Non-vanishing of modular $L$-functions on a disc

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Introduction. Non-vanishing of $L$-functions on a disc has been studied in various contexts in the recent years. In the context of Dirichlet $L$-functions P. Elliott [6] proved that there are infinitely many Dirichlet $L$-functions $L(s,\chi_p)$ ($\chi_p$ is a Dirichlet character mod $p$ (prime)) which are uniformly bounded below by $c(\log p)^{1/2}$ in the disc $|s - 1/2| \leq (\log p)^{-(1+\epsilon)}$, and so do not vanish there. This result has been improved by R. Balasubramanian in [2]. He proved that the number of Dirichlet $L$-functions $L(s,\chi_p)$ that do not vanish in the disc $|s - 1/2| \leq (\log p)^{-(1+\epsilon)}$ is bounded below by $cp(\log p)^{-2}$. Also, in [3] R. Balasubramanian and K. Murty studied non-vanishing of Dirichlet $L$-functions in the disc $|s - \sigma_j| \leq 2(\log p)^{-1}$, where $\sigma_j = 1/2 + j/\log p$ and $2 \leq j \leq (\log p)/2 - 2$. They proved that for a positive proportion of the characters $\chi_p \pmod{p}$, $L(s,\chi_p)$ does not have a real zero in the region $1/2 + c/\log p \leq \text{Re}(s) < 1$. Here, $c > 0$ is an absolute constant and $p$ is a sufficiently large prime.

In this paper we prove an analogue of the above results in the context of modular $L$-functions. We are interested in the zeros of $L_f(s,\chi)$ in the critical strip $k/2 < \text{Re}(s) < (k + 1)/2$, where $L_f(s,\chi)$ is the twisted $L$-function associated with the newform $f$ and Dirichlet character $\chi$. Generalized Riemann Hypothesis predicts that $L_f(s,\chi)$ is non-zero in this strip. One of the known results in the subject is given by K. Murty and T. Stefanicki [7]. They proved that at least $Y^{2/3-\epsilon}$ quadratic twists $L_f(s,\chi_d)$ ($|d| \leq Y$, $d \equiv 1 \pmod{4}$) attached to holomorphic newforms and $Y^{2/3-\epsilon}$ attached to Maass newforms do not vanish inside the disc $|s - s_0| < (\log Y)^{-(1+\epsilon)}$ for any $\epsilon > 0$ and any point $s_0$ inside the critical strip (the exponent $2/3$ can in fact be improved now to 1 using improved character sum estimates of Heath-Brown as in the work of Perelli and Pomykała [8]).

Here, we prove the following theorem.

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Theorem 1. Let \( s_0 = \sigma_0 + it_0 \) be a point in the strip \( k/2 < \text{Re}(s) < (k+1)/2 \) and let \( C_N \) be the disc with center \( s_0 \) and radius \( r_N = o(1) \) (i.e. \( r_N \to 0 \) as \( N \to \infty \)). Suppose that \( \chi \) is a fixed primitive Dirichlet character \( \bmod q \) such that \((q,N) = 1\). Then there are positive constants \( C_{\sigma_0,k} \) (depending only on \( k \) and \( \sigma_0 \)) and \( C_{\sigma_0,q,k,r_N} \) (depending on \( q, k, s_0 \) and \( r_N \)) such that for prime \( N > C_{\sigma_0,q,k,r_N} \) there exist at least \( C_{\sigma_0,k}N(\log N)^{-1} \) newforms \( f \) of weight \( k \) and level \( N \) for which \( L_f(s,\chi) \neq 0 \) for all \( s \in C_N \).

The methodology of the proof is based on a comparison of mean values. In Sections 3 and 4, we derive asymptotic formulae for \( L_f(s_f,\chi) \) and \( |L_f(s_f,\chi)|^2 \) on average, where \( s_f \) is an arbitrary point in the disc \( C_N \). To do this first we derive the asymptotic formulae for a fixed point \( s_0 \) in the critical strip (Lemmas 5 and 7). These are analogues of the results given by W. Duke [4] for the center of critical strip. Then an application of Cauchy’s integral formula gives us the asymptotic formulae on a disc (Propositions 1 and 2). This technique has already been applied by P. Elliott, B. Balasubramanian and B. Balasubramanian–K. Murty for Dirichlet \( L \)-functions.

Finally we have to deal with the contribution of oldforms; we apply the technique developed by the author in [1] to overcome this difficulty. In Section 5 we finish off the proof of Theorem 1 by an application of the Cauchy–Schwarz inequality.

Finally, with a slight modification of our previous results, we establish asymptotic formulae for \( L_f(s_f,\chi) \) and \( |L_f(s_f,\chi)|^2 \) on average, where \( s_f \) is an arbitrary point in the disc \( C_N \) with center on the critical line \( s = k/2 + it \), and as a result we prove the following non-vanishing theorem.

Theorem 2. Let \( s_0 = k/2 + it_0 \) and let \( C_N \) be the disc with center \( s_0 \) and radius \( r_N = 1/(\log N)^{4+\varepsilon} \) (\( \varepsilon > 0 \)). Suppose that \( \chi \) is a fixed primitive Dirichlet character \( \bmod q \) such that \((q,N) = 1\). Then there are positive constants \( C_k \) (depending only on \( k \)) and \( C_{t_0,q,k,\varepsilon} \) (depending on \( q, k, t_0 \) and \( \varepsilon \)) such that for prime \( N > C_{t_0,q,k,\varepsilon} \) there exists at least \( C_k N(\log N)^{-2\varepsilon} \) newforms \( f \) of weight \( k \) and level \( N \) for which \( L_f(s,\chi) \neq 0 \) for all \( s \in C_N \).

