overline{z}p^{-s})$ the decomposition of its local polynomials. In this case

$$D_{m+1}(N) = \begin{vmatrix} z^{mN} & z^{(m-2)N} & \dots & \overline{z}^{mN} \\ z^{m(m-1)} & z^{(m-2)(m-1)} & \dots & \overline{z}^{m(m-1)} \\ z^{m(m-2)} & z^{(m-2)(m-2)} & \dots & \overline{z}^{m(m-2)} \\ \vdots & \vdots & & \vdots \\ z^{m} & z^{m-2} & \dots & \overline{z}^{m} \\ 1 & 1 & \dots & 1 \end{vmatrix}.$$

From long and not completely elementary calculations involving the Gauss polynomials, which we do not report here, it is possible to verify that

$$D_{m+1}(N) = \Big(\prod_{j=1}^{m-1}(z^j - \overline{z}^j)^{m-j}\Big)\Big(\prod_{j=0}^{m-1}(z^{N-j} - \overline{z}^{N-j})\Big).$$

Setting $z =: e^{i\theta}$, one gets

$$h_k = \prod_{j=1}^{m} \frac{\sin(k+j)\theta}{\sin j\theta}. \tag{18}$$

For $m = 1$, (18) is the well known trigonometric expression for the p-part of the coefficients of $L_f(s)$.

Appendix. Writing $f(s) = L(s,\kappa)$ with κ a primitive character modulo q_0, we want prove that $f \in \mathcal{C}_1$, so we have to study the functional equation of $f \otimes \chi$ where χ is a primitive character modulo q. Let υ be the character modulo q_1 ($q_1 \mid q_0 q$) that induces $\kappa\chi$. Then the identity $f \otimes \chi = L(s,\upsilon)\prod_{p \mid q_0 q}(1-\upsilon(p)p^{-s})$ holds. It follows that $f\otimes\chi$ satisfies the functional equation

$$f \otimes \chi(1-s) = i^{-\nu_\upsilon}\varepsilon_\upsilon q_1^{(2s-1)/2}\pi^{-(2s-1)/2}\frac{\Gamma((s+\nu_\upsilon)/2)}{\Gamma((1-s+\nu_\upsilon)/2)}\prod_{p\mid q_0 q}\frac{1-\upsilon(p)p^{s-1}}{1-\overline{\upsilon}(p)p^{-s}}\,\bar{f}\otimes\overline{\chi}(s)$$

where ν_υ is the parity of υ and $\varepsilon_\upsilon = \tau(\upsilon)/\sqrt{q_1}$ (phase of the Gauss sum). We write the functional equation selecting the following components:

$$f \otimes \chi(1-s) = q^{(2s-1)/2}\alpha_\upsilon \Psi_{\nu_\upsilon}(s)\widetilde{\Psi}(\kappa,\chi,s)\bar{f}\otimes\overline{\chi}(s),$$

where

$$\alpha_\upsilon := i^{-\nu_\upsilon}\varepsilon_\upsilon,$$

$$\Psi_{\nu_\upsilon}(s) := \Big(\frac{q_0}{\pi}\Big)^{(2s-1)/2}\frac{\Gamma((s+\nu_\upsilon)/2)}{\Gamma((1-s+\nu_\upsilon)/2)},$$

$$\widetilde{\Psi}(\kappa,\chi,s) := \left(\frac{q_1}{q_0 q}\right)^{(2s-1)/2} \prod_{p\mid q_0 q} \frac{1-\upsilon(p)p^{s-1}}{1-\overline{\upsilon}(p)p^{-s}}.$$

