ACTA ARITHMETICA
XCII.4 (2000)

A Schinzel theorem on continued fractions in function fields

by

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1. Introduction and statement of result. It is well known that the expansion of the continued fraction of a positive quadratic irrational number α is periodical. The problem of evaluating and estimating the period $p(\alpha)$ of the continued fraction of α is closely related to many number theoretic problems such as Pell's equations and the fundamental unit of the real quadratic field $\mathbb{Q}(\alpha)$. In 1961, A. Schinzel [8] proved an interesting result. Let a, b, c be integers with $a \geq 1$, and let $f(x) = a^2x^2 + bx + c$ be a quadratic polynomial with discriminant $d = b^2 - 4a^2c$. We define two sets

$$\widetilde{\mathbb{Z}} = \{n \in \mathbb{Z} : n \geq 1 \text{ and } f(n) \text{ is square-free}\},$$
$$E = \{n \in \widetilde{\mathbb{Z}} : d \nmid (2a^2n + b)^2\}.$$

Schinzel's result says that

$$\text{(I)} \ \overline{\lim_{n \in \widetilde{\mathbb{Z}} - E}} \, p(\sqrt{f(n)}) < \infty, \qquad \text{(II)} \ \overline{\lim_{n \in E}} \, p(\sqrt{f(n)}) = \infty.$$

Later, S. Louboutin [6] and A. Farhane [2] presented more exact and effective lower bounds of $p(\sqrt{f(n)})$ for $n \in E$. In this paper we show an analogue of Schinzel's result in the function field case.

Let $\mathbb{F}_q$ be the finite field with q elements where q is odd, $k = \mathbb{F}_q(t)$ be the rational function field over $\mathbb{F}_q$, $R = \mathbb{F}_q[t]$ be its polynomial ring, $k_\infty = \mathbb{F}_q((1/t))$ be the completion of k at the infinite place $\infty = (1/t)$. Each element $\alpha \in k_\infty - k$ is a (formal) power series

$$\alpha = \sum_{n \geq l} c_n \left(\frac{1}{t}\right)^n, \qquad c_n \in \mathbb{F}_q \ (n \geq l), \qquad \operatorname{sgn} \alpha = c_l \in \mathbb{F}_q^*.$$

Let v_∞ be the normal exponential valuation in k_∞ with $v_\infty(1/t) = 1$. In this paper we prefer to use the notation $\deg \alpha = -v_\infty(\alpha) = -l$ since it is

2000 *Mathematics Subject Classification*: 11D41, 11R27, 11R58.
Key words and phrases: continued fraction, function field.

just the ordinary degree if α is a polynomial in R. We define

$$[\alpha] = \sum_{n=l}^{0} c_n \left(\frac{1}{t}\right)^n \in R, \qquad \{\alpha\} = \sum_{n\ge\sup\{1,l\}} c_n \left(\frac{1}{t}\right)^n.$$

Then we have the continued fraction expansion of α,

$$\alpha = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

where $a_0 = [\alpha]$ and $\alpha_i = 1/\{a_{i-1}\}$, $a_i = [\alpha_i]$ for $i \ge 1$ so that

$$\alpha_i = [a_i, a_{i+1}, \ldots], \qquad \deg \alpha_i = \deg a_i \ge 1 \qquad \text{for } i \ge 1.$$

Suppose that $F(t)$ is a square-free polynomial in R such that $\deg F(t) \ge 1$ and $\sqrt{F(t)} \in k_\infty$. (Note: $F(t)$ is called *square-free* if there is no $A \in R$ with $\deg A \ge 1$ such that $A^2 \mid F(t)$. By Hensel's lemma, $\sqrt{F(t)} \in k_\infty$ if and only if $2 \mid \deg F(t)$ and $\operatorname{sgn} F(t) \in \mathbb{F}_q^{*2}$.) The subfield $K = k(\sqrt{F(t)})$ of k_∞ is called a "*real*" *quadratic function field*. It is well known that the continued fraction of $\sqrt{F(t)}$ is periodical. In fact, we need a generalized version of the period in the study of the algebraic structure of the real quadratic function field K. For each $\alpha \in k_\infty$, α is algebraic over k with degree two if and only if the continued fraction of $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$ is infinite and quasi-periodical, which means that there exist $c \in \mathbb{F}_q^*$ and integers $n_0 \ge 0$ and $l \ge 1$ such that

$$a_{n+l} = c^{(-1)^n} a_n \qquad \text{for all } n \ge n_0.$$

The smallest positive integer l satisfying this condition is called the *quasi-period* of the continued fraction $\alpha = [a_0, a_1, \ldots]$ and denoted by $p(\alpha)$. If $\alpha = \sqrt{F(t)}$ and $l = p(\sqrt{F(t)})$, then the fundamental unit of the real quadratic function field $K = k(\sqrt{F(t)})$ is $P_{l-1} + Q_{l-1}\sqrt{F(t)}$ where

$$P_{l-1}/Q_{l-1} = [a_0, a_1, \ldots, a_{l-1}].$$

The continued fraction method of studying the ideal class number and units of real quadratic function fields was initiated by E. Artin [1]. For recent work we refer to D. Hayes [4] and C. D. González [3].