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2. Preliminaries. In this section we review some basic facts concerning modular forms and set up our notation.

Let \( S_k(N) \) be the space of cusp forms of weight \( k \) for \( \Gamma_0(N) \) with trivial character. The space \( S_k(N) \) has an inner product (Petersson inner product)

\[
\langle f, g \rangle = \int_{\Gamma_0(N) \backslash \mathbb{H}} f(z)\overline{g(z)}y^k \frac{dx \, dy}{y^2}
\]
where $\mathcal{H}$ denotes the upper half-plane. For any $f \in S_k(N)$ let
\[
f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz), \quad e(z) = e^{2\pi i z},
\]
be the Fourier expansion of $f$ at $i\infty$.

Let $\chi$ be a primitive Dirichlet character mod $q$ with $(q,N) = 1$. Then the twisted $L$-function associated with $f$ and $\chi$ is defined by
\[
L_f(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a_f(n)}{n^s}.
\]

The twisted $L$-function is given by an absolutely convergent series on the half-plane $\Re(s) > (k+1)/2$ and it has an analytic continuation to the whole plane. Moreover, if $f$ is a newform (in Atkin–Lehner sense), then $L_f(s,\chi)$ has an Euler product valid on $\Re(s) > (k+1)/2$ and it satisfies the following functional equation:
\[
(1) \quad \left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma(s)L_f(s,\chi) = \varepsilon_\chi \left(\frac{q\sqrt{N}}{2\pi}\right)^{k-s} \Gamma(k-s)L_f(k-s,\chi)
\]
where $\varepsilon_\chi = \varepsilon_f \chi(N)^{1/2}q^{-1}$ with $\varepsilon_f = \pm 1$ (the root number of $f$) which depends only on $f$ and $\tau(\chi)$ is the Gauss sum.

Let $\{T_p \ (p \nmid N), \ U_q \ (q \nmid N)\}$ be the collection of the classical Hecke operators and let $W_q \ (q \mid N)$ be the “$W$ operator” of Atkin and Lehner. In 1983 A. Pizer introduced the operators $C_q$ on $S_k(N)$ for $q \nmid N$, such that the action of $C_q$ on the new part of $S_k(N)$ is the same as the action of the classical $U_q$ operators. More precisely he defined $C_q$ as
\[
C_q = \begin{cases} 
U_q + W_q U_q W_q + q^{k/2-1} W_q & \text{if } q \parallel N, \\
U_q + W_q U_q W_q & \text{if } q^2 \nmid N.
\end{cases}
\]
Then he showed that $T_p \ (p \nmid N), \ C_q \ (q \mid N)$ form a commuting family of Hermitian operators. Using this, he proved ([9], Theorem 3.10) the following result:

**Theorem.** There exists a basis $f_i(z)$ ($1 \leq i \leq \dim S_k(N)$) of $S_k(N)$ such that each $f_i(z)$ is an eigenform for all the $T_p$ and $C_q$ operators with $p \nmid N$ and $q \mid N$. Let $f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz)$ be an element of this basis. Then $a_f(1) \neq 0$ and assuming $f(z)$ is normalized so that $a_f(1) = 1$, we have $f \mid T_p = a_f(p)f$ for all $p \nmid N$, $f \mid C_q = a_f(q)f$ for all $q \mid N$, and $a_f(nm) = a_f(n)a_f(m)$ whenever $(n,m) = 1$. Furthermore $f(z)$ is an eigenform for all $W_q$ operators, $q \mid N$. Finally, if $g(z) \in S_k(N)$ is an eigenform for all the $T_p$ and $C_q$ operators with $p \nmid N$ and $q \mid N$, then $g(z) = cf_i(z)$ for some $c \in \mathbb{C}^*$ and some unique $i$, $1 \leq i \leq \dim S_k(N)$. 
Now let \( \mathcal{F}_N \) be the set of all normalized \((a_f(1) = 1)\) newforms in \( S_k(N) \) and let \( \mathcal{P}_N \) be the basis of \( S_k(N) \) given by the above theorem. The elements of \( \mathcal{P}_N \) form an orthogonal basis (with respect to the Petersson inner product) for \( S_k(N) \), any \( f \in \mathcal{P}_N \) has real Fourier coefficient and \( L_f(s, \chi) \) satisfies the functional equations (1). Moreover, we can show that the action of \( C_q \) on \( S_k(N)^{\text{new}} \) is the same as the action of \( U_q \) (see [9], Remark 2.9). This shows that \( \mathcal{F}_N \subset \mathcal{P}_N \).

For the Fourier coefficients of a newform \( f \) we have the Deligne bound
\[
|a_f(n)| \leq d(n)^{(k-1)/2}
\]
where \( d(n) \) is the divisor function. For \( N \) prime, we have the following estimation of the Fourier coefficients of \( f \in \mathcal{P}_N \).

