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Non-vanishing of modular L-functions on a disc

by

AMIR AKBARY (Montréal)

Introduction. Non-vanishing of L-functions on a disc has been studied in various contexts in the recent years. In the context of Dirichlet L-functions P. Elliott [6] proved that there are infinitely many Dirichlet L-functions $L(s,\chi_p)$ (χ_p is a Dirichlet character mod p (prime)) which are uniformly bounded below by $c(\log p)^{1/2}$ in the disc $|s-1/2| \le (\log p)^{-(1+\varepsilon)}$, and so do not vanish there. This result has been improved by R. Balasubramanian in [2]. He proved that the number of Dirichlet L-functions $L(s,\chi_p)$ that do not vanish in the disc $|s-1/2| \le (\log p)^{-(1+\varepsilon)}$ is bounded below by $cp(\log p)^{-2}$. Also, in [3] R. Balasubramanian and K. Murty studied non-vanishing of Dirichlet L-functions in the disc $|s-\sigma_j| \le 2(\log p)^{-1}$, where $\sigma_j = 1/2 + j/\log p$ and $2 \le j \le (\log p)/2 - 2$. They proved that for a positive proportion of the characters $\chi_p \pmod p$, $L(s,\chi_p)$ does not have a real zero in the region $1/2 + c/\log p \le \mathrm{Re}(s) < 1$. Here, $c > 0$ is an absolute constant and p is a sufficiently large prime.

In this paper we prove an analogue of the above results in the context of modular L-functions. We are interested in the zeros of $L_f(s,\chi)$ in the critical strip $k/2 < \mathrm{Re}(s) < (k+1)/2$, where $L_f(s,\chi)$ is the twisted L-function associated with the newform f and Dirichlet character χ. Generalized Riemann Hypothesis predicts that $L_f(s,\chi)$ is non-zero in this strip. One of the known results in the subject is given by K. Murty and T. Stefanicki [7]. They proved that at least $Y^{2/3-\varepsilon}$ quadratic twists $L_f(s,\chi_d)$ ($|d| \le Y$, $d \equiv 1 \pmod 4$) attached to holomorphic newforms and $Y^{2/3-\varepsilon}$ attached to Maass newforms do not vanish inside the disc $|s-s_0| < (\log Y)^{-(1+\varepsilon)}$ for any $\varepsilon > 0$ and any point s_0 inside the critical strip (the exponent $2/3$ can in fact be improved now to 1 using improved character sum estimates of Heath-Brown as in the work of Perelli and Pomykała [8]).

Here, we prove the following theorem.

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THEOREM 1. *Let $s_0 = \sigma_0 + it_0$ be a point in the strip $k/2 < \mathrm{Re}(s) < (k+1)/2$ and let C_N be the disc with center s_0 and radius $r_N = o(1)$ (i.e. $r_N \to 0$ as $N \to \infty$). Suppose that χ is a fixed primitive Dirichlet character mod q such that $(q, N) = 1$. Then there are positive constants $C_{\sigma_0,k}$ (depending only on k and σ_0) and C_{s_0,q,k,r_N} (depending on q, k, s_0 and r_N) such that for prime $N > C_{s_0,q,k,r_N}$ there exist at least $C_{\sigma_0,k}N(\log N)^{-1}$ newforms f of weight k and level N for which $L_f(s,\chi) \neq 0$ for all $s \in C_N$.*

The methodology of the proof is based on a comparison of mean values. In Sections 3 and 4, we derive asymptotic formulae for $L_f(s_f,\chi)$ and $|L_f(s_f,\chi)|^2$ on average, where s_f is an arbitrary point in the disc C_N. To do this first we derive the asymptotic formulae for a fixed point s_0 in the critical strip (Lemmas 5 and 7). These are analogues of the results given by W. Duke [4] for the center of critical strip. Then an application of Cauchy's integral formula gives us the asymptotic formulae on a disc (Propositions 1 and 2). This technique has already been applied by P. Elliott, B. Balasubramanian and B. Balasubramanian–K. Murty for Dirichlet L-functions. Finally we have to deal with the contribution of oldforms; we apply the technique developed by the author in [1] to overcome this difficulty. In Section 5 we finish off the proof of Theorem 1 by an application of the Cauchy–Schwarz inequality.

Finally, with a slight modification of our previous results, we establish asymptotic formulae for $L_f(s_f,\chi)$ and $|L_f(s_f,\chi)|^2$ on average, where s_f is an arbitrary point in the disc C_N with center on the critical line $s = k/2+it$, and as a result we prove the following non-vanishing theorem.

THEOREM 2. *Let $s_0 = k/2 + it_0$ and let C_N be the disc with center s_0 and radius $r_N = 1/(\log N)^{4+\varepsilon}$ ($\varepsilon > 0$). Suppose that χ is a fixed primitive Dirichlet character mod q such that $(q, N) = 1$. Then there are positive constants C_k (depending only on k) and $C_{t_0,q,k,\varepsilon}$ (depending on q, k, t_0 and ε) such that for prime $N > C_{t_0,q,k,\varepsilon}$ there exists at least $C_kN(\log N)^{-2}$ newforms f of weight k and level N for which $L_f(s,\chi) \neq 0$ for all $s \in C_N$.*

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2. Preliminaries. In this section we review some basic facts concerning modular forms and set up our notation.