Now we state the result of this paper. Let A, B, C be polynomials in $R = \mathbb{F}_q[t]$, $\deg A \ge 0$, $f(x) = A^2x^2 + Bx + C$, $D = B^2 - 4A^2C$, and

$$\widetilde{R} = \{N \in R : f(N) \text{ is square-free and } \sqrt{f(N)} \in k_\infty\},$$
$$E = \{N \in \widetilde{R} : D \nmid (2A^2N + B)^2\}.$$

Let $p(\sqrt{f(N)})$ be the quasi-period of the continued fraction of $\sqrt{f(N)}$.

THEOREM 1.1.

$$\overline{\lim_{N\in\widetilde{R}-E}}\, p(\sqrt{f(N)}) < \infty.$$

THEOREM 1.2.

$$\lim_{N\in E} p(\sqrt{f(N)}) = \infty.$$

Before proving Theorems 1.1 and 1.2, in Section 2 we introduce several basic facts on continued fractions in function fields. Most results are just simple analogues of facts in the theory of ordinary continued fractions in the real number field case (see Hua's book [5], for instance), so we present the proof of some particular facts for the reader's convenience and omit the proof of others. In Sections 3 and 4 we prove Theorems 1.1 and 1.2.

2. Continued fractions in function fields. From now on we fix the following notations: $k = \mathbb{F}_q(t)$ $(2\nmid q)$, $R = \mathbb{F}_q[t]$, $k_\infty = \mathbb{F}((1/t))$. For $\alpha \in k_\infty - k$,

$$\alpha = c_m\left(\frac{1}{t}\right)^m + c_{m+1}\left(\frac{1}{t}\right)^{m+1} + \ldots, \qquad c_i \in \mathbb{F}_q\ (i \geq m),\ c_m \in \mathbb{F}_q^*;$$

we set $\deg\alpha = -m$, $\operatorname{sgn}\alpha = c_n$. We have the continued fraction

$$\alpha = [a_0, a_1, \ldots, a_n, \ldots]. \tag{2.1}$$

For $n \geq 0$, the nth convergent of the continued fraction (2.1) is

$$P_n/Q_n = [a_0, a_1, \ldots, a_n]$$

which can be calculated recursively by

$$\begin{array}{llll} P_0 = a_0, & P_1 = a_1a_0 + 1, & P_n = a_nP_{n-1} + P_{n-2} & (n \geq 2), \\ Q_0 = 1, & Q_1 = a_1, & Q_n = a_nQ_{n-1} + Q_{n-2} & (n \geq 2). \end{array} \tag{2.2}$$

We have the following basic facts.

FACT 1. *For $n \geq 1$,*

$$P_nQ_{n-1} - P_{n-1}Q_n = (-1)^{n+1}, \tag{2.3}$$

$$\alpha = \frac{\alpha_{n+1}P_n + P_{n-1}}{\alpha_{n+1}Q_n + Q_{n-1}} \tag{2.4}$$

where $\alpha_n = [a_n, a_{n+1}, \ldots]$.

Let $A, B, C \in R$, $\deg A \geq 0$, $f(x) = A^2x + Bx + C$, $D = B^2 - 4A^2C$, and $\sqrt{f(N)} \in k_\infty$ $(N \in R)$.

FACT 2. *For $\alpha = \sqrt{f(N)} = [a_0, a_1, \ldots, a_n, \ldots]$, we have*

$$\alpha_n = \frac{U_n + \sqrt{f(N)}}{V_n} \qquad (n \geq 0) \tag{2.5}$$

where U_n, V_n *are polynomials in* R *and can be calculated recursively by*

$$U_0 = 0, \quad V_0 = 1, \quad a_0 = [\sqrt{f(N)}],$$

$$U_{n+1} = a_n V_n - U_n, \quad V_{n+1} = \frac{f(N) - U_{n+1}^2}{V_n}, \tag{2.6}$$

$$a_{n+1} = \left[\frac{U_{n+1} + \sqrt{f(N)}}{V_{n+1}}\right] \quad (n \geq 0).$$

Moreover, $p(\sqrt{f(N)}) = 2k$ *if and only if* k *is the smallest integer such that* $U_k/U_{k+1} \in \mathbb{F}_q^*$, *while* $p(\sqrt{f(N)}) = 2k+1$ *if and only if* k *is the smallest integer such that* $V_k/V_{k+1} \in \mathbb{F}_q^*$.

