A problem of Galambos on Engel expansions

by

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1. Introduction. Given $x$ in $(0, 1]$, let $x = [d_1(x), d_2(x), \ldots]$ denote the Engel expansion of $x$, that is,
\[
x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \cdots + \frac{1}{d_1(x)d_2(x)\ldots d_n(x)} + \cdots,
\]
where $\{d_j(x), j \geq 1\}$ is a sequence of positive integers satisfying $d_1(x) \geq 2$ and $d_{j+1}(x) \geq d_j(x)$ for $j \geq 1$. (See [3].) In [3], János Galambos proved that for almost all $x \in (0, 1]$,
\[
\lim_{n \to \infty} \frac{d_1(x)}{n} = e.
\]
He conjectured ([3], P132) that the Hausdorff dimension of the set where (2) fails is one. In this paper, we prove this conjecture:

**Theorem.** $\dim_H \{x \in (0, 1] : (2) \text{ fails} \} = 1$.

We use $L^1$ to denote the one-dimensional Lebesgue measure on $(0, 1]$ and $\dim_H$ to denote the Hausdorff dimension.

2. Proof of Theorem. The aim of this section is to prove the main result of this paper.

By Egoroff’s Theorem, there exists a Borel set $A \subset (0, 1]$ with $L^1(A) \geq 1/2$ such that $\{d_n^{1/n}(x), n \geq 1\}$ converges to $e$ uniformly on $A$. In particular, there exists a positive number $N$ such that
\[
2 \leq d_n^{1/n}(x) \leq 3 \quad \text{for all } n \geq N \text{ and } x \in A.
\]

Choose a positive integer $M$ satisfying $M \geq N$. For any $x = [d_1, d_2, \ldots] = [d_1, d_2, \ldots, d_M, d_{M+1}, d_{M+2}, \ldots, d_{kM+1}, d_{kM+2}, \ldots, d_{(k+1)M}, \ldots]$, we construct a new point $\bar{x} \in (0, 1]$ as follows:
\[
\bar{x} = [\overline{d_1}, \overline{d_2}, \ldots],
\]

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where \( d_{k(M+1)+l} = d_{kM+l} \) for all \( k \geq 0 \) and \( 0 \leq l \leq M \). That is,

\[
\mathfrak{P} = [d_1, d_2, \ldots, d_M, d_M, d_{M+1}, d_{M+2}, \ldots, \\
\quad d_{kM+1}, d_{kM+2}, \ldots, d_{(k+1)M}, d_{(k+1)M}, \ldots].
\]

**Lemma 1.** \( \{\mathfrak{P} : x \in A\} \subset \{x \in (0, 1] : (2) \text{ fails}\}. \)

**Proof.** Note that for any \( k \geq 1 \), \( d_{k(M+1)}(\mathfrak{P}) = d_{kM}(x) \). We have

\[
\lim_{k \to \infty} d_{k(M+1)}^{1/(k(M+1))}(\mathfrak{P}) = \lim_{k \to \infty} (d_{kM}^{1/(kM)}(x))^{kM/(k(M+1))} = e^{M/(M+1)},
\]

and this proves the assertion.

For any \( x = [d_1(x), d_2(x), \ldots] \in (0, 1], y = [d_1(y), d_2(y), \ldots] \in (0, 1] \), define

\[
\varrho(x, y) = \inf \{|j : d_j(x) \neq d_j(y)| \} \quad (\inf \emptyset = \infty).
\]

For any \( x, y \in (0, 1], x \neq y \), suppose \( \varrho(x, y) = k \). Without loss of generality, assume \( d_k(x) < d_k(y) \). Then \( x > y \) and \( x \in (B, C), y \in (D, E) \) with

\[
\begin{align*}
B &= \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \cdots + \frac{1}{d_1(x)d_2(x) \cdots d_{k-1}(x)d_k(x)} \\
&\quad + \frac{1}{d_1(x)d_2(x) \cdots d_k(x)d_{k+1}(x)},
\end{align*}
\]

\[
\begin{align*}
C &= \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \cdots + \frac{1}{d_1(x)d_2(x) \cdots d_{k-1}(x)(d_k(x)-1)} ,
\end{align*}
\]

\[
\begin{align*}
D &= \frac{1}{d_1(y)} + \frac{1}{d_1(y)d_2(y)} + \cdots + \frac{1}{d_1(y)d_2(y) \cdots d_{k-1}(y)d_k(y)} ,
\end{align*}
\]

\[
\begin{align*}
E &= \frac{1}{d_1(y)} + \frac{1}{d_1(y)d_2(y)} + \cdots + \frac{1}{d_1(y)d_2(y) \cdots d_{k-1}(y)d_k(y)} \\
&\quad + \frac{1}{d_1(y)d_2(y) \cdots d_k(y)(d_{k+1}(y)-1)},
\end{align*}
\]

hence

\[
\frac{1}{d_1(x)d_2(x) \cdots d_k(x)d_{k+1}(x)} \leq |x - y| \leq \frac{1}{d_1(x)d_2(x) \cdots d_{k-1}(x)},
\]

where \( d_0(x) \equiv 1 \).