**Lemma 1.** Suppose \( N \) is prime and \( f \in \mathcal{P}_N \). Then
\[
|a_f(n)| \leq c_0 n^{k/2}
\]
where \( c_0 \) is an absolute constant independent of \( f \).

**Proof.** Propositions 3.6 and 3.4 of [9] imply that if \( f \in \mathcal{P}_N - \mathcal{F}_N \), then
\[
f(z) = h(z) \pm N^{k/2} h(Nz)
\]
where \( h \) is the normalized newform of weight \( k \) and level 1 associated with \( f \). Now the result follows from the Deligne bound for the newforms (see [1], Lemma 2.2, for the details).

Finally, since \( \mathcal{P}_N \) forms an orthogonal basis of \( S_k(N) \), the Fourier coefficients of its elements are semi-orthogonal in the following sense:

**Lemma 2.** Let \( \omega_f = \Gamma(k-1)/(4\pi)^{k-1} (f,f) \) and let \( \delta_{m,n} \) be the Kronecker delta. For \( m \) and \( n \) positive integers we have the inequality
\[
\left| \sum_{f \in \mathcal{P}_N} \omega_f \frac{a_f(m)}{\sqrt{m}^{k-1}} \frac{a_f(n)}{\sqrt{n}^{k-1}} - \delta_{m,n} \right| \leq M d(N)^{1/2-k} (m,n)^{1/2} \sqrt{(mn)^{k-1}}
\]
where \( M \) is a constant depending only on \( k \) and \( d(N) \) is the number of divisors of \( N \).

**Proof.** See [4], Lemma 1.

**3. Mean estimation.** In this section we will find an asymptotic formula for
\[
\sum_{f \in \mathcal{P}_N} \omega_f L_f(s_f, \chi)
\]
where \( s_f \) is a variable point in the disc with center \( s_0 = \sigma_0 + it_0 \) \((k/2 < \sigma_0 < (k+1)/2)\) and radius \( r_N = o(1) \).
Lemma 3. For any $x > 0$ and $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ where $(k - 1)/2 \leq \sigma_0 \leq (k + 1)/2$, let

$$W(s_0, x) = \frac{1}{2\pi i} \int_{(5/4)} \Gamma(s + s_0) x^{-s} \frac{ds}{s}$$

and

$$A_{f, \chi}(x, s_0) = \sum_{n \geq 1} \chi(n) a_f(n) n^{-s_0} W(s_0, 2\pi n / x)$$

where $\chi$ is a fixed primitive Dirichlet character mod $q$ with $(q, N) = 1$. Then

$$\Gamma(s_0) L_f(s_0, \chi) = A_{f, \chi}(x, s_0) + \varepsilon_{\chi} \left( \frac{q\sqrt{N}}{2\pi} \right)^{k-2s_0} A_{f, \chi}\left( \frac{q^2 N}{x}, k - s_0 \right)$$

where $\varepsilon_{\chi}$ is the root number of $L_f(s, \chi)$.

Proof. From the definition of $W(s_0, x)$ it is clear that

$$A_{f, \chi}(x, s_0) = \frac{1}{2\pi i} \int_{(5/4)} L_f(s + s_0, \chi) \left( \frac{x}{2\pi} \right)^s \Gamma(s + s_0) \frac{ds}{s}.$$ 

Changing the line of integration from $5/4$ to $-5/4$ and using the functional equation (1) yields

$$A_{f, \chi}(x, s_0) = \Gamma(s_0) L_f(s_0, \chi)$$

$$+ \varepsilon_{\chi} \left( \frac{q\sqrt{N}}{2\pi} \right)^{k-2s_0} \frac{1}{2\pi i} \int_{(-5/4)} L_f(k - s - s_0, \chi) \left( \frac{2\pi x}{q^2 N} \right)^s \Gamma(k - s - s_0) \frac{ds}{s}.$$ 

Now changing variables $s \mapsto -s$ implies the result. 

Lemma 4. Under the assumptions of Lemma 3,

$$W(s_0, x) \ll x^{\sigma_0 - 1} e^{-x} \quad \text{as } x \to \infty,$$

$$W(s_0, x) \ll k \quad \text{as } x \to 0.$$

Proof. We have

$$W(s_0, x) = \frac{1}{2\pi i} \int_{(5/4)} \left( \int_0^\infty e^{-t} t^{s+s_0-1} dt \right) x^{-s} \frac{ds}{s} = \int_0^\infty t^{s_0-1} e^{-t} dt.$$ 

Therefore

$$|W(s_0, x)| = \left| \int_x^\infty t^{s_0-1} e^{-t} dt \right| \leq \int_x^\infty t^{\sigma_0-1} e^{-t} dt.$$ 

Now the first result follows from the estimation of the last integral using integration by parts. The second result is clear since $|W(s_0, x)| \leq \Gamma(\sigma_0)$ as $x \to 0$. 

Lemma 5. Let $\chi$ be a fixed primitive Dirichlet character mod $q$ with $(q, N) = 1$ and let $s_0 = \sigma_0 + it_0$ be a point in the strip $(k - 1)/2 < \Re(s) \leq (k + 1)/2$. Then
\[
\sum_{f \in P_N} \omega_f L_f(s_0, \chi) = 1 + O\left(\frac{1}{|\Gamma(s_0)|} N^{1/2-\sigma_0} (\log N)^{k-\sigma_0}\right)
+ O\left(\frac{1}{|\Gamma(s_0)|} N^{(k-1)/2-\sigma_0} (\log N)^{k-\sigma_0-1}\right)
\]
for $N$ prime. The implied constant depends only on $q$ and $k$.

