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# On sums of two $k$th powers: an asymptotic formula for the mean square of the error term

by

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**1. Introduction.** For a fixed natural number $k \geq 2$ we consider the arithmetic function $r_k(n)$ which counts the number of ways to write the positive integer $n$ as a sum of the $k$th powers of two integers taken absolutely:

$$r_k(n) = \#\{(u_1, u_2) \in \mathbb{Z}^2 : |u_1|^k + |u_2|^k = n\}.$$

To study the average order of this arithmetic function, one is interested in the Dirichlet summatory function

$$A_k(t) = \sum_{1 \leq n \leq t^{k/2}} r_k(n),$$

where $t$ is a large real variable. For the special case $k = 2$, Gauß proved that

$$A_2(t) = \pi t + P_2(t),$$

with $P_2(t) \ll t^{1/2}$. Since then the question of the exact order of $P_2(t)$ has been called the circle problem of Gauß. For an exposition of its history, see e.g. the textbook of Krätzel [10]. At present the sharpest upper bound is

$$P_2(t) = O(t^{23/73}(\log t)^{315/146}), \tag{1.1}$$

due to Huxley [7], [8]. In the opposite direction the best results to date are

$$P_2(t) = \Omega_-((t \log t)^{1/4} (\log\log t)^{(\log 2)/4} \exp(-c\sqrt{\log\log\log t})) \qquad (c > 0),$$

and

$$P_2(t) = \Omega_+(t^{1/4} \exp(c'(\log\log t)^{1/4}(\log\log\log t)^{-3/4})) \qquad (c' > 0),$$

due to Hafner [5], and Corrádi and Kátai [2], respectively. They refined earlier work of Hardy [6], resp. Gangadharan [3]. It is usually conjectured that

$$\inf\{\theta \in \mathbb{R} : P_2(t) \ll_\theta t^\theta\} = 1/4.$$

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This hypothesis is supported by the mean square result

$$(1.2)\qquad \int_0^X (P_2(t))^2\,dt = CX^{3/2} + O(X(\log X)^2), \qquad C = \frac{1}{3\pi^2}\sum_{n=1}^{\infty}\frac{(r_2(n))^2}{n^{3/2}},$$

which has been established (in this sharp form) by Kátai [9].

For $k \ge 3$, the asymptotic formula for $A_k(t)$ contains a second main term which comes from the points of the boundary curve where the curvature vanishes. It turns out that

$$(1.3)\qquad A_k(t) = \frac{2\Gamma^2(1/k)}{k\Gamma(2/k)}t + B_k\Phi_k(t)t^{1/2-1/(2k)} + P_k(t),$$

where

$$B_k = 2^{3-1/k}\pi^{-1-1/k}k^{1/k}\Gamma\left(1+\frac{1}{k}\right),$$

$$\Phi_k(t) = \sum_{n=1}^{\infty} n^{-1-1/k}\sin\left(2\pi n\sqrt{t} - \frac{\pi}{2k}\right).$$

A thorough account on the history (which goes back to van der Corput [19]) and the diverse aspects of this problem can be found in the textbook of Krätzel [10]. Using Huxley's deep method in its sharpest form, Kuba [12] proved that the new error term $P_k(t)$ again satisfies the estimate (1.1). Quite recently Nowak [14] was able to show that this analogy partially extends to the order of the mean square, i.e.

$$(1.4)\qquad \int_0^X (P_k(t))^2\,dt \ll X^{3/2},$$

for a large real parameter $X$ (the $\ll$-constant possibly depending on $k$). Concerning lower estimates it is known that

$$P_k(t) = \begin{cases} \Omega_-((t\log t)^{1/4}) & \text{for } k \ge 3,\\ \Omega_+((t\log\log t)^{1/4}) & \text{for } k = 3,\end{cases}$$

due to Nowak [15] and Nowak, Schoissengeier, Wooley and the author [13], which corresponds to the results of Hardy [6], resp. Gangadharan [3]. See also earlier works of Krätzel [11] and Schnabel [17] where somewhat weaker estimates were obtained.

The proof of the estimate (1.2) uses the fact that the generating function $\sum_{n=1}^{\infty} r_2(n)/n^s$ ($\mathrm{Re}\, s > 1$) of the number of lattice points on the circle satisfies a functional equation. For general Dirichlet series satisfying a functional equation with multiple gamma factors, Redmond [16] proved a mean-square asymptotic formula for the error term of the summatory function. In the case

$k \geq 3$, such a handy functional equation is not available. Using a different method we improve (1.4) to

$$\int_0^X (P_k(t))^2\,dt \sim C_k X^{3/2} \qquad (C_k > 0)$$

with an explicitly given error term.