Let $S_k(N)$ be the space of cusp forms of weight k for $\Gamma_0(N)$ with trivial character. The space $S_k(N)$ has an inner product (*Petersson inner product*)

$$\langle f, g\rangle = \int\limits_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{g(z)}y^k\,\frac{dx\,dy}{y^2}$$

where $\mathcal{H}$ denotes the upper half-plane. For any $f \in S_k(N)$ let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz), \qquad e(z) = e^{2\pi i z},$$

be the Fourier expansion of f at $i\infty$.

Let χ be a primitive Dirichlet character mod q with $(q, N) = 1$. Then the *twisted L-function* associated with f and χ is defined by

$$L_f(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n) a_f(n)}{n^s}.$$

The twisted L-function is given by an absolutely convergent series on the half-plane $\mathrm{Re}(s) > (k+1)/2$ and it has an analytic continuation to the whole plane. Moreover, if f is a newform (in Atkin–Lehner sense), then $L_f(s, \chi)$ has an Euler product valid on $\mathrm{Re}(s) > (k+1)/2$ and it satisfies the following functional equation:

$$(1) \qquad \left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_f(s, \chi) = \varepsilon_\chi \left(\frac{q\sqrt{N}}{2\pi}\right)^{k-s} \Gamma(k-s) L_f(k-s, \overline{\chi})$$

where $\varepsilon_\chi = \varepsilon_f \chi(N) \tau(\chi)^2 q^{-1}$ with $\varepsilon_f = \pm 1$ (the root number of f) which depends only on f and $\tau(\chi)$ is the Gauss sum.

Let $\{T_p\ (p \nmid N),\ U_q\ (q \mid N)\}$ be the collection of the classical Hecke operators and let $W_q\ (q \mid N)$ be the "W operator" of Atkin and Lehner. In 1983 A. Pizer introduced the operators C_q on $S_k(N)$ for $q \mid N$, such that the action of C_q on the new part of $S_k(N)$ is the same as the action of the classical U_q operators. More precisely he defined C_q as

$$C_q = \begin{cases} U_q + W_q U_q W_q + q^{k/2-1} W_q & \text{if } q \parallel N, \\ U_q + W_q U_q W_q & \text{if } q^2 \mid N. \end{cases}$$

Then he showed that $T_p\ (p \nmid N)$, $C_q\ (q \mid N)$ form a commuting family of Hermitian operators. Using this, he proved ([9], Theorem 3.10) the following result:

THEOREM. *There exists a basis $f_i(z)$ $(1 \le i \le \dim S_k(N))$ of $S_k(N)$ such that each $f_i(z)$ is an eigenform for all the T_p and C_q operators with $p \nmid N$ and $q \mid N$. Let $f(z) = \sum_{n=1}^{\infty} a_f(n) e(z)$ be an element of this basis. Then $a_f(1) \ne 0$ and assuming $f(z)$ is normalized so that $a_f(1) = 1$, we have $f \mid T_p = a_f(p) f$ for all $p \nmid N$, $f \mid C_q = a_f(q) f$ for all $q \mid N$, and $a_f(nm) = a_f(n) a_f(m)$ whenever $(n, m) = 1$. Furthermore $f(z)$ is an eigenform for all W_q operators, $q \mid N$. Finally, if $g(z) \in S_k(N)$ is an eigenform for all the T_p and C_q operators with $p \nmid N$ and $q \mid N$, then $g(z) = c f_i(z)$ for some $c \in \mathbb{C}^*$ and some unique i, $1 \le i \le \dim S_k(N)$.*

Now let $\mathcal{F}_N$ be the set of all normalized ($a_f(1) = 1$) newforms in $S_k(N)$ and let $\mathcal{P}_N$ be the basis of $S_k(N)$ given by the above theorem. The elements of $\mathcal{P}_N$ form an orthogonal basis (with respect to the Petersson inner product) for $S_k(N)$, any $f \in \mathcal{P}_N$ has real Fourier coefficient and $L_f(s,\chi)$ satisfies the functional equations (1). Moreover, we can show that the action of C_q on $S_k(N)^{\mathrm{new}}$ is the same as the action of U_q (see [9], Remark 2.9). This shows that $\mathcal{F}_N \subset \mathcal{P}_N$.

For the Fourier coefficients of a newform f we have the Deligne bound

$$|a_f(n)| \le \mathbf{d}(n) n^{(k-1)/2}$$

where $\mathbf{d}(n)$ is the divisor function. For N prime, we have the following estimation of the Fourier coefficients of $f \in \mathcal{P}_N$.

LEMMA 1. *Suppose N is prime and $f \in \mathcal{P}_N$. Then*

$$|a_f(n)| \le c_0 n^{k/2}$$

where c_0 is an absolute constant independent of f.