Proof. Choosing $x = q^2 N \log N$ in Lemma 3 gives
\[
A_{f, \pi}(Nq^2 x, k - s_0) = \sum_{n \geq 1} \chi(n) a_f(n) n^{\sigma_0 - k} W(k - s_0, 2\pi n \log N).
\]
Using Lemmas 4 and 1 we have
\[
\left| A_{f, \pi}(Nq^2 x, k - s_0) \right| \leq \sum_{n \geq 1} |a_f(n)| |n^{\sigma_0 - k}| W(k - s_0, 2\pi n \log N)|
\leq \sum_{n \geq 1} c_0 n^{k/2} n^{\sigma_0 - k} (2\pi n \log N)^{k-\sigma_0-1} e^{-2\pi n \log N}
= c_0 (2\pi \log N)^{k-\sigma_0-1} \sum_{n \geq 1} n^{k/2-1} (N^2 \pi)^n.
\]
Therefore from Lemma 3 we get
\[
\Gamma(s_0) \sum_{f \in P_N} \omega_f L_f(s_0, \chi) = \sum_{f \in P_N} \omega_f A_{f, \pi}(x, s_0)
+ \left(\sum_{f \in P_N} \omega_f\right) O_q,k(N^{-6+k/2-\sigma_0} (\log N)^{k-\sigma_0-1}).
\]
From this, we have
\[
\Gamma(s_0) \sum_{f \in P_N} \omega_f L_f(s_0, \chi) - \Gamma(s_0)
= \sum_{n \geq 1} \chi(n) \left(\sum_{f \in P_N} \omega_f a_f(n) n^{(k-1)/2} - \delta_{1,n}\right) W\left(s_0, \frac{2\pi n}{q^2 N \log N}\right) n^{(k-1)/2-s_0}
+ W\left(s_0, \frac{2\pi}{q^2 N \log N}\right) - \Gamma(s_0)
+ \left(\sum_{f \in P_N} \omega_f\right) O_q,k(N^{-6+k/2-\sigma_0} (\log N)^{k-\sigma_0-1}).
\]
Note that
\[
W\left(s_0, \frac{2\pi}{q^2 N \log N}\right) - \Gamma(s_0) = \int_0^{2\pi/(q^2 N \log N)} t^{s_0 - 1} e^{-t} \, dt
= O_{q,k}(N \log N)^{-\sigma_0}.
\]
Also, from Lemma 2 for \(m = n = 1\) it follows that
\[
\sum_{f \in P_N} \omega_f = 1 + O(N^{1/2-k}).
\]
By applying \(m = 1\) in Lemma 2 and using the above identities, we have
\[
\left|\Gamma(s_0)\left(\sum_{f \in P_N} \omega_f L_f(s_0, \chi) - 1\right)\right|
\leq M_1 \frac{N^{1/2-k}}{(N \log N)^{\sigma_0 - 1}} \sum_{n \geq 1} n^{k-2} e^{-2\pi n/(q^2 N \log N)} + M_2 (N \log N)^{-\sigma_0}
+ M_3 N^{-6+k/2-\sigma_0} (\log N)^{k-\sigma_0-1}
\]
where \(M_1, M_2, M_3\) are constants depending on \(q\) and \(k\). This proves the desired result.

**Proposition 1.** Let \(s_0 = \sigma_0 + it_0\) be a point in the strip \(k/2 < \text{Re}(s) < (k+1)/2\) and let \(\Gamma\) and \(C_N\) be the circles with center \((\sigma_0, t_0)\) and radius \(R = \frac{1}{2} \min\{(k+1)/2 - \sigma_0, \sigma_0 - k/2\}\) and \(r_N = o(1)\) respectively. Then for \(N\) prime
\[
\sum_{f \in P_N} \omega_f L_f(s_f, \chi) = 1 + O_{q,k}\left(\frac{1}{|\Gamma(s_0)|} N^{-1/2}\right) + O_{q,k,s_0}\left(\frac{r_N}{R - r_N} N^{-1/2}\right)
\]
where \(s_f\) is an arbitrary point in \(C_N\).

**Proof.** By Cauchy’s integral formula for any \(s_f \in C_N\), we have
\[
L_f(s_f, \chi) - L_f(s_0, \chi) = \frac{1}{2\pi i} \int_{\Gamma} L_f(w, \chi) \frac{1}{w - s_f} - \frac{1}{w - s_0} \, dw
\]
where \(\Gamma\) is traversed in the counter clockwise direction. Therefore
\[
\sum_{f \in P_N} \omega_f L_f(s_f, \chi) = \sum_{f \in P_N} \omega_f L_f(s_0, \chi)
+ \frac{1}{2\pi i} \int_{\Gamma} \left(\sum_{f \in P_N} \omega_f L_f(w, \chi)\right) \frac{s_f - s_0}{(w - s_f)(w - s_0)} \, dw.
\]
Now using Lemma 5 yields
(3) \[ \left| \frac{1}{2\pi i} \int_{C} \left( \sum_{f \in P} \omega_f L_f(w, \chi) \right) \frac{s_f - s_0}{(w - s_f)(w - s_0)} \, dw \right| \leq \frac{r_N}{R - r_N} O_{q,k,s_0}(N^{-1/2}). \]

Note that here we used the fact that \( \frac{1}{2\pi i} \int_{C} \frac{s_f - s_0}{(w - s_f)(w - s_0)} \, dw = 0. \)

Applying (3) and Lemma 5 in (2) completes the proof. 