*Notation.* For any fixed natural number $k$ let $q$ be defined by $1/k+1/q = 1$, i.e. $q = k/(k-1)$. Further, let $|\cdot|_q$ denote the $q$-norm in $\mathbb{R}^2$, i.e.

$$|(w_1, w_2)|_q = (|w_1|^q + |w_2|^q)^{1/q}.$$

THEOREM. *For any fixed integer $k \geq 3$, the error term $P_k(t)$ defined in* (1.3) *satisfies*

$$\int_0^X (P_k(t))^2\,dt = C_k X^{3/2} + O(X^{3/2-\alpha_k+\varepsilon})$$

*for any $\varepsilon > 0$, where*

$$C_k = \frac{16}{3\pi^2(k-1)} \sum_{\substack{a,b,c,d\in\mathbb{N} \\ |(a,b)|_q=|(c,d)|_q}} (abcd)^{-1+q/2}|(a,b)|_q^{-2q+1}$$

*and*

$$\alpha_k = \frac{1}{12(q+1/6+k(k-1)^4)}. \tag{1.5}$$

REMARKS. 1. The convergence of the above series will be a by-result of our proof: see (3.15) ff.

2. The constant $\alpha_k$ can certainly be improved by a more elaborate analysis. The author did not invest much effort to obtain the optimal $\alpha_k$ in reach of the present method.

3. It is natural to compare our constant $C_k$ with the constant $C$ of (1.2) which may be written in the form

$$C = \frac{1}{3\pi^2} \sum_{\substack{a,b,c,d\in\mathbb{Z} \\ 0<|(a,b)|_2=|(c,d)|_2}} |(a,b)|_2^{-3}.$$

We notice that this latter sum ranges also over the pairs $(a,b)$, $(c,d)$ with one vanishing component, in contrast to the series for $C_k$, $k \geq 3$. The reason for this is that the Lamé's curve has curvature 0 at its points of intersection with the coordinate axes: These give rise to the second main term in (1.3).

## 2. Some lemmas

Lemma 1 (see Vaaler [18]). *For arbitrary* $w \in \mathbb{R}$ *and* $H \in \mathbb{N}$, *let*

$$\psi(w) = w - [w] - 1/2, \qquad \psi_H^*(w) = -\frac{1}{\pi} \sum_{h=1}^{H} \frac{\sin(2\pi hw)}{h} \tau\left(\frac{h}{H+1}\right),$$

*where*

$$\tau(x) = \pi x(1-x)\cot(\pi x) + x \qquad \text{for } 0 < x < 1.$$

*Then*

$$|\psi(w) - \psi_H^*(w)| \le \frac{1}{H+1} \sum_{h=1}^{H} \left(1 - \frac{h}{H+1}\right) \cos(2\pi hw) + \frac{1}{2H+2}.$$

Lemma 2. *Let* $f(w)$ *be a real-valued function with continuous derivatives up to the fourth order on the interval* $[A,B]$. *Let* $L$ *and* $U$ *be real parameters not less than* 1 *such that* $B - A \asymp L$,

$$f^{(j)}(w) \ll UL^{1-j} \qquad \text{for } w \in [A,B],\ j = 1,2,3,4,$$

*and*, *for some* $C > 0$,

$$f''(w) \ge CUL^{-1} \qquad \text{for } w \in [A,B].$$

*Suppose that* $f'(A)$ *and* $f'(B)$ *are integers*, *and denote by* $\phi$ *the inverse function of* $f'$. *Then*

$$\sum_{A \le n \le B} e(f(n)) = e\left(\frac{1}{8}\right) \sideset{}{''}\sum_{f'(A) \le m \le f'(B)} \frac{e(f(\phi(m)) - m\phi(m))}{\sqrt{f''(\phi(m))}} + O(\log(1+U)),$$

*where* $e(u) = e^{2\pi iu}$ *for real* $u$, *and* $\sum''$ *indicates that the terms corresponding to* $m = f'(A)$, *resp.* $m = f'(B)$ *are weighted with the factor* $1/2$. *The implied* $O$*-constant depends at most on* $C$ *and on the constants implied in the order symbols in the suppositions.*

To prove Lemma 2, let us first state the following result.