Proof. Propositions 3.6 and 3.4 of [9] imply that if $f \in \mathcal{P}_N - \mathcal{F}_N$, then

$$f(z) = h(z) \pm N^{k/2} h(Nz)$$

where h is the normalized newform of weight k and level 1 associated with f. Now the result follows from the Deligne bound for the newforms (see [1], Lemma 2.2, for the details). ∎

Finally, since $\mathcal{P}_N$ forms an orthogonal basis of $S_k(N)$, the Fourier coefficients of its elements are semi-orthogonal in the following sense:

LEMMA 2. *Let $\omega_f = \Gamma(k-1)/((4\pi)^{k-1}\langle f, f\rangle)$ and let $\delta_{m,n}$ be the Kronecker delta. For m and n positive integers we have the inequality*

$$\Big|\sum_{f\in\mathcal{P}_N} \omega_f \frac{a_f(m)}{\sqrt{m^{k-1}}}\cdot\frac{a_f(n)}{\sqrt{n^{k-1}}} - \delta_{m,n}\Big| \le M\mathbf{d}(N)N^{1/2-k}(m,n)^{1/2}\sqrt{(mn)^{k-1}}$$

where M is a constant depending only on k and $\mathbf{d}(N)$ is the number of divisors of N.

Proof. See [4], Lemma 1. ∎

3. Mean estimation. In this section we will find an asymptotic formula for

$$\sum_{f\in\mathcal{P}_N} \omega_f L_f(s_f,\chi)$$

where s_f is a variable point in the disc with center $s_0 = \sigma_0 + it_0$ ($k/2 < \sigma_0 < (k+1)/2$) and radius $r_N = o(1)$.

LEMMA 3. *For any $x > 0$ and $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ where $(k-1)/2 \le \sigma_0 \le (k+1)/2$, let*

$$W(s_0, x) = \frac{1}{2\pi i} \int\limits_{(5/4)} \Gamma(s+s_0) x^{-s} \, \frac{ds}{s}$$

and

$$\mathcal{A}_{f,\chi}(x, s_0) = \sum_{n\ge 1} \chi(n) a_f(n) n^{-s_0} W(s_0, 2\pi n/x)$$

where χ is a fixed primitive Dirichlet character mod q with $(q, N) = 1$. Then

$$\Gamma(s_0) L_f(s_0, \chi) = \mathcal{A}_{f,\chi}(x, s_0) + \varepsilon_\chi \left(\frac{q\sqrt{N}}{2\pi}\right)^{k-2s_0} \mathcal{A}_{f,\overline{\chi}}\left(\frac{q^2 N}{x}, k - s_0\right)$$

where ε_χ is the root number of $L_f(s, \chi)$.

Proof. From the definition of $W(s_0, x)$ it is clear that

$$\mathcal{A}_{f,\chi}(x, s_0) = \frac{1}{2\pi i} \int\limits_{(5/4)} L_f(s+s_0, \chi) \left(\frac{x}{2\pi}\right)^s \Gamma(s+s_0) \, \frac{ds}{s}.$$

Changing the line of integration from $5/4$ to $-5/4$ and using the functional equation (1) yields

$$\begin{aligned}\mathcal{A}_{f,\chi}(x, s_0) &= \Gamma(s_0) L_f(s_0, \chi) \\ &+ \varepsilon_\chi \left(\frac{q\sqrt{N}}{2\pi}\right)^{k-2s_0} \frac{1}{2\pi i} \int\limits_{(-5/4)} L_f(k-s-s_0, \overline{\chi}) \left(\frac{2\pi x}{q^2 N}\right)^s \Gamma(k-s-s_0) \, \frac{ds}{s}.\end{aligned}$$

Now changing variables $s \mapsto -s$ implies the result. ∎

LEMMA 4. *Under the assumptions of Lemma 3,*

$$\begin{aligned} W(s_0, x) &\ll x^{\sigma_0 - 1} e^{-x} && \text{as } x \to \infty, \\ W(s_0, x) &\ll_k 1 && \text{as } x \to 0. \end{aligned}$$

Proof. We have

$$W(s_0, x) = \frac{1}{2\pi i} \int\limits_{(5/4)} \Big(\int\limits_0^\infty e^{-t} t^{s+s_0-1} \, dt\Big) x^{-s} \, \frac{ds}{s} = \int\limits_x^\infty t^{s_0-1} e^{-t} \, dt.$$

Therefore

$$|W(s_0, x)| = \Big|\int\limits_x^\infty t^{s_0-1} e^{-t} \, dt\Big| \le \int\limits_x^\infty t^{\sigma_0-1} e^{-t} \, dt.$$

Now the first result follows from the estimation of the last integral using integration by parts. The second result is clear since $|W(s_0, x)| \le \Gamma(\sigma_0)$ as $x \to 0$. ∎

Lemma 5. *Let χ be a fixed primitive Dirichlet character mod q with $(q,N)=1$ and let $s_0=\sigma_0+it_0$ be a point in the strip $(k-1)/2<\mathrm{Re}(s)\le (k+1)/2$. Then*

$$\sum_{f\in\mathcal{P}_N}\omega_f L_f(s_0,\chi)=1+O\bigg(\frac{1}{|\Gamma(s_0)|}N^{1/2-\sigma_0}(\log N)^{k-\sigma_0}\bigg) + O\bigg(\frac{1}{|\Gamma(s_0)|}N^{(k-12)/2-\sigma_0}(\log N)^{k-\sigma_0-1}\bigg)$$

for N prime. The implied constant depends only on q and k.