4. Mean square estimation. In this section we are going to find an asymptotic formula for the average values of \( |L_f(s_f, \chi)|^2 \) where \( s_f \) is a variable point in a disc with center \( s_0 = \sigma_0 + it_0 \) \( (k/2 < \sigma_0 < (k + 1)/2) \) and radius \( r_N = \omega(1) \). We start with writing \( |L_f(s_0, \chi)|^2 \) as a sum of two convergent series.

Let \( |L_f(s_0, \chi)|^2 = \sum_{l \geq 1} b_f(l) l^{-\sigma_0} \) so that

(4) \[ b_f(l) = \sum_{mn = l} \chi(n) \overline{\chi(m)} a_f(n) a_f(m) \left( \frac{m}{n} \right)^{it_0}. \]

For \( x > 0 \) and \( s_0 = \sigma_0 + it_0 \in \mathbb{C} \) where \( (k - 1)/2 \leq \sigma_0 \leq (k + 1)/2 \), define

(5) \[ B_f(x, s_0) = \sum_{l \geq 1} \frac{b_f(l)}{l^{\sigma_0}} Z(s_0, l/x) \]

where

(6) \[ Z(s_0, x) = \frac{1}{2\pi i} \int_{(5/4)} (2\pi)^{-2s} \Gamma(s + s_0) \Gamma(s + \overline{s_0}) x^{-s} ds/s. \]

Using Deligne’s bound in (4) and standard estimates for \( Z(s_0, x) \) shows that (5) is absolutely convergent.

Lemma 6. Let \( f \in P_N \) and suppose that \( \chi \) is a primitive Dirichlet character mod \( q \) with \( (q, N) = 1 \). For any \( x > 0 \) we have

\[ |\Gamma(s_0) L_f(s_0, \chi)|^2 = B_f(x, s_0) + \left( \frac{q^2 N}{4\pi^2} \right)^{k-2\sigma_0} B_f \left( \frac{(q^2 N)^2}{x}, k - \sigma_0 \right). \]

Proof. From (6) we have

\[ B_f(x, s_0) = \frac{1}{2\pi i} \int_{(5/4)} (2\pi)^{-2s} \Gamma(s + s_0) \Gamma(s + \overline{s_0}) L_f(s + s_0, \chi) L_f(s + \overline{s_0}, \overline{\chi}) x^s ds/s. \]

By changing the line of integration from \( 5/4 \) to \(-5/4\) and using the functional
equation (1) we get
\[ B_f(x, s_0) = |\Gamma(s_0) L_f(s_0, \chi)|^2 + \left( \frac{q^2 N}{4\pi^2} \right)^{k-2\sigma_0} \int_{(-5/4)} (2\pi)^{2s} \Gamma(k-s-s_0) \Gamma(k-s-s_0) \times L_f(k-s-s_0, \chi) L_f(k-s-s_0, \chi) \left( \frac{x}{(q^2 N)^2} \right)^s \frac{ds}{s}. \]

Now changing variables \( s \mapsto -s \) yields the result. \( \blacksquare \)

We estimate \( B_f(x, s_0) \) on average. From (4) and (5) it follows that
\[
\sum_{f \in \mathcal{P}} \omega_f B_f(x, s_0) = \sum_{l \geq 1} b_f(l) l^{-\sigma_0} Z(s_0, l/x) = \sum_{m, n \geq 1} \chi(n) \overline{\chi}(m) Z(s_0, mn/x) \frac{(m/n)^{it_0}}{(mn)^{\sigma_0 - (k-1)/2}} \times \sum_{f \in \mathcal{P}_N} \omega_f \frac{a_f(m)}{\sqrt{m^{k-1}}} \frac{a_f(n)}{\sqrt{n^{k-1}}} = \sum_{n \geq 1} |\chi(n)|^2 Z(s_0, n^2/x) \frac{1}{n^{2\sigma_0-k+1}} + R
\]

where
\[
R \ll N^{1/2-k} \sum_{m, n \geq 1} Z(s_0, mn/x) (m, n)^{1/2} (mn)^{-\sigma_0+k-1}.
\]

Note that here we are using the inequality \(|Z(s_0, x)| \leq Z(s_0, x)\). This is true since by writing \( \Gamma \) functions in terms of integrals in (6) and interchanging the order of integration, we have
\[
Z(s_0, x) = \int_0^{t_{s_0}} e^{-t} \left( \int_{4\pi^2 t_1}^{\infty} e^{-t_2} t_{s_0}^{-1} dt_2 \right) dt_1.
\]

Applying the triangle inequality in the above identity implies the desired inequality.

Using the definition of \( Z(s_0, x) \), the first term in (7) is equal to
\[
\frac{1}{2\pi i} \int_{(5/4)} L(2s + 2\sigma_0 - k + 1, \chi_0) (2\pi)^{-2s} \Gamma(s + s_0) \Gamma(s + \sigma_0) x^s \frac{ds}{s}
\]

where \( \chi_0 \) is the principal character mod \( q \) and \( L(s, \chi_0) = \zeta(s) \prod_{p \mid q} (1 - 1/p^s) \).