The next result means that the convergents are the best approximations of $\alpha \in k_\infty$.

LEMMA 2.1. *For* $\alpha \in k_\infty - k$, $P, Q \in R$, $Q \neq 0$ *and* $(P, Q) = 1$, *the following statements are equivalent to each other*:

(1) $P/Q = P_n/Q_n$ *or* $P/Q = -P_n/Q_n$ *for some* $n \geq 0$;
(2) $\deg(P^2 - \alpha^2 Q^2) < \deg \alpha$;
(3) $\deg(P - \alpha Q) < -\deg Q$ *or* $\deg(P + \alpha Q) < -\deg Q$.

Proof. Let $P/Q = [c_0, c_1, \ldots, c_n]$ be the finite continued fraction of P/Q, and $P'/Q' = [c_0, c_1, \ldots, c_{n-1}]$. There is $\beta \in k_\infty$ such that

$$\alpha = [c_0, \ldots, c_n, \beta] = \frac{\beta P + P'}{\beta Q + Q'}. \tag{2.7}$$

Namely,

$$\beta = \frac{-\alpha Q' + P'}{\alpha Q - P}.$$

From (2.7) and (2.3) we have

$$\alpha - \frac{P}{Q} = \frac{(-1)^n}{Q(\beta Q + Q')}, \quad \deg(P - \alpha Q) = -\deg(\beta Q + Q').$$

Therefore

$$P/Q = P_n/Q_n \Leftrightarrow \deg \beta \geq 1 \Leftrightarrow \deg(P - \alpha Q) < -\deg Q,$$
$$P/Q = -P_n/Q_n \Leftrightarrow \deg(P + \alpha Q) < -\deg Q.$$

So we proved the equivalence (1)⇔(3). The equivalence (2)⇔(3) is easy to prove.

The following result is analogous to Theorem H in [8].

LEMMA 2.2. *Suppose that* $\xi = [b_0, b_1, \ldots] \in k_\infty - k$ *and* $\xi_\nu = [b_\nu, b_{\nu+1}, \ldots]$. *Let* $p, r, s \in R$, $\deg r < \deg s$ *and* $rs = d \neq 0$. *If*

$$\xi' = \frac{p\xi + r}{s}, \quad \frac{p[b_0, b_1, \ldots, b_{\nu-1}] + r}{s} = [d_0, d_1, \ldots, d_{\mu-1}],$$

then $\xi' = [d_0, d_1, \ldots, d_{\mu-1}, \xi'_\mu]$ *where* $\xi'_\mu = (p'\xi_\nu + r')/s'$, $p', r', s' \in R$, $p's' = \pm d$, $\deg r' < \deg s'$.

P r o o f. For a non-singular matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ over R and $\alpha \in k_\infty$ we define the action of A on α by

$$A(\alpha) = \frac{a\alpha + b}{c\alpha + d}.$$

Let $[b_0, \ldots, b_m] = A_m/B_m$ and $[d_0, \ldots, d_l] = C_l/D_l$. From (2.4) we have

$$\xi' = \frac{C_{\mu-1}\xi'_\mu + C_{\mu-2}}{D_{\mu-1}\xi'_\mu + D_{\mu-2}} = \begin{pmatrix} C_{\mu-1} & C_{\mu-2} \\ D_{\mu-1} & D_{\mu-2} \end{pmatrix} (\xi'_\mu),$$

$$\xi' = \begin{pmatrix} p & r \\ o & s \end{pmatrix} (\xi) = \begin{pmatrix} p & r \\ o & s \end{pmatrix} \begin{pmatrix} A_{\nu-1} & A_{\nu-2} \\ B_{\nu-1} & B_{\nu-2} \end{pmatrix} (\xi_\nu).$$

Therefore $\xi'_\mu = \left(\begin{smallmatrix} p' & r' \\ t' & s' \end{smallmatrix}\right)(\xi_\nu)$ where

$$(2.8) \qquad \begin{pmatrix} p' & r' \\ t' & s' \end{pmatrix} = \begin{pmatrix} C_{\mu-1} & C_{\mu-2} \\ D_{\mu-1} & D_{\mu-2} \end{pmatrix}^{-1} \begin{pmatrix} p & r \\ o & s \end{pmatrix} \begin{pmatrix} A_{\nu-1} & A_{\nu-2} \\ B_{\nu-1} & B_{\nu-2} \end{pmatrix}$$
$$= \pm \begin{pmatrix} -D_{\mu-2} & C_{\mu-2} \\ D_{\mu-1} & -C_{\mu-1} \end{pmatrix} \begin{pmatrix} p & r \\ o & s \end{pmatrix} \begin{pmatrix} A_{\nu-1} & A_{\nu-2} \\ B_{\nu-1} & B_{\nu-2} \end{pmatrix}$$
$$= \pm \begin{pmatrix} -pD_{\mu-2} & -rD_{\mu-2} + sC_{\mu-2} \\ pD_{\mu-1} & rD_{\mu-1} - sC_{\mu-1} \end{pmatrix} \begin{pmatrix} A_{\nu-1} & A_{\nu-2} \\ B_{\nu-1} & B_{\nu-2} \end{pmatrix}$$