Proof. Choosing $x=q^2N\log N$ in Lemma 3 gives

$$\mathcal{A}_{f,\overline{\chi}}\bigg(\frac{Nq^2}{x},k-s_0\bigg)=\sum_{n\ge1}\overline{\chi}(n)a_f(n)n^{s_0-k}W(k-s_0,2\pi n\log N).$$

Using Lemmas 4 and 1 we have

$$\begin{aligned}\bigg|\mathcal{A}_{f,\overline{\chi}}\bigg(\frac{1}{\log N},k-s_0\bigg)\bigg| &\le \sum_{n\ge1}|a_f(n)|n^{\sigma_0-k}|W(k-s_0,2\pi n\log N)|\\ &\le \sum_{n\ge1}c_0n^{k/2}n^{\sigma_0-k}(2\pi n\log N)^{k-\sigma_0-1}e^{-2\pi n\log N}\\ &= c_0(2\pi\log N)^{k-\sigma_0-1}\sum_{n\ge1}\frac{n^{k/2-1}}{(N^{2\pi})^n}.\end{aligned}$$

Therefore from Lemma 3 we get

$$\begin{aligned}\Gamma(s_0)\sum_{f\in\mathcal{P}_N}\omega_f L_f(s_0,\chi)&=\sum_{f\in\mathcal{P}_N}\omega_f\mathcal{A}_{f,\chi}(x,s_0)\\ &\quad+\Big(\sum_{f\in\mathcal{P}_N}\omega_f\Big)O_{q,k}(N^{-6+k/2-\sigma_0}(\log N)^{k-\sigma_0-1}).\end{aligned}$$

From this, we have

$$\begin{aligned}&\Gamma(s_0)\sum_{f\in\mathcal{P}_N}\omega_f L_f(s_0,\chi)-\Gamma(s_0)\\ &\quad=\sum_{n\ge1}\chi(n)\bigg(\sum_{f\in\mathcal{P}_N}\omega_f\frac{a_f(n)}{n^{(k-1)/2}}-\delta_{1,n}\bigg)W\bigg(s_0,\frac{2\pi n}{q^2N\log N}\bigg)n^{(k-1)/2-s_0}\\ &\qquad+W\bigg(s_0,\frac{2\pi}{q^2N\log N}\bigg)-\Gamma(s_0)\\ &\qquad+\Big(\sum_{f\in\mathcal{P}_N}\omega_f\Big)O_{q,k}(N^{-6+k/2-\sigma_0}(\log N)^{k-\sigma_0-1}).\end{aligned}$$

Note that

$$W\left(s_0, \frac{2\pi}{q^2 N\log N}\right) - \Gamma(s_0) = \int\limits_0^{2\pi/(q^2N\log N)} t^{s_0-1}e^{-t}\,dt$$
$$= O_{q,k}((N\log N)^{-\sigma_0}).$$

Also, from Lemma 2 for $m = n = 1$ it follows that

$$\sum_{f\in\mathcal{P}_N} \omega_f = 1 + O(N^{1/2-k}).$$

By applying $m = 1$ in Lemma 2 and using the above identities, we have

$$\Big|\Gamma(s_0)\Big(\sum_{f\in\mathcal{P}_N} \omega_f L_f(s_0,\chi) - 1\Big)\Big|$$
$$\le M_1 \frac{N^{1/2-k}}{(N\log N)^{\sigma_0-1}} \sum_{n\ge 1} n^{k-2} e^{-2\pi n/(q^2 N\log N)} + M_2 (N\log N)^{-\sigma_0}$$
$$+ M_3 N^{-6+k/2-\sigma_0} (\log N)^{k-\sigma_0-1}$$

where M_1, M_2 and M_3 are constants depending on q and k. This proves the desired result. ∎

Proposition 1. *Let $s_0 = \sigma_0 + it_0$ be a point in the strip $k/2 < \mathrm{Re}(s) < (k+1)/2$ and let Γ and C_N be the circles with center (σ_0, t_0) and radius $R = \frac{1}{2}\min\{(k+1)/2 - \sigma_0, \sigma_0 - k/2\}$ and $r_N = o(1)$ respectively. Then for N prime*

$$\sum_{f\in\mathcal{P}_N} \omega_f L_f(s_f,\chi) = 1 + O_{q,k}\left(\frac{1}{|\Gamma(s_0)|} N^{-1/2}\right) + O_{q,k,s_0}\left(\frac{r_N}{R - r_N} N^{-1/2}\right)$$

where s_f is an arbitrary point in C_N.

Proof. By Cauchy's integral formula for any $s_f \in C_N$, we have

$$L_f(s_f,\chi) - L_f(s_0,\chi) = \frac{1}{2\pi i}\int\limits_{\Gamma} L_f(w,\chi)\left(\frac{1}{w - s_f} - \frac{1}{w - s_0}\right) dw$$

where Γ is traversed in the counter clockwise direction. Therefore

$$(2)\quad \sum_{f\in\mathcal{P}_N} \omega_f L_f(s_f,\chi) = \sum_{f\in\mathcal{P}_N} \omega_f L_f(s_0,\chi)$$
$$+ \frac{1}{2\pi i}\int\limits_{\Gamma}\Big(\sum_{f\in\mathcal{P}_N} \omega_f L_f(w,\chi)\Big)\frac{s_f - s_0}{(w - s_f)(w - s_0)}\,dw.$$

Now using Lemma 5 yields

$$(3)\qquad \left|\frac{1}{2\pi i}\int_{\Gamma}\Big(\sum_{f\in\mathcal{P}_N}\omega_f L_f(w,\chi)\Big)\frac{s_f-s_0}{(w-s_f)(w-s_0)}\,dw\right| \le \frac{r_N}{R-r_N}O_{q,k,s_0}(N^{-1/2}).$$

Note that here we used the fact that

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{s_f-s_0}{(w-s_f)(w-s_0)}\,dw=0.$$

Applying (3) and Lemma 5 in (2) completes the proof. ∎

4. Mean square estimation. In this section we are going to find an asymptotic formula for the average values of $|L_f(s_f,\chi)|^2$ where s_f is a variable point in a disc with center $s_0=\sigma_0+it_0$ $(k/2<\sigma_0<(k+1)/2)$ and radius $r_N=o(1)$. We start with writing $|L_f(s_0,\chi)|^2$ as a sum of two convergent series.