Now we assume that \( \sigma_0 \neq k/2 \), since the integrand has simple poles at \( s = 0 \).
and $s = k/2 - \sigma_0$, by moving the line of integration from $5/4$ to $-1/2$, the integral is equal to

$$
|\Gamma(s_0)|^2 \prod_{p|q} \left(1 - \frac{1}{p^{2\sigma_0-k+1}}\right) \zeta(2\sigma_0 - k + 1)
$$

$$
+ \prod_{p|q}(1 - 1/p)(2\pi)^{2\sigma_0-k} \frac{\Gamma(k/2 + it_0)\Gamma(k/2 - it_0)x^{k/2-\sigma_0}}{k - 2\sigma_0} + O_{\sigma_0,q,k}(x^{-1/2}).
$$

Now in (7) we estimate the remainder term $R$. We calculate

$$
\sum_{m,n \geq 1} Z(\sigma_0, mn/x)(m,n)^{1/2}(mn)^{-\sigma_0+k-1}.
$$

It is

$$
\frac{1}{2\pi i} \int_{((k+1)/2)}^{} (2\pi)^{-2s}(\Gamma(s + \sigma_0))^2 x^s \left(\sum_{m,n \geq 1} (m,n)^{1/2}(mn)^{-s+\sigma_0-k+1}\right) ds.
$$

Note that since the integrand does not have any pole in the strip $5/4 < \text{Re}(s) < (k + 1)/2$, we can move the line of integration from $5/4$ to $(k + 1)/2$. From [4], Lemma 4, we know that

$$
\sum_{m,n \geq 1} (m,n)^{1/2}(mn)^{-s+\sigma_0-k+1} = \frac{\zeta(2s + 2\sigma_0 - 2k + 3/2)\zeta(s + \sigma_0 - k + 1)^2}{\zeta(2s + 2\sigma_0 - 2k + 2)}.
$$

Applying this identity to the above integral and moving the line of integration from $(k + 1)/2$ to $k - \sigma_0 - \varepsilon$ ($\varepsilon > 0$) yields

$$
\sum_{m,n \geq 1} Z(\sigma_0, mn/x)(m,n)^{1/2}(mn)^{-s+\sigma_0-k+1} \sim C_{\sigma_0,k}x^{k-\sigma_0} \log x
$$

and by (8), $R \ll N^{1/2-k}x^{k-\sigma_0} \log x$. Therefore we have

$$
\sum_{f \in P_N} \omega_f B_f(x, s_0)
$$

$$
= |\Gamma(s_0)|^2 \prod_{p|q} \left(1 - \frac{1}{p^{2\sigma_0-k+1}}\right) \zeta(2\sigma_0 - k + 1)
$$

$$
+ \prod_{p|q}(1 - 1/p)(2\pi)^{2\sigma_0-k} \frac{\Gamma(k/2 + it_0)\Gamma(k/2 - it_0)x^{k/2-\sigma_0}}{k - 2\sigma_0} + O_{\sigma_0,q,k}(x^{-1/2}) + O_{\sigma_0,k}(N^{1/2-k}x^{k-\sigma_0} \log x).
$$
Lemma 7. Let $\chi$ be a fixed primitive Dirichlet character mod $q$ with $(q,N) = 1$ and let $s_0 = \sigma_0 + it_0$ where $k/2 < \sigma_0 \leq (k + 1)/2$. Then

$$\sum_{f \in \mathcal{P}_N} \omega_f |L_f(s_0, \chi)|^2 = \prod_{p|q} \left(1 - \frac{1}{p^{2\sigma_0-k+1}}\right) \zeta(2\sigma_0 - k + 1) + c_1 N^{k/2 - \sigma_0} + O_{s_0,q,k}(N^{-1/2})$$

for $N$ prime. Here, $c_1$ depends on $s_0$, $q$ and $k$.

Proof. Choosing $x = q^2 N$ in Lemma 6 and applying (11) in it, proves the lemma.

Proposition 2. Under the assumptions of Proposition 1,

$$\sum_{f \in \mathcal{P}_N} \omega_f |L_f(s_f, \chi)|^2 = \prod_{p|q} \left(1 - \frac{1}{p^{2\sigma_0-k+1}}\right) \zeta(2\sigma_0 - k + 1) + c_1 N^{k/2 - \sigma_0}$$

$$+ O_{s_0,q,k}(N^{-1/2}) + O_{\sigma_0,k} \left(\frac{r_N}{R - r_N}\right)$$

$$+ O_{s_0,q,k} \left(\frac{r_N}{R - r_N} N^{k/2 - \sigma_0 + R}\right).$$

Here, $c_1$ depends on $s_0$, $q$ and $k$.