and

$$\begin{aligned} t' &= \pm(pD_{\mu-1}A_{\nu-1} + rD_{\mu-1}B_{\nu-1} - sC_{\mu-1}B_{\nu-1}) \\ &= \pm sD_{\mu-1}B_{\nu-1}\left(\frac{pA_{\nu-1}/B_{\nu-1} + r}{s} - \frac{C_{\mu-1}}{D_{\mu-1}}\right) = 0, \\ s' &= \pm(pD_{\mu-1}A_{\nu-2} + rD_{\mu-1}B_{\nu-2} - sC_{\mu-1}B_{\nu-2}), \\ r' &= \mp(pD_{\mu-2}A_{\nu-2} + rD_{\mu-2}B_{\nu-2} - sC_{\mu-2}B_{\nu-2}), \end{aligned}$$

so that $\deg r' < \deg s'$. From (2.8) we have

$$p's' = \begin{vmatrix} p' & r' \\ o & s' \end{vmatrix} = \pm \begin{vmatrix} p & r \\ o & s \end{vmatrix} = \pm d.$$

This completes the proof of Lemma 2.2 (cf. Theorem 3 in Chapter IV of [7]).

3. Proof of Theorem 1.1. Suppose that $N \in \widetilde{R} - E$, which means that $f(N) = A^2N^2 + BN + C$ is square-free, $\sqrt{f(N)} \in k_\infty$ and

$$(3.1) \qquad D = B^2 - 4A^2C \mid (2A^2N + B)^2.$$

Let

$$(3.2) \qquad \xi' = \sqrt{f(N)} = \frac{1}{2A}\sqrt{(2A^2N + B)^2 - D}.$$

From (3.1) we know that there exist $U, V, W \in R$ such that

$$2A^2N + B = UVW, \qquad D = UV^2. \tag{3.3}$$

The formula (3.2) becomes

$$\xi' = \frac{V}{2A}\xi \tag{3.4}$$

where

$$\xi = \sqrt{U^2W^2 - U} = [UW, \overline{2W, 2UW}] = [UW, \xi''], \tag{3.5}$$

$$\xi'' = [\overline{2W, 2UW}] = [2W, 2UW, \xi'']. \tag{3.6}$$

Now we use Lemma 2.2 repeatedly to get the continued fraction expansion of $\xi' = \sqrt{f(N)}$. Let

$$\eta_1 = \frac{VUW}{2A} = \frac{2A^2N + B}{2A} = [a_0, a_1, \dots, a_{\mu_1 - 1}]. \tag{3.7}$$

From (3.5) and Lemma 2.2 we have

$$\xi' = [a_0, a_1, \dots, a_{\mu_1 - 1}, \xi_1'] \tag{3.8}$$

where

$$\xi_1' = \frac{p_1\xi'' + r_1}{s_1}, \qquad p_1, r_1, s_1 \in R,\ p_1 s_1 = \pm 2AV,\ \deg r_1 < \deg s_1. \tag{3.9}$$

In general, for each $i \geq 1$ we have

$$\xi_i' = \frac{p_i\xi'' + r_i}{s_i}, \qquad p_i, r_i, s_i \in R,\ p_i s_i = \pm 2AV,\ \deg r_i < \deg s_i. \tag{3.10}$$

Let

$$\begin{aligned}\eta_{i+1} &= \frac{p_i[2W, 2UW] + r_i}{s_i} = \frac{p_i(2W + 1/(2UW)) + r_i}{s_i} \\ &= [a_{\mu_i}, a_{\mu_i + 1}, \dots, a_{\mu_{i+1} - 1}].\end{aligned} \tag{3.11}$$

From (3.6) and Lemma 2.2 we have

$$\xi_i' = [a_{\mu_i}, a_{\mu_i + 1}, \dots, a_{\mu_{i+1} - 1}, \xi_{i+1}'] \tag{3.12}$$

where

$$\begin{aligned}\xi_{i+1}' = \frac{p_{i+1}\xi'' + r_{i+1}}{s_{i+1}}, \qquad p_{i+1}, r_{i+1}, s_{i+1} \in R,\ p_{i+1}s_{i+1} = \pm 2AV, \\ \deg r_{i+1} < \deg s_{i+1}.\end{aligned} \tag{3.13}$$