Let $|L_f(s_0,\chi)|^2=\sum_{l\ge1}b_f(l)l^{-\sigma_0}$ so that

$$(4)\qquad b_f(l)=\sum_{mn=l}\chi(n)\overline{\chi}(m)a_f(n)a_f(m)\Big(\frac{m}{n}\Big)^{it_0}.$$

For $x>0$ and $s_0=\sigma_0+it_0\in\mathbb{C}$ where $(k-1)/2\le\sigma_0\le(k+1)/2$, define

$$(5)\qquad \mathcal{B}_f(x,s_0)=\sum_{l\ge1}\frac{b_f(l)}{l^{\sigma_0}}Z(s_0,l/x)$$

where

$$(6)\qquad Z(s_0,x)=\frac{1}{2\pi i}\int_{(5/4)}(2\pi)^{-2s}\Gamma(s+s_0)\Gamma(s+\overline{s}_0)x^{-s}\,\frac{ds}{s}.$$

Using Deligne's bound in (4) and standard estimates for $Z(s_0,x)$ shows that (5) is absolutely convergent.

Lemma 6. *Let $f\in\mathcal{P}_N$ and suppose that χ is a primitive Dirichlet character mod q with $(q,N)=1$. For any $x>0$ we have*

$$|\Gamma(s_0)L_f(s_0,\chi)|^2=\mathcal{B}_f(x,s_0)+\Big(\frac{q^2N}{4\pi^2}\Big)^{k-2\sigma_0}\mathcal{B}_f\Big(\frac{(q^2N)^2}{x},k-\overline{s}_0\Big).$$

Proof. From (6) we have

$$\mathcal{B}_f(x,s_0)=\frac{1}{2\pi i}\int_{(5/4)}(2\pi)^{-2s}\Gamma(s+s_0)\Gamma(s+\overline{s}_0)L_f(s+s_0,\chi)L_f(s+\overline{s}_0,\overline{\chi})x^{s}\,\frac{ds}{s}.$$

By changing the line of integration from $5/4$ to $-5/4$ and using the functional equation (1) we get

$$\mathcal{B}_f(x,s_0) = |\Gamma(s_0)L_f(s_0,\chi)|^2 + \left(\frac{q^2N}{4\pi^2}\right)^{k-2\sigma_0} \int_{(-5/4)} (2\pi)^{2s}\Gamma(k-s-s_0)\Gamma(k-s-\bar{s}_0) \times L_f(k-s-s_0,\overline{\chi})L_f(k-s-\bar{s}_0,\chi)\left(\frac{x}{(q^2N)^2}\right)^s \frac{ds}{s}.$$

Now changing variables $s \mapsto -s$ yields the result. ∎

We estimate $\mathcal{B}_f(x,s_0)$ on average. From (4) and (5) it follows that

$$(7)\qquad \sum_{f\in\mathcal{P}_N} \omega_f \mathcal{B}_f(x,s_0) = \sum_{f\in\mathcal{P}_N} \omega_f \sum_{l\ge 1} b_f(l) l^{-\sigma_0} Z(s_0,l/x)$$
$$= \sum_{m,n\ge 1} \chi(n)\overline{\chi}(m) Z(s_0,mn/x) \frac{\left(\frac{m}{n}\right)^{it_0}}{(mn)^{\sigma_0-(k-1)/2}} \times \sum_{f\in\mathcal{P}_N} \omega_f \frac{a_f(m)}{\sqrt{m^{k-1}}}\cdot\frac{a_f(n)}{\sqrt{n^{k-1}}}$$
$$= \sum_{n\ge 1} |\chi(n)|^2 Z(s_0,n^2/x)\frac{1}{n^{2\sigma_0-k+1}} + R$$

where

$$(8)\qquad R \ll N^{1/2-k} \sum_{m,n\ge 1} Z(\sigma_0,mn/x)(m,n)^{1/2}(mn)^{-\sigma_0+k-1}.$$

Note that here we are using the inequality $|Z(s_0,x)| \le Z(\sigma_0,x)$. This is true since by writing Γ functions in terms of integrals in (6) and interchanging the order of integration, we have

$$Z(s_0,x) = \int_0^\infty t_1^{s_0-1}e^{-t_1}\Big(\int_{4\pi^2x/t_1}^\infty e^{-t_2}t_2^{\bar{s}_0-1}\,dt_2\Big)\,dt_1.$$

Applying the triangle inequality in the above identity implies the desired inequality.