Proof. We have

$$\left| \sum_{f \in \mathcal{P}_N} \omega_f |L_f(s_f, \chi)|^2 - \sum_{f \in \mathcal{P}_N} \omega_f |L_f(s_0, \chi)|^2 \right|$$

$$\leq \sum_{f \in \mathcal{P}_N} \omega_f |L_f(s_f, \chi)|^2 - |L_f(s_0, \chi)|^2$$

$$\leq \sum_{f \in \mathcal{P}_N} \omega_f |L_f^2(s_f, \chi) - L_f^2(s_0, \chi)|.$$

By applying Cauchy’s integral formula and Lemma 7, the last expression equals to

$$\sum_{f \in \mathcal{P}_N} \omega_f \left| \frac{1}{2\pi i} \int \frac{s_f - s_0}{(w - s_f)(w - s_0)} \frac{dw}{L_f^2(w, \chi)} \right|$$

$$\leq \frac{r_N}{R - r_N} (O_{\sigma_0,k}(1) + O_{s_0,q,k}(N^{k/2 - \sigma_0 + R})).$$

This shows that
Now applying Lemma 7 in (12) completes the proof. ■

5. Proof of Theorem 1. We need the following estimation of $\omega_f$.

**Proposition 3.** For $N$ prime we have

$$\omega_f \ll_k \begin{cases} \frac{(\log N)}{N}, & f \in \mathcal{F}_N, \\ \frac{1}{N}, & f \in \mathcal{P}_N - \mathcal{F}_N. \end{cases}$$

**Proof.** See [4], Proposition 4, for the case $f \in \mathcal{F}_N$. If $f \in \mathcal{P}_N - \mathcal{F}_N$ then

$$f(z) = h(z) \pm N^{k/2}h(Nz)$$

as mentioned in the proof of Lemma 1. Now the result follows from the fact that

$$\langle f, f \rangle = \langle h(z) \pm N^{k/2}h(Nz), h(z) \pm N^{k/2}h(Nz) \rangle$$

is bounded below by a constant multiple of $N$ (see [1], Proposition 5.3 for the details). ■

Now we can prove our theorem. Set

$$\mathcal{E}_N = \{ f \in \mathcal{P}_N : L_f(s, \chi) \neq 0 \text{ for all } s \in \mathcal{C}_N \}.$$

Proposition 1 shows that $\mathcal{E}_N \neq \emptyset$ for large $N$. Now if $f \in \mathcal{P}_N - \mathcal{E}_N$ we choose $s_f$ such that $L_f(s_f, \chi) = 0$. With this choice of $s_f$ for elements of $\mathcal{P}_N - \mathcal{E}_N$ and arbitrary choice of $s_f$ in $\mathcal{C}_N$ for elements of $\mathcal{E}_N$ and applying the Cauchy–Schwarz inequality and (13), we get

$$\left| \sum_{f \in \mathcal{P}_N} \omega_f |L_f(s_f, \chi)|^2 \right|^2 = \left| \sum_{f \in \mathcal{E}_N} \omega_f |L_f(s_f, \chi)|^2 \right|^2 \leq \left( \sum_{f \in \mathcal{E}_N \cap \mathcal{F}_N} \omega_f + \sum_{f \in \mathcal{E}_N \cap \mathcal{F}_N} \omega_f \right) \left( \sum_{f \in \mathcal{P}_N} \omega_f |L_f(s_f, \chi)|^2 \right) \ll \left( \sharp \{ f \in \mathcal{F}_N : L_f(s, \chi) \neq 0 \text{ for all } s \in \mathcal{C}_N \} \frac{\log N}{N} \right) \sum_{f \in \mathcal{P}_N} \omega_f |L_f(s_f, \chi)|^2.$$

Theorem 1 follows by applying Propositions 1 and 2 in (14). ■
6. Proof of Theorem 2. We first establish the analogues of Proposition 1, Lemma 7 and Proposition 2 for a point $s_0$ on the critical line $\sigma = k/2$.

Proposition 1'. Let $N$ be prime, and let $\Gamma$ and $C_N$ be the circles with center $(k/2, t_0)$ and radius $R_N$ and $r_N$ respectively. Suppose that $0 < r_N < R_N < 1/2$, and

$$\frac{r_N}{R_N} N^{R_N/2} (\log N)^{R_N} = o\left(\frac{N^{1/2}}{\log N}\right).$$

Then

$$\sum_{f \in \mathcal{P}_N} \omega_f L_f(s_f, \chi) = 1 + O_{q,k}\left(\frac{1}{\Gamma(k/2 + t_0)} N^{-1/2} \log N\right) + O_{q,k,t_0}\left(\frac{r_N}{R_N - r_N} N^{R_N-1/2} (\log N)^{R_N+1}\right)$$

where $s_f$ is an arbitrary point in $C_N$.

Proof. It is similar to the proof of Proposition 1. ■

Lemma 7'. Let $\chi$ be a fixed primitive Dirichlet character mod $q$ with $(q, N) = 1$ and let $s_0 = k/2 + it_0$. Then

$$\sum_{f \in \mathcal{P}_N} \omega_f |L_f(k/2 + it_0, \chi)|^2 = \prod_{p | q} \left(1 - \frac{1}{p}\right) \log N + c_1 + O_{t_0,q,k}(N^{-1/2} \log N)$$

for $N$ prime. Here, $c_1$ depends on $t_0, q$ and $k$.