Since A, B, C and $D = B^2 - 4A^2C$ are fixed polynomials, we choose N such that

$$\deg N > \max\{\deg(A^2/B), \deg(D/A)\}. \tag{3.14}$$

From (3.3) we know that

$$\deg(UVW) = \deg(A^2N) > \deg(AD) = \deg(AUV^2)$$

so that $\deg W > \deg(AV) = \deg(p_i s_i) \ge \deg s_i$ $(i \ge 1)$. Therefore by (3.11),

$$\deg a_{\mu_i} = \deg[\eta_{i+1}] = \deg \frac{p_i W}{s_i} \ge 1 \quad (i \ge 1)$$

and by (3.8) and (3.11) we have the continued fraction expansion of $\xi' = \sqrt{f(N)}$:

$$\sqrt{f(N)} = [a_0, \ldots, a_{\mu_1 - 1}, a_{\mu_1}, \ldots, a_{\mu_2 - 1}, \ldots, a_{\mu_i}, a_{\mu_i + 1}, \ldots, a_{\mu_{i+1}}, \ldots]. \tag{3.15}$$

The total number of tuples (p, r, s) satisfying $ps = \pm 2AN$ and $\deg r < \deg s$ is at most $M = q^{2 \deg AN + 1}$, thus from (3.10) we know that there exist l and j, $1 \le l < j \le M$, such that $\xi'_l = \xi'_j$. From the expansion (3.15) we know that

$$p(\sqrt{f(N)}) \le \sum_{i=l}^{j-1} (\mu_{i+1} - \mu_i). \tag{3.16}$$

We use Lemma 2.2 again to estimate μ_l. Let

$$\frac{2Wp_i + r_i}{s_i} = [c_0, c_1, \ldots, c_{t-1}].$$

By Lemma 2.2 and (3.11) we have

$$\eta_{i+1} = \left[c_0, c_1, \ldots, c_{t-1}, \frac{p'(2UW) + r'}{s'}\right]$$

where $p's' = \pm 2AV$ and $\deg r' < \deg s'$, so that $\deg(\sqrt{p'(2UW) + r'}/s') \ge 1$. Let

$$\frac{p'(2UW) + r'}{s'} = [c_t, c_{t+1}, \ldots, c_{t+\lambda-1}].$$

Then $\eta_{i+1} = [c_0, c_1, \ldots, c_{t-1}, c_t, \ldots, c_{t+\lambda-1}]$ and $\mu_{i+1} - \mu_i = t + \lambda$ by (3.11). From the recursive formula for Q_i in (2.2) and $\deg c_i \ge 1$ $(0 \le i \le t + \lambda - 1)$ we know that

$$t \le \deg s_i \le \deg(AV), \quad \lambda \le \deg s' \le \deg(AV).$$

Therefore $\mu_{i+1} - \mu_i = t + \lambda \le 2\deg(AV)$ and by (3.16),

$$p(\sqrt{f(N)}) \le (j - l) 2 \deg(AV) \le 2M \deg(AV) \le 2 \deg \sqrt{(AV) q^{2 \deg(AV+1)}}$$

provided the formula (3.14) is satisfied. Since there are only finitely many N such that $\deg N \le \max\{\deg(A^2/B), \deg(D/A)\}$, this completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2. Now we assume that $f(N) = A^2N^2 + BN + C$ and $D = B^2 - 4A^2C \nmid (2A^2N + B)^2$. Let $l = p(\sqrt{f(N)})$ and

$$\sqrt{f(N)} = [a_0, a_1, \ldots, a_n, \ldots],$$

$$\alpha_i = [a_i, a_{i+1}, \ldots], \qquad P_i/Q_i = [a_0, a_1, \ldots, a_i],$$
$$\varphi_i = P_i + Q_i\sqrt{f(N)}, \qquad \overline{\varphi}_i = P_i - Q_i\sqrt{f(N)}.$$

Then $\varepsilon = \varphi_{l-1}$ is the fundamental unit of the quadratic function field $K = k(\sqrt{f(N)})$. Let

$$G = (A, B), \qquad \beta = DG^{-2}, \qquad U = 2AG^{-1}, \qquad V = BG^{-1}.$$

We choose

$$A_1 = AUN + V = (2A^2N + B)G^{-1}, \qquad B_1 = U = 2AG^{-1}$$

and

$$X = A_1 + B_1\sqrt{f(N)}.$$

It is easy to see that

$$\mathrm{Norm}(X) = X\overline{X} = A_1^2 - B_1^2 f(N) = \beta$$

where Norm denotes the norm for $\mathbb{F}_q(t, \sqrt{f(N)})/\mathbb{F}_q(t)$. For each $k \geq 1$, let

$$X^k = A_k + B_k\sqrt{f(N)}, \qquad A_k, B_k \in R.$$

The polynomials A_k and B_k can be calculated recursively by

$$A_{k+1} = A_1A_k + B_1B_kf(N), \qquad B_{k+1} = A_1B_k + B_1A_k. \tag{4.1}$$

Let

$$D_k = (A_k, B_k), \quad A'_k = A_k/D_k, \quad B'_k = B_k/D_k, \quad N_k = A'^2_k - B'_kf(N). \tag{4.2}$$