Let

\[
\varepsilon = \frac{6 \log 3}{M \log 2}, \quad c = \frac{1}{34M(M+1)}.
\]

**Lemma 2.** For any \( x, y \in A \),

\[
|\mathfrak{P} - \mathfrak{P}| \geq c|x - y|^{1+2\varepsilon}.
\]

**Proof.** Without loss of generality, assume \( x > y \). Suppose \( \varrho(x, y) = k \).
(a) If \( k \leq 2M \), then by (3), (4) and (6), we have

\[
|x - y| \geq \frac{1}{d_1(x)d_2(x) \ldots d_k(x)d_{k+1}(x)d_{k+2}(x)} \geq \left( \frac{1}{3^{2M}} \right)^{2M+2} \geq c|x - y|^{1+2\varepsilon}.
\]

(b) If \( pM < k \leq (p+1)M \) for some \( p \geq 2 \), then by (4) and (6), we have

\[
|x - y| \geq \frac{1}{d_1(x)d_2(x) \ldots d_k(x)} \geq \frac{1}{3^{2M}} \geq c|x - y|^{1+2\varepsilon}.
\]

For \( 1 \leq j \leq p - 1 \), by (3), we have

\[
d_{jM+k}(x) \leq 3^{jM+k} \leq 3^{2jM+1} \leq \left( \prod_{l=1}^{M} 2^{jM+l+1} \right)^{\varepsilon} \leq \left( \prod_{l=1}^{M} d_{jM+l}(x) \right)^{\varepsilon},
\]

thus

\[
\left( \prod_{l=1}^{M} \frac{1}{d_{jM+l}(x)} \right) \frac{1}{d_{jM+k}(x)} \geq \left( \prod_{l=1}^{M} \frac{1}{d_{jM+l}(x)} \right)^{1+\varepsilon}, \quad 1 \leq j \leq p - 1.
\]

For \( j = 0 \),

\[
\left( \prod_{l=1}^{M} \frac{1}{d_l(x)} \right) \frac{1}{d_{M+k}(x)} \geq \frac{1}{3^{M(M+1)}}.
\]

On the other hand,

\[
d_k(x)d_{k+1}(x) \leq 3^{2k+1} \leq 3^{3k} \leq (2M2^{M+1} \ldots 2^{k-1})^\varepsilon
\]

\[
\leq (d_M(x) \ldots d_{k-1}(x))^\varepsilon,
\]

hence

\[
\frac{1}{d_k(x)d_{k+1}(x)} \geq \left( \frac{1}{d_1(x)d_2(x) \ldots d_{k-1}(x)} \right)^\varepsilon.
\]

Combining (10), (11) and (13), we have

\[
|x - y| \geq \frac{1}{3^{M(M+1)}} \left[ \prod_{j=1}^{p-1} \left( \prod_{l=1}^{M} \frac{1}{d_{jM+l}(x)} \right)^{1+\varepsilon} \right] \left( \prod_{j=pM+1}^{k-1} \frac{1}{d_j(x)} \right) \frac{1}{d_k(x)} \cdot \frac{1}{d_{k+1}(x)}
\]

\[
\geq \frac{1}{3^{M(M+1)}} \left( \prod_{i=1}^{k-1} \frac{1}{d_i(x)} \right)^{1+2\varepsilon} \geq c|x - y|^{1+2\varepsilon}.
\]

**Proof of Theorem.** Consider a map \( f : A \to (0, 1] \) defined by \( f(x) = x \).

Note that \( f : A \to f(A) \) is bijective. Lemma 2 implies that the inverse map of \( f \) is \( 1/(1 + 2\varepsilon) \)-Hölder. By Lemma 1 and [2], Proposition 2.3, we have
\[ 1 = \dim_H A = \dim_H[f^{-1}(f(A))] \leq (1 + 2\varepsilon) \dim_H f(A) \leq (1 + 2\varepsilon) \dim_H \{x \in (0, 1] : (2) \text{ fails}\}. \]

Hence
\[ \dim_H \{x \in (0, 1] : (2) \text{ fails}\} \geq \frac{1}{1 + \frac{12}{M} \log 3 / \log 2}. \]

Since \( M > N \) is arbitrary, we have
\[ \dim_H \{x \in (0, 1] : (2) \text{ fails}\} = 1, \]
and this completes the proof of Theorem.

References


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