Proof. The proof is exactly similar to the proof of Lemma 7. The result follows by observing that

$$\frac{1}{2\pi i} \int \left(\frac{s}{s_0}\right) L(2s + 1, \chi_0)(2\pi)^{-2s} \Gamma(s + k/2 + it_0) \Gamma(s + k/2 - it_0) x^s ds$$

has a double pole at $s = k/2$ which contributes $\log N$ to the main term (see [1], Proposition 4.2 for the details). ■

Lemma 8. Let $\Gamma$ be a circle with center $(k/2, t_0)$ and radius $0 < R_N < 1/2$, and let $w$ be a point on (or inside) $\Gamma$. Then if $\sigma = \text{Re}(w) \geq k/2$,

$$\sum_{f \in \mathcal{P}_N} \omega_f |L_f(w, \chi)|^2 \ll_{k,q,t_0} (\log N)^4$$

and if $\sigma = \text{Re}(w) \leq k/2$,

$$\sum_{f \in \mathcal{P}_N} \omega_f |L_f(w, \chi)|^2 \ll_{k,q,t_0} N^{k-2\sigma} (\log N)^4.$$
Proof. First we assume that $\sigma = \text{Re}(w) \geq k/2$. Choosing $x = q^2N \log N$ in Lemma 3 gives

$$\Gamma(w)L_f(w, \chi) = \sum_{n \geq 1} \frac{\chi(n)a_f(n)}{nw} W\left(w, \frac{2\pi n}{q^2N \log N}\right) + O_{q,k}(N^{-6+k/2-\sigma}(\log N)^{k-\sigma+1}).$$

Now by applying the upper bound of Lemma 1 for $a_f(n)$ and the upper bound of Lemma 4 for $W(w, \cdot)$, we deduce that

$$\sum_{n > q^2N(\log N)^2} \frac{\chi(n)a_f(n)}{nw} W\left(w, \frac{2\pi n}{q^2N \log N}\right) = O_{q,k}(N^{-5+k/2-\sigma}(\log N)^{k-\sigma}).$$

Therefore

$$\Gamma(w)L_f(w, \chi) = \sum_{n \leq q^2N(\log N)^2} \frac{\chi(n)}{nw} W\left(w, \frac{2\pi n}{q^2N \log N}\right) a_f(n) + O_{q,k}(N^{-5}(\log N)^{k/2}).$$

We know that for complex numbers $c_n$,

$$\left| \sum_{f \in \mathcal{P}_N} \omega_f \left( \sum_{n \leq X} c_n a_f(n) \right)^2 \right| = (1 + O(N^{-1}X \log X)) \sum_{n \leq X} n^{k-1}|c_n|^2$$

with an absolute implied constant (see [5], Theorem 1). Applying this identity for

$$X = Nq^2(\log N)^2, \quad c_n = \frac{\chi(n)}{nw} W\left(w, \frac{2\pi n}{q^2N \log N}\right),$$

and using Lemma 4 imply that

$$\sum_{f \in \mathcal{P}_N} \omega_f \left( \sum_{n \leq q^2N(\log N)^2} c_n a_f(n) \right)^2 = O_{q,k}\left( (\log N)^3 \sum_{n \leq q^2N(\log N)^2} \frac{1}{n^{2\sigma-k+1}} \right) = O_{q,k}(\log N)^4.$$

This together with (15) proves the lemma.

If $\sigma = \text{Re}(w) < k/2$ the assertion results from the functional equation of $|L_f(w, \chi)|^2$. 

Proposition 2. Let $N$ be prime, and let $\Gamma$ and $C_N$ be the circles with center $(k/2, t_0)$ and radius $R_N$ and $r_N$ respectively. Suppose that $0 < r_N < R_N < 1/2$ and

$$\frac{r_NN^{2R_N}}{R_N} = o\left( \frac{1}{(\log N)^3} \right).$$
Then
\[
\sum_{f \in P_N} \omega_f |L_f(s_f, \chi)|^2 = \prod_{p | q} \left(1 - \frac{1}{p}\right) \log N + c_1
\]
\[+ O_{t_0, q, k}(N^{-1/2} \log N) + O_{t_0, q, k} \left(\frac{r_N N^{2R_N} (\log N)^4}{R_N - r_N}\right)\]
where \(s_f\) is an arbitrary point in \(C_N\) and \(c_1\) depends on \(t_0\), \(q\) and \(k\).

**Proof.** From the proof of Proposition 2, we know that
\[
\sum_{f \in P_N} \omega_f |L_f(s_f, \chi)|^2 = \sum_{f \in P_N} \omega |L_f(k/2 + it, \chi)|^2
\]
\[+ \sum_{f \in P_N} \omega_f \left| \frac{1}{2\pi i} \int_I L_f^2(w, \chi) \frac{s_f - s_0}{(w - s_f)(w - s_0)} \, dw \right|.
\]
The result follows by applying Lemma 7’ in the above identity and the fact that by Lemma 8,
\[
\sum_{f \in P_N} \omega_f \left| \frac{1}{2\pi i} \int_I L_f^2(w, \chi) \frac{s_f - s_0}{(w - s_f)(w - s_0)} \, dw \right| \leq \frac{r_N}{R_N - r_N} O_{t_0, q, k}(N^{2R_N} (\log N)^4). \]

Now in Propositions 1’ and 2’, let \(R_N = 1/\log N\) and \(r_N = 1/(\log N)^{4+\varepsilon}\).
We then proceed in a way similar to the proof of Theorem 1 and finally Theorem 2 follows by applying Propositions 1’ and 2’ in (14).

**References**


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