Then $(A'_k, B'_k) = 1$. Finally we choose

$$M = \left[\frac{\deg f(N)}{2\deg\beta}\right] - 1. \tag{4.3}$$

Lemma 4.1. *For $1 \leq k \leq M$, there exists i_k such that $A'_k/B'_k = P_{i_k}/Q_{i_k}$.*

Proof. We have

$$\begin{aligned}\deg(A'^2_k - B'^2_k f(N)) &= \deg(A_k^2 - B_k^2 f(N)) - 2\deg D_k \\ &\leq \deg(A_k^2 - B_k^2 f(N)) = \deg(\mathrm{Norm}(X^k)) = k\deg\beta \\ &\leq M\deg\beta < \tfrac{1}{2}\deg f(N).\end{aligned}$$

Then the conclusion follows from Lemma 2.1 and $\deg(A'_k + B'_k\sqrt{f(N)}) \geq 0$.

Lemma 4.2. *$i_{k+1} - i_k \geq 2$ for $1 \leq k \leq M - 1$.*

Proof. From the condition $D \nmid (2A^2N + B)^2$ we know that there is an irreducible polynomial $P = P(t)$ in R such that $v_P(D) > 2v_P(2A^2N + B)$, which means that $v_P(\beta) > 2v_P(A_1)$ where v_P is the normal P-adic exponential valuation. From $\beta = \mathrm{Norm}(X) = A_1^2 - B_1^2 f(N)$ we have $v_P(A_1^2) = v_P(B_1^2 f(N))$. Then $(A_1, B_1) = 1$ and $f(N) \in \widetilde{R}$ imply that

$$v_P(B_1) = v_P(f(N)) = v_P(A_1) = 0. \tag{4.4}$$

Now we prove that $v_P(A_k) = v_P(B_k) = 0$ for all $k \geq 1$ by induction. This is true for $k = 1$ from (4.4). Assume that $v_P(B_i) = v_P(A_i) = 0$ for some $i \geq 1$. Since $v_P(f(N)) = 0$, $v_P(\beta) > 2v_P(A_1) = 0$ and $\beta^i = A_i^2 - B_i^2 f(N)$, we know that $A_i^2 \equiv B_i^2 f(N) \pmod{P}$, which means that $\left(\frac{f(N)}{P}\right) = 1$ and P is split in the quadratic field $K = \mathbb{F}_q(t)(\sqrt{f(N)})$ where $\left(\frac{}{P}\right)$ means the Legendre symbol. Let $P = p\widetilde{p}$ be the decomposition in K. Then $X = A_1 + B_1\sqrt{f(N)}$ has to be divisible by p or $\widetilde{p}$. We can assume that

$$A_1 + B_1\sqrt{f(N)} \equiv 0 \pmod{p}.$$

From $v_P(B_1) = 0$ we know that $v_p(B_1) = 0$ so that

$$\frac{A_1}{B_1} \equiv -\sqrt{f(N)} \pmod{p}.$$

Moreover, $A_i + B_i\sqrt{f(N)} = (A_1 + B_1\sqrt{f(N)})^i \equiv 0 \pmod{p}$. From the assumption $v_P(B_i) = 0$ we have

$$A_i/B_i \equiv -\sqrt{f(N)} \pmod{p}.$$

Then from (4.1) we see that

$$\frac{B_{i+1}}{B_1 B_i} = \frac{A_1}{B_1} + \frac{A_i}{B_i} \equiv -2\sqrt{f(N)} \not\equiv 0 \pmod{p},$$

which means that $v_p(B_{i+1}) = 0$ so that $v_P(B_{i+1}) = 0$. Finally we deduce that $v_P(A_{i+1}) = 0$ from $\beta^{i+1} = A_{i+1}^2 - B_{i+1}^2 f(N)$. Thus we have proved that $v_P(A_k) = v_P(B_k) = 0$ for all $k \geq 1$. Then $v_P(D_k) = 0$ since $D_k = (A_k, B_k)$. From (2.3) and (2.4) we have

$$\sqrt{f(N)} - \frac{P_n}{Q_n} = \frac{(-1)^{n+1}}{Q_n(\alpha_{n+1}Q_n + Q_{n-1})}.$$

