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A problem of Galambos on Engel expansions

by

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1. Introduction. Given x in $(0,1]$, let $x=[d_1(x),d_2(x),\ldots]$ denote the Engel expansion of x, that is,

$$x=\frac{1}{d_1(x)}+\frac{1}{d_1(x)d_2(x)}+\ldots+\frac{1}{d_1(x)d_2(x)\ldots d_n(x)}+\ldots, \tag{1}$$

where $\{d_j(x),\ j\ge 1\}$ is a sequence of positive integers satisfying $d_1(x)\ge 2$ and $d_{j+1}(x)\ge d_j(x)$ for $j\ge 1$. (See [3].) In [3], János Galambos proved that for almost all $x\in(0,1]$,

$$\lim_{n\to\infty} d_n^{1/n}(x)=e. \tag{2}$$

He conjectured ([3], P132) that the Hausdorff dimension of the set where (2) fails is one. In this paper, we prove this conjecture:

THEOREM. $\dim_{\mathrm H}\{x\in(0,1]:(2)\ \textit{fails}\}=1$.

We use L^1 to denote the one-dimensional Lebesgue measure on $(0,1]$ and $\dim_{\mathrm H}$ to denote the Hausdorff dimension.

2. Proof of Theorem. The aim of this section is to prove the main result of this paper.

By Egoroff's Theorem, there exists a Borel set $A\subset(0,1]$ with $L^1(A)\ge 1/2$ such that $\{d_n^{1/n}(x),\ n\ge 1\}$ converges to e uniformly on A. In particular, there exists a positive number N such that

$$2\le d_n^{1/n}(x)\le 3 \quad \text{for all } n\ge N \text{ and } x\in A. \tag{3}$$

Choose a positive integer M satisfying $M\ge N$. For any $x=[d_1,d_2,\ldots]=[d_1,d_2,\ldots,d_M,d_{M+1},d_{M+2},\ldots,d_{kM+1},d_{kM+2},\ldots,d_{(k+1)M},\ldots]$, we construct a new point $\overline{x}\in(0,1]$ as follows:

$$\overline{x}=[\overline{d}_1,\overline{d}_2,\ldots],$$

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where $\overline{d}_{k(M+1)+l} = d_{kM+l}$ for all $k \ge 0$ and $0 \le l \le M$. That is,

$$\overline{x} = [d_1, d_2, \ldots, d_M, d_M, d_{M+1}, d_{M+2}, \ldots, d_{kM+1}, d_{kM+2}, \ldots, d_{(k+1)M}, d_{(k+1)M}, \ldots]. \tag{4}$$

LEMMA 1. $\{\overline{x} : x \in A\} \subset \{x \in (0,1] : (2)$ *fails*$\}$.

Proof. Note that for any $k \ge 1, d_{k(M+1)}(\overline{x}) = d_{kM}(x)$. We have

$$\lim_{k\to\infty} d_{k(M+1)}^{1/(k(M+1))}(\overline{x}) = \lim_{k\to\infty} (d_{kM}^{1/(kM)}(x))^{kM/(k(M+1))} = e^{M/(M+1)}, \tag{5}$$

and this proves the assertion.

For any $x = [d_1(x), d_2(x), \ldots] \in (0,1]$, $y = [d_1(y), d_2(y), \ldots] \in (0,1]$, define

$$\varrho(x,y) = \inf\{j : d_j(x) \ne d_j(y)\} \qquad (\inf \emptyset = \infty).$$

For any $x, y \in (0,1]$, $x \ne y$, suppose $\varrho(x,y) = k$. Without loss of generality, assume $d_k(x) < d_k(y)$. Then $x > y$ and $x \in (B, C]$, $y \in (D, E]$ with

$$B = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \ldots + \frac{1}{d_1(x)d_2(x)\ldots d_{k-1}(x)d_k(x)} + \frac{1}{d_1(x)d_2(x)\ldots d_k(x)d_{k+1}(x)},$$

$$C = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \ldots + \frac{1}{d_1(x)d_2(x)\ldots d_{k-1}(x)(d_k(x)-1)},$$

$$D = \frac{1}{d_1(y)} + \frac{1}{d_1(y)d_2(y)} + \ldots + \frac{1}{d_1(y)d_2(y)\ldots d_{k-1}(y)d_k(y)},$$

$$E = \frac{1}{d_1(y)} + \frac{1}{d_1(y)d_2(y)} + \ldots + \frac{1}{d_1(y)d_2(y)\ldots d_{k-1}(y)d_k(y)} + \frac{1}{d_1(y)d_2(y)\ldots d_k(y)(d_{k+1}(y)-1)},$$

hence

$$\frac{1}{d_1(x)d_2(x)\ldots d_k(x)d_{k+1}(x)} \le |x-y| \le \frac{1}{d_1(x)d_2(x)\ldots d_{k-1}(x)}, \tag{6}$$

where $d_0(x) \equiv 1$.