Using the definition of $Z(s_0,x)$, the first term in (7) is equal to

$$\frac{1}{2\pi i}\int_{(5/4)} L(2s+2\sigma_0-k+1,\chi_0)(2\pi)^{-2s}\Gamma(s+s_0)\Gamma(s+\bar{s}_0)x^s\,\frac{ds}{s}$$

where χ_0 is the principal character mod q and $L(s,\chi_0) = \zeta(s)\prod_{p|q}(1-1/p^s)$. Now we assume that $\sigma_0 \ne k/2$, since the integrand has simple poles at $s = 0$

and $s = k/2 - \sigma_0$, by moving the line of integration from $5/4$ to $-1/2$, the integral is equal to

$$(9)\qquad |\Gamma(s_0)|^2 \prod_{p|q}\left(1 - \frac{1}{p^{2\sigma_0-k+1}}\right)\zeta(2\sigma_0 - k + 1)$$
$$+ \frac{\prod_{p|q}(1-1/p)(2\pi)^{2\sigma_0-k}}{k-2\sigma_0}\Gamma(k/2+it_0)\Gamma(k/2-it_0)x^{k/2-\sigma_0} + O_{\sigma_0,q,k}(x^{-1/2}).$$

Now in (7) we estimate the remainder term R. We calculate

$$\sum_{m,n\geq 1} Z(\sigma_0, mn/x)(m,n)^{1/2}(mn)^{-\sigma_0+k-1}.$$

It is

$$\frac{1}{2\pi i}\int_{((k+1)/2)}(2\pi)^{-2s}(\Gamma(s+\sigma_0))^2 x^s\Big(\sum_{m,n\geq 1}(m,n)^{1/2}(mn)^{-(s+\sigma_0-k+1)}\Big)\frac{ds}{s}.$$

Note that since the integrand does not have any pole in the strip $5/4 < \mathrm{Re}(s) < (k+1)/2$, we can move the line of integration from $5/4$ to $(k+1)/2$. From [4], Lemma 4, we know that

$$\sum_{m,n\geq 1}(m,n)^{1/2}(mn)^{-(s+\sigma_0-k+1)} = \frac{\zeta(2s+2\sigma_0-2k+3/2)\zeta(s+\sigma_0-k+1)^2}{\zeta(2s+2s_0-2k+2)}.$$

Applying this identity to the above integral and moving the line of integration from $(k+1)/2$ to $k-\sigma_0-\varepsilon$ ($\varepsilon > 0$) yields

$$(10)\qquad \sum_{m,n\geq 1} Z(\sigma_0, mn/x)(m,n)^{1/2}(mn)^{-(s+\sigma_0-k+1)} \sim C_{\sigma_0,k}x^{k-\sigma_0}\log x$$

and by (8), $R \ll N^{1/2-k}x^{k-\sigma_0}\log x$. Therefore we have

$$(11)\qquad \sum_{f\in\mathcal{P}_N}\omega_f\mathcal{B}_f(x,s_0)$$
$$= |\Gamma(s_0)|^2\prod_{p|q}\left(1-\frac{1}{p^{2\sigma_0-k+1}}\right)\zeta(2\sigma_0-k+1)$$
$$+ \frac{\prod_{p|q}(1-1/p)(2\pi)^{2\sigma_0-k}}{k-2\sigma_0}\Gamma(k/2+it_0)\Gamma(k/2-it_0)x^{k/2-\sigma_0}$$
$$+ O_{\sigma_0,q,k}(x^{-1/2}) + O_{\sigma_0,k}(N^{1/2-k}x^{k-\sigma_0}\log x).$$

LEMMA 7. *Let χ be a fixed primitive Dirichlet character mod q with $(q,N)=1$ and let $s_0=\sigma_0+it_0$ where $k/2<\sigma_0\le(k+1)/2$. Then*

$$\sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_0,\chi)|^2 = \prod_{p\mid q}\left(1-\frac{1}{p^{2\sigma_0-k+1}}\right)\zeta(2\sigma_0-k+1)+c_1N^{k/2-\sigma_0}+O_{s_0,q,k}(N^{-1/2})$$

for N prime. Here, c_1 depends on s_0, q and k.

Proof. Choosing $x=q^2N$ in Lemma 6 and applying (11) in it, proves the lemma. ∎

PROPOSITION 2. *Under the assumptions of Proposition* 1,

$$\begin{aligned}\sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_f,\chi)|^2 = \prod_{p\mid q}\left(1-\frac{1}{p^{2\sigma_0-k+1}}\right)&\zeta(2\sigma_0-k+1)+c_1N^{k/2-\sigma_0}\\ &+O_{s_0,q,k}(N^{-1/2})+O_{\sigma_0,k}\left(\frac{r_N}{R-r_N}\right)\\ &+O_{s_0,q,k}\left(\frac{r_N}{R-r_N}N^{k/2-\sigma_0+R}\right).\end{aligned}$$

Here, c_1 depends on s_0, q and k.