Therefore

$$\begin{aligned}\deg \overline{\varphi}_n &= \deg(P_n - Q_n\sqrt{f(N)}) = -\deg(\alpha_{n+1}Q_n + Q_{n-1}) \\ &= -\deg(a_{n+1}Q_n + Q_{n-1}) = -\deg Q_{n+1}.\end{aligned}$$

From $\deg(A_1 - B_1\sqrt{f(N)}) < 0$ and $D_k \mid D_{k+1}$ (by definition) we know that

$$\begin{aligned}-\deg Q_{i_k+1} &= \deg \overline{\varphi}_{i_k} = \deg(P_{i_k} - Q_{i_k}\sqrt{f(N)}) \\ &= \deg(A_k' - B_k'\sqrt{f(N)}) \\ &= -\deg D_k + \deg(A_k - B_k\sqrt{f(N)}) \\ &= -\deg D_k + k\deg(A_1 - B_1\sqrt{f(N)}) \\ &> -\deg D_{k+1} + (k+1)\deg(A_1 - B_1\sqrt{f(N)}) \\ &= \deg \overline{\varphi}_{i_{k+1}} = -\deg Q_{i_{k+1}+1}.\end{aligned}$$

Therefore $\deg Q_{i_{k+1}+1} > \deg Q_{i_k+1}$ which means that $i_{k+1} > i_k$. If $i_{k+1} = i_k + 1$, then

$$P_{i_{k+1}} Q_{i_k} - P_{i_k} Q_{i_{k+1}} = P_{i_k+1} Q_{i_k} - P_{i_k} Q_{i_k+1} = (-1)^{i_k+1},$$

so that $v_P(P_{i_{k+1}} Q_{i_k} - P_{i_k} Q_{i_{k+1}}) = 0$. On the other hand,

$$\begin{aligned} v_P(P_{i_{k+1}} Q_{i_k} - P_{i_k} Q_{i_{k+1}}) &= v_P(A'_{k+1} B'_k - A'_k B'_{k+1}) \\ &= v_P(A_{k+1} B_k - A_k B_{k+1}) \\ &\qquad\qquad (\text{since } v_P(D_k) = v_P(D_{k+1}) = 0) \\ &= v_P((A_k A_1 + B_k B_1 f(N)) B_k \\ &\quad - A_k(A_k B_1 + A_1 B_k)) \\ &= v_P(B_1(A_k^2 - B_k^2 f(N))) = v_P(\beta^k) \\ &\qquad\qquad (\text{since } v_P(B_1) = 0) \\ &= k v_P(\beta) > 0 \quad (\text{for } k \ge 1). \end{aligned}$$

This contradiction shows that $i_{k+1} - i_k \ge 2$, which completes the proof of Lemma 4.2.

LEMMA 4.3. *Let*

$$M_1 = \left[\frac{\deg f(N) - 2\deg U}{2\deg \beta}\right] - 1 \ (\le M).$$

Then $\deg \varphi_{i_k} < \deg \varepsilon$ *for all* $1 \le k \le M_1$ *where* $\varepsilon = \varphi_{l-1}$ *is the fundamental unit in* $K = \mathbb{F}_q(t, \sqrt{f(N)})$.

Proof. If $\deg \varphi_{i_s} \ge \deg \varepsilon$ for some s $(1 \le s \le M_1)$, let s be the smallest one satisfying this condition. Then

$$\varphi_{i_s} \overline{\varepsilon} = (P_{i_s} + Q_{i_s} \sqrt{f(N)})(P_{l-1} - Q_{l-1}\sqrt{f(N)}) = A' + B'\sqrt{f(N)}$$

where $A', B' \in R$, and

$$\deg(A'^2 - B'^2 f(N)) = \deg(\mathrm{Norm}(\varphi_{i_s})) = \deg(\mathrm{Norm}(\varphi_{i_s}\overline{\varepsilon})) < \tfrac{1}{2}\deg f(N).$$

From Lemma 2.1 and $\deg(\varphi_{i_s}\overline{\varepsilon}) = \deg \varphi_{i_s} - \deg \varepsilon \ge 0$ we know that there exists j such that $\varphi_{i_s}\overline{\varepsilon} = \alpha \varphi_j$ for some $\alpha \in \mathbb{F}_q^*$. Therefore $D_s^{-1} X^s = \varphi_{i_s} = \alpha' \varepsilon \varphi_j$ $(\alpha' \in \mathbb{F}_q^*)$. If $j \ge i_1$, then

$$s \deg X \ge \deg(D_s^{-1} X^s) = \deg(\varepsilon \varphi_j) \ge \deg(\varepsilon \varphi_{i_1}) = \deg(\varepsilon X) > \deg X > 0.$$