Let

$$\varepsilon = \frac{6\log 3}{M\log 2}, \qquad c = \frac{1}{3^{4M(M+1)}}.$$

LEMMA 2. *For any* $x, y \in A$,

$$|\overline{x} - \overline{y}| \ge c|x-y|^{1+2\varepsilon}. \tag{7}$$

Proof. Without loss of generality, assume $x > y$. Suppose $\varrho(x,y) = k$.

(a) If $k \le 2M$, then by (3), (4) and (6), we have

$$|\overline{x}-\overline{y}| \ge \frac{1}{d_1(\overline{x})d_2(\overline{x})\dots d_k(\overline{x})d_{k+1}(\overline{x})d_{k+2}(\overline{x})} \ge \left(\frac{1}{3^{2M}}\right)^{2M+2} \ge c|x-y|^{1+2\varepsilon}.$$

(b) If $pM < k \le (p+1)M$ for some $p \ge 2$, then by (4) and (6), we have

$$(8) \qquad |\overline{x}-\overline{y}| \ge \prod_{j=0}^{p-1}\left[\left(\prod_{l=1}^{M}\frac{1}{d_{jM+l}(x)}\right)\frac{1}{d_{jM+M}(x)}\right]\prod_{j=pM+1}^{k+1}\frac{1}{d_j(x)}.$$

For $1 \le j \le p-1$, by (3), we have

$$(9) \qquad d_{jM+M}(x) \le 3^{jM+M} \le 3^{2jM} \le \left(\prod_{l=1}^{M}2^{jM+l}\right)^{\varepsilon} \le \left(\prod_{l=1}^{M}d_{jM+l}(x)\right)^{\varepsilon},$$

thus

$$(10) \qquad \left(\prod_{l=1}^{M}\frac{1}{d_{jM+l}(x)}\right)\frac{1}{d_{jM+M}(x)} \ge \left(\prod_{l=1}^{M}\frac{1}{d_{jM+l}(x)}\right)^{1+\varepsilon}, \quad 1 \le j \le p-1.$$

For $j = 0$,

$$(11) \qquad \left(\prod_{l=1}^{M}\frac{1}{d_l(x)}\right)\frac{1}{d_M(x)} \ge \frac{1}{3^{M(M+1)}}.$$

On the other hand,

$$(12) \qquad \begin{aligned} d_k(x)d_{k+1}(x) &\le 3^{2k+1} \le 3^{3k} \le (2^M 2^{M+1}\dots 2^{k-1})^{\varepsilon} \\ &\le (d_M(x)\dots d_{k-1}(x))^{\varepsilon}, \end{aligned}$$

hence

$$(13) \qquad \frac{1}{d_k(x)d_{k+1}(x)} \ge \left(\frac{1}{d_1(x)d_2(x)\dots d_{k-1}(x)}\right)^{\varepsilon}.$$

Combining (10), (11) and (13), we have

$$\begin{aligned} &|\overline{x}-\overline{y}| \\ &\quad \ge \frac{1}{3^{M(M+1)}}\left[\prod_{j=1}^{p-1}\left(\prod_{l=1}^{M}\frac{1}{d_{jM+l}(x)}\right)^{1+\varepsilon}\right]\left(\prod_{j=pM+1}^{k-1}\frac{1}{d_j(x)}\right)\frac{1}{d_k(x)}\cdot\frac{1}{d_{k+1}(x)} \\ &\quad \ge \frac{1}{3^{M(M+1)}}\left(\prod_{i=1}^{k-1}\frac{1}{d_i(x)}\right)^{1+2\varepsilon} \ge c|x-y|^{1+2\varepsilon}. \end{aligned}$$

Proof of Theorem. Consider a map $f : A \to (0,1]$ defined by $f(x) = \overline{x}$. Note that $f : A \to f(A)$ is bijective. Lemma 2 implies that the inverse map of f is $1/(1+2\varepsilon)$-Hölder. By Lemma 1 and [2], Proposition 2.3, we have

$$1 = \dim_{\mathrm{H}} A = \dim_{\mathrm{H}}[f^{-1}(f(A))] \le (1+2\varepsilon)\dim_{\mathrm{H}} f(A)$$
$$\le (1+2\varepsilon)\dim_{\mathrm{H}}\{x \in (0,1] : (2) \text{ fails}\}.$$

Hence

$$\dim_{\mathrm{H}}\{x \in (0,1] : (2) \text{ fails}\} \ge \frac{1}{1+\frac{12}{M}\cdot\frac{\log 3}{\log 2}}.$$

Since $M > N$ is arbitrary, we have

$$\dim_{\mathrm{H}}\{x \in (0,1] : (2) \text{ fails}\} = 1,$$

and this completes the proof of Theorem.

References

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