Proof. We have

$$\begin{aligned}\Big|\sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_f,\chi)|^2-\sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_0,\chi)|^2\Big| &\le \sum_{f\in\mathcal{P}_N}\omega_f\big||L_f(s_f,\chi)|^2-|L_f(s_0,\chi)|^2\big|\\ &\le \sum_{f\in\mathcal{P}_N}\omega_f|L_f^2(s_f,\chi)-L_f^2(s_0,\chi)|.\end{aligned}$$

By applying Cauchy's integral formula and Lemma 7, the last expression equals to

$$\begin{aligned}\sum_{f\in\mathcal{P}_N}\omega_f\left|\frac{1}{2\pi i}\int_{\Gamma}L_f^2(w,\chi)\frac{s_f-s_0}{(w-s_f)(w-s_0)}\,dw\right| \le \frac{r_N}{R-r_N}(O_{\sigma_0,k}(1)+O_{s_0,q,k}(N^{k/2-\sigma_0+R})).\end{aligned}$$

This shows that

$$(12)\qquad \sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_f,\chi)|^2 = \sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_0,\chi)|^2+\frac{r_N}{R-r_N}(O_{\sigma_0,k}(1)+O_{s_0,q,k}(N^{k/2-\sigma_0+R})).$$

Now applying Lemma 7 in (12) completes the proof. ■

5. Proof of Theorem 1. We need the following estimation of ω_f.

PROPOSITION 3. *For N prime we have*

$$(13)\qquad \omega_f \ll_k \begin{cases}(\log N)/N, & f\in\mathcal{F}_N,\\ 1/N, & f\in\mathcal{P}_N-\mathcal{F}_N.\end{cases}$$

Proof. See [4], Proposition 4, for the case $f\in\mathcal{F}_N$. If $f\in\mathcal{P}_N-\mathcal{F}_N$ then

$$f(z)=h(z)\pm N^{k/2}h(Nz)$$

as mentioned in the proof of Lemma 1. Now the result follows from the fact that

$$\langle f,f\rangle=\langle h(z)\pm N^{k/2}h(Nz),h(z)\pm N^{k/2}h(Nz)\rangle$$

is bounded below by a constant multiple of N (see [1], Proposition 5.3 for the details). ■

Now we can prove our theorem. Set

$$\mathcal{E}_N=\{f\in\mathcal{P}_N : L_f(s,\chi)\neq 0 \text{ for all } s \text{ in } C_N\}.$$

Proposition 1 shows that $\mathcal{E}_N\neq\emptyset$ for large N. Now if $f\in\mathcal{P}_N-\mathcal{E}_N$ we choose s_f such that $L_f(s_f,\chi)=0$. With this choice of s_f for elements of $\mathcal{P}_N-\mathcal{E}_N$ and arbitrary choice of s_f in C_N for elements of $\mathcal{E}_N$ and applying the Cauchy–Schwarz inequality and (13), we get

$$\begin{aligned}(14)\qquad \Big|\sum_{f\in\mathcal{P}_N}\omega_f L_f(s_f,\chi)\Big|^2 &= \Big|\sum_{f\in\mathcal{E}_N}\omega_f L_f(s_f,\chi)\Big|^2\\ &\le \Big(\sum_{f\in\mathcal{E}_N\cap\mathcal{F}_N}\omega_f+\sum_{f\in\mathcal{E}_N-\mathcal{F}_N}\omega_f\Big)\Big(\sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_f,\chi)|^2\Big)\\ &\ll \bigg(\sharp\{f\in\mathcal{F}_N : L_f(s,\chi)\neq 0 \text{ for all } s\in C_N\}\frac{\log N}{N}\\ &\qquad + 2\dim S_k(1)\frac{1}{N}\bigg)\sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_f,\chi)|^2.\end{aligned}$$

Theorem 1 follows by applying Propositions 1 and 2 in (14). ■

6. Proof of Theorem 2. We first establish the analogues of Proposition 1, Lemma 7 and Proposition 2 for a point s_0 on the critical line $\sigma = k/2$.

PROPOSITION 1′. *Let N be prime, and let Γ and C_N be the circles with center $(k/2, t_0)$ and radius R_N and r_N respectively. Suppose that $0 < r_N < R_N < 1/2$, and*

$$\frac{r_N}{R_N} N^{R_N} (\log N)^{R_N} = o\bigg(\frac{N^{1/2}}{\log N}\bigg).$$

Then

$$\sum_{f \in \mathcal{P}_N} \omega_f L_f(s_f, \chi) = 1 + O_{q,k}\bigg(\frac{1}{\Gamma(k/2 + t_0)} N^{-1/2} \log N\bigg) + O_{q,k,t_0}\bigg(\frac{r_N}{R_N - r_N} N^{R_N - 1/2} (\log N)^{R_N + 1}\bigg)$$

where s_f is an arbitrary point in C_N.

Proof. It is similar to the proof of Proposition 1. ∎

LEMMA 7′. *Let χ be a fixed primitive Dirichlet character mod q with $(q, N) = 1$ and let $s_0 = k/2 + it_0$. Then*

$$\sum_{f \in \mathcal{P}_N} \omega_f |L_f(k/2 + it_0, \chi)|^2 = \prod_{p \mid q} \bigg(1 - \frac{1}{p}\bigg) \log N + c_1 + O_{t_0,q,k}(N^{-1/2} \log N)$$

for N prime. Here, c_1 depends on t_0, q and k.