Thus $s \ge 2$ and

$$\deg \varphi_{i_{s-1}} \ge \deg(D_{s-1} D_s^{-1} \varphi_{i_{s-1}}) = \deg(D_s^{-1} X^{s-1}) \ge \deg \varepsilon,$$

which contradicts the definition of s. Therefore $j < i_1$. Since

$$\deg(\varphi_j \overline{\varphi}_j) = \deg(\varphi_{i_s} \overline{\varphi}_{i_s}) = \deg(\beta^s D_s^{-2}), \quad \overline{\varphi}_j = (-1)^{j+1}(\alpha_{j+1} Q_j + Q_{j-1})^{-1}$$

and $\deg \alpha_{j+1} = \deg a_{j+1}$, we know that $\deg \overline{\varphi}_j = -\deg Q_{j+1} = -\deg Q_{i_1} = -\deg U$ and

$$\deg(\beta^s U) \ge \deg(\beta^s D_s^{-2} U) \ge \deg(\beta^s D_s^{-2} \overline{\varphi}_j^{-1}) = \deg \varphi_j \ge \tfrac{1}{2} \deg f(N).$$

Therefore $s \ge (\deg f(N) - 2\deg U)/(2\deg \beta)$, which contradicts $s \le M_1$. This completes the proof of Lemma 4.3.

Now Theorem 1.2 is a direct consequence since Lemma 4.2 says that $i_{k+1} - i_k \ge 2$ for $1 \le k \le M_1 - 1$ and Lemma 4.3 says that $1 \le i_k \le p(\sqrt{f(N)})$ for all $1 \le k \le M_1$, so that

$$p(\sqrt{f(N)}) \ge 2M_1 - 1 = 2\left[\frac{\deg f(N) - 2\deg U}{2\deg \beta}\right] - 1 \to \infty \quad \text{as } \deg N \to \infty.$$

This completes the proof of Theorem 1.2.

Example. Let $f(x) = x^2 + C$ where $C = 2a + 1$, $a \in R$, $\deg a \ge 1$. We have $A = 1$, $B = 0$, $\beta = -4C = D$ and $U = 1$. For each $k \ge 2$, we choose $N = C^k + a$ so that $f(N) = (C^k + a)^2 + C$. Since $D = -4C \nmid (2A^2N + B)^2 = 4(C^k + a)^2$, N belongs to the set E for all $k \ge 2$. We use Fact 2 to determine the quasi-period $p(\sqrt{f(N)})$ as follows:

$$U_0 = 0, \quad V_0 = 1, \quad a_0 = [\sqrt{f(N)}] = C^k + a,$$
$$U_1 = C^k + a, \quad V_1 = C, \quad a_1 = \left[\frac{2C^k + 2a}{C}\right] = \left[\frac{2C^k + C - 1}{C}\right] = 2C^{k-1} + 1,$$
$$U_2 = C^k + a + 1, \quad V_2 = -2C^{k-1}, \quad a_2 = \left[\frac{2C^k + C}{-2C^{k-1}}\right] = -C,$$
$$U_3 = C^k - a - 1, \quad V_3 = \frac{C + (2C^k - 1)C}{-2C^{k-1}} = -C^2,$$
$$a_3 = \left[\frac{2C^k - 1}{-C^2}\right] = -2C^{k-2},$$
$$U_4 = C^k + a + 1, \quad V_4 = 2C^{k-2}, \quad a_4 = C^2,$$
$$U_5 = C^k - a - 1, \quad V_5 = C^3, \quad a_5 = 2C^{k-3},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

In general, $V_{2i+1} = \alpha C^{i+1}$, $V_{2i} = \alpha' C^{k-i}$, where $\alpha, \alpha' \in \mathbb{F}_q^*$. If $k = 2n$, then $V_{2n}/V_{2n+1} \in \mathbb{F}_q^*$ and $p(\sqrt{f(N)}) = 4n + 1 = 2k - 1$. If $k = 2n$, then $V_{2n-1}/V_{2n} \in \mathbb{F}_q^*$ and $p(\sqrt{f(N)}) = 4n - 1 = 2k - 1$. Therefore $p(\sqrt{f(N)}) = 2k - 1$ for all $k \ge 2$. On the other hand,

$$2M_1 - 1 = 2\left[\frac{\deg f(N) - 2\deg U}{2\deg \beta}\right] - 1 = \frac{2\deg(C^k + a)}{\deg C} - 1 = 2k - 1.$$

Therefore $p(\sqrt{f(N)}) = 2M_1 - 1$, which shows that our general estimate $p(\sqrt{f(N)}) \geq 2M_1 - 1$ is the best one.

Acknowledgments. I would like to thank an anonymous referee for several suggestions that improved the exposition of this paper.

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Received on 22.5.1997
and in revised form on 12.4.1999 (3194)