Here $|\alpha_\upsilon| = 1$, $\Psi_{\nu_\upsilon}(s)$ is a holomorphic function in $\sigma > 0$ that depends only on the parity of υ, with a $|t|^\sigma$ behaviour on the vertical lines by the Stirling formula, and $\widetilde{\Psi}(\kappa,\chi,s)$ is a holomorphic function in $\sigma > 0$, bounded on the vertical strips but depending on the character χ. Verifying that $f \in \mathcal{C}_1$ means then proving that $\widetilde{\Psi}(\kappa,\chi,s)$ is bounded uniformly in t and χ for large and fixed σ; we prove this for $\sigma > 0$. In fact

$$(19)\qquad |\widetilde{\Psi}(\kappa,\chi,s)| \le \left(\frac{q_1}{q_0 q}\right)^{(2\sigma-1)/2} \prod_{\substack{p\mid q_0 q\\ p\nmid q_1}} \frac{1+p^{\sigma-1}}{1-p^{-\sigma}} \le \left(\frac{1}{M}\right)^{(2\sigma-1)/2} \prod_{p\mid M} \frac{1+p^{\sigma-1}}{1-p^{-\sigma}}$$

since $(1+p^{\sigma-1})/(1-p^{-\sigma}) > 1$ and $M := q_0 q/q_1$ is an integer. If we assume $\sigma \ge 1$, (19) implies that

$$(20)\qquad |\widetilde{\Psi}(\kappa,\chi,s)| \le \left(\frac{1}{M}\right)^{(2\sigma-1)/2} \prod_{p\mid M} p^{\sigma-1} \prod_{p\mid M} \frac{1+p^{1-\sigma}}{1-p^{-\sigma}} \le \frac{c(\varepsilon)}{M^{1/2-\varepsilon}},$$

where we have used $(1+p^{1-\sigma})/(1-p^{-\sigma}) \le 4$ for all p. Estimate (20) is particularly interesting because it is uniform in the character κ also.

The bound (20) holds in $\sigma > 1$, and it is sufficient to prove that $L(s,\kappa) \in \mathcal{C}_1$, but we further observe that an estimate uniform in χ but not in κ is still possible for $0 < \sigma$; in fact, we will prove that $M \mid \mathrm{MCD}(q_0^2, q^2)$, thus from (19) we have

$$|\widetilde{\Psi}(\kappa,\chi,s)| \le \max(1, q_0^{1-2\sigma}) \prod_{p\mid q_0} \frac{1+p^{\sigma-1}}{1-p^{-\sigma}},$$

which is independent of χ.

For a proof of $M \mid \mathrm{MCD}(q_0^2, q^2)$, let $q_0 = \prod_p p^{a_p}$, $q = \prod_p p^{b_p}$, $q_1 = \prod_p p^{c_p}$ be the p-parts of the moduli and $\kappa = \prod_p \kappa_{p^{a_p}}$, $\chi = \prod_p \chi_{p^{b_p}}$ and $\upsilon = \prod_p \upsilon_{p^{c_p}}$ be the p-parts of the characters. Then $\kappa_{p^{a_p}}$, $\chi_{p^{b_p}}$ and $\upsilon_{p^{c_p}}$ are primitive and $\upsilon_{p^{c_p}}$ induces $\kappa_{p^{a_p}}\chi_{p^{b_p}}$. We prove that if $a_p \ne b_p$, then $c_p = \max(a_p, b_p)$. In fact let $a_p < b_p$ and by contradiction $c_p < b_p$. Then $\overline{\kappa}_{p^{a_p}}$ is a character modulo p^{a_p} so $\overline{\kappa}_{p^{a_p}}\upsilon_{p^{c_p}}$ is a character modulo $\max(p^{a_p}, p^{c_p}) < p^{b_p}$, hence it induces a character mod p^{b_p} that cannot be primitive. This is a contradiction since $\chi_{p^{b_p}}$ is the induced character. It follows that

$$M = \prod_{p} \frac{p^{a_p} p^{b_p}}{p^{c_p}} = \prod_{p \mid q_0} \frac{p^{a_p} p^{b_p}}{p^{c_p}} \prod_{p \nmid q_0} \frac{p^{b_p}}{p^{c_p}} = \prod_{p \mid q_0} p^{a_p + b_p - c_p},$$

but $a_p \neq b_p$ implies $a_p + b_p - c_p = \min(a_p, b_p)$ and $a_p = b_p$ implies $a_p + b_p - c_p \leq 2a_p$, hence $M \mid q_0^2$. In a similar way we prove that $M \mid q^2$.

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