Proof. The proof is exactly similar to the proof of Lemma 7. The result follows by observing that

$$\frac{1}{2\pi i} \int\limits_{(5/4)} L(2s + 1, \chi_0)(2\pi)^{-2s} \Gamma(s + k/2 + it_0)\Gamma(s + k/2 - it_0) x^s \, \frac{ds}{s}$$

has a double pole at $s = k/2$ which contributes $\log N$ to the main term (see [1], Proposition 4.2 for the details). ∎

LEMMA 8. *Let Γ be a circle with center $(k/2, t_0)$ and radius $0 < R_N < 1/2$, and let w be a point on (or inside) Γ. Then if $\sigma = \mathrm{Re}(w) \ge k/2$,*

$$\sum_{f \in \mathcal{P}_N} \omega_f |L_f(w, \chi)|^2 \ll_{k,q,t_0} (\log N)^4$$

and if $\sigma = \mathrm{Re}(w) \le k/2$,

$$\sum_{f \in \mathcal{P}_N} \omega_f |L_f(w, \chi)|^2 \ll_{k,q,t_0} N^{k - 2\sigma} (\log N)^4.$$

Proof. First we assume that $\sigma = \mathrm{Re}(w) \geq k/2$. Choosing $x = q^2 N \log N$ in Lemma 3 gives

$$\Gamma(w) L_f(w,\chi) = \sum_{n\geq 1} \frac{\chi(n) a_f(n)}{n^w} W\biggl(w, \frac{2\pi n}{q^2 N \log N}\biggr) + O_{q,k}(N^{-6+k/2-\sigma}(\log N)^{k-\sigma+1}).$$

Now by applying the upper bound of Lemma 1 for $a_f(n)$ and the upper bound of Lemma 4 for $W(w,\cdot)$, we deduce that

$$\sum_{n>q^2 N(\log N)^2} \frac{\chi(n) a_f(n)}{n^w} W\biggl(w, \frac{2\pi n}{q^2 N \log N}\biggr) = O_{q,k}(N^{-5+k/2-\sigma}(\log N)^{k-\sigma}).$$

Therefore

$$\Gamma(w) L_f(w,\chi) = \sum_{n\leq q^2 N(\log N)^2} \frac{\chi(n)}{n^w} W\biggl(w, \frac{2\pi n}{q^2 N \log N}\biggr) a_f(n) + O_{q,k}(N^{-5}(\log N)^{k/2}). \tag{15}$$

We know that for complex numbers c_n,

$$\sum_{f\in\mathcal{P}_N} \omega_f \Bigl| \sum_{n\leq X} c_n a_f(n)\Bigr|^2 = (1 + O(N^{-1} X \log X)) \sum_{n\leq X} n^{k-1} |c_n|^2$$

with an absolute implied constant (see [5], Theorem 1). Applying this identity for

$$X = Nq^2(\log N)^2, \qquad c_n = \frac{\chi(n)}{n^w} W\biggl(w, \frac{2\pi n}{q^2 N \log N}\biggr),$$

and using Lemma 4 imply that

$$\sum_{f\in\mathcal{P}_N} \omega_f \Bigl| \sum_{n\leq q^2 N(\log N)^2} c_n a_f(n)\Bigr|^2 = O_{q,k}\biggl((\log N)^3 \sum_{n\leq q^2 N(\log N)^2} \frac{1}{n^{2\sigma-k+1}}\biggr) = O_{q,k}((\log N)^4).$$

This together with (15) proves the lemma.

If $\sigma = \mathrm{Re}(w) < k/2$ the assertion results from the functional equation of $|L_f(w,\chi)|^2$. ■

PROPOSITION 2′. *Let N be prime, and let Γ and C_N be the circles with center $(k/2, t_0)$ and radius R_N and r_N respectively. Suppose that $0 < r_N < R_N < 1/2$ and*

$$\frac{r_N N^{2R_N}}{R_N} = o\biggl(\frac{1}{(\log N)^3}\biggr).$$

Then

$$\sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_f,\chi)|^2=\prod_{p|q}\left(1-\frac{1}{p}\right)\log N+c_1 +O_{t_0,q,k}(N^{-1/2}\log N)+O_{t_0,q,k}\left(\frac{r_N N^{2R_N}(\log N)^4}{R_N-r_N}\right)$$

where s_f is an arbitrary point in C_N and c_1 depends on t_0, q and k.

Proof. From the proof of Proposition 2, we know that

$$\sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_f,\chi)|^2=\sum_{f\in\mathcal{P}_N}\omega|L_f(k/2+it,\chi)|^2 +\sum_{f\in\mathcal{P}_N}\omega_f\left|\frac{1}{2\pi i}\int_{\Gamma}L_f^2(w,\chi)\frac{s_f-s_0}{(w-s_f)(w-s_0)}\,d\omega\right|.$$

The result follows by applying Lemma 7′ in the above identity and the fact that by Lemma 8,

$$\sum_{f\in\mathcal{P}_N}\omega_f\left|\frac{1}{2\pi i}\int_{\Gamma}L_f^2(w,\chi)\frac{s_f-s_0}{(w-s_f)(w-s_0)}\,d\omega\right| \le\frac{r_N}{R_N-r_N}O_{t_0,q,k}(N^{2R_N}(\log N)^4).\ \blacksquare$$

Now in Propositions 1′ and 2′, let $R_N=1/\log N$ and $r_N=1/(\log N)^{4+\varepsilon}$. We then proceed in a way similar to the proof of Theorem 1 and finally Theorem 2 follows by applying Propositions 1′ and 2′ in (14). ■

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Department of Mathematics and Statistics
Concordia University
1455 de Maisonneuve Blvd. West
Montréal, Quebec, H3G 1M8
Canada
E–mail: akbary@cicma.concordia.ca

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