Lemma 3. *Let* $F(w)$ *be a real-valued function with continuous derivatives up to the fourth order on the interval* $[A,B]$. *Let* $L$ *and* $U$ *be real parameters not less than* 1 *such that* $B - A \asymp L$,

$$F^{(j)}(w) \ll UL^{1-j} \qquad \text{for } w \in [A,B],\ j = 1,2,3,4,$$

*and*, *for some* $C > 0$,

$$F''(w) \ge CUL^{-1} \qquad \text{for } w \in [A,B].$$

*Suppose that there exists a value* $c \in [A, B]$ *for which* $F'(c) = 0$. *Then*

$$\int\limits_A^B e(F(w))\,dw$$

$$= \begin{cases} \frac{1}{2}(F''(A))^{-1/2} e\Big(\frac{1}{8} + F(A)\Big) + O\Big(\frac{1}{|F'(B)|}\Big) + O\Big(\frac{1}{U}\Big) & \textit{if } c = A, \\ \frac{1}{2}(F''(B))^{-1/2} e\Big(\frac{1}{8} + F(B)\Big) + O\Big(\frac{1}{|F'(A)|}\Big) + O\Big(\frac{1}{U}\Big) & \textit{if } c = B, \\ (F''(c))^{-1/2} e\Big(\frac{1}{8} + F(c)\Big) + O\Big(\frac{1}{|F'(A)|} + \frac{1}{|F'(B)|}\Big) + O\Big(\frac{1}{U}\Big) & \textit{else.} \end{cases}$$

Proof. For $A < c < B$, this is explicitly contained in Lemma 3.4 of Graham and Kolesnik [4]. The case $c = B$ can be reduced to $c = A$ by the substitution $w \mapsto A + B - w$. Finally, to deal with the case $c = A$, it suffices to have a close look at the proof of Lemma 3.4 in [4]: Here $F$ is approximated by its quadratic Taylor expansion $q(w)$ at the stationary point $c$. If $c = A$, the integral $\int_A^B e(q(w))\,dw$ can be evaluated by an obvious variant of Lemma 3.3 in [4], viz.

$$\int\limits_0^X e(Hw^2)\,dw = \frac{e(1/8)}{2\sqrt{2H}} + O\Big(\frac{1}{HX}\Big).$$

The ingenious estimation of the remainder integral then works exactly as in [4].

*Proof of Lemma 2.* Again by [4], Lemma 3.5 (after taking conjugates),

$$\sum_{A \le n \le B} e(f(n)) = \sum_{m=f'(A)-1}^{f'(B)+1} \int\limits_A^B e(f(w) - mw)\,dw + O(\log(1+U)).$$

To each of the integrals on the right-hand side we apply Lemma 3. (The first and last one are estimated as $O(1)$ by the first derivative test.) The main term produces no difficulties, and the error terms are readily

$$\ll \frac{1}{f'(B) - f'(A)} + \sum_{m=f'(A)+1}^{f'(B)-1} \Big(\frac{1}{m - f'(A)} + \frac{1}{f'(B) - m}\Big) + \frac{1}{U}(f'(B) - f'(A) + 1)$$
$$\ll \log(1+U).$$

Lemma 4. *Let* $s$, $t$, $u$, $v$ *be natural numbers with*

$$|(s,t)|_q - |(u,v)|_q \neq 0.$$

*Then*

$$||(s,t)|_q - |(u,v)|_q| \gg M^{-(q-1+k(k-1)^4)},$$

*where* $M = \max(s,t,u,v)$ *and the implied* $\gg$ *constant depends at most on* $k$.

Proof. By the mean value theorem we have

$$|(s,t)|_q^q - |(u,v)|_q^q \ll M^{q-1}||(s,t)|_q - |(u,v)|_q|. \tag{2.1}$$

The left-hand side of (2.1) can be written as

$$L := a^{1/r} + b^{1/r} - c^{1/r} - d^{1/r} \neq 0,$$

with natural numbers $a = s^k$, $b = t^k$, $c = u^k$, $d = v^k$, and $r := k-1$. Consider the field extension

$$\mathbb{F} = \mathbb{Q}(a^{1/r}, b^{1/r}, c^{1/r}, d^{1/r}, e^{2\pi i/r}).$$

The corresponding Galois group $G = \mathrm{Gal}(\mathbb{F}/\mathbb{Q})$ contains at most $r^5$ elements $\chi$. It is clear that

$$\Big|\prod_{\chi\in G}\chi(L)\Big| \ge 1,$$

since the left-hand side is the modulus of the norm of a nonzero algebraic integer. Furthermore, for every $\chi \in G$,

$$|\chi(L)| \le |\chi(a^{1/r})| + |\chi(b^{1/r})| + |\chi(c^{1/r})| + |\chi(d^{1/r})| \le 4M^{k/r}.$$

Consequently,

$$|L| \ge \prod_{\substack{\chi\in G\\ \chi\neq \mathrm{id}}} |\chi(L)|^{-1} \gg M^{-r^4},$$

which establishes Lemma 4.

**3. Proof of the Theorem.** We use $\varepsilon$ to denote an arbitrary small positive quantity which need not be the same at each occurrence. The constants implied in the symbols $\ll$ and $O$ may depend on $\varepsilon$ and $k$. We start from formulae (3.57), (3.58) of Krätzel [10], p. 148, and the asymptotic expansion below:

$$P_k(t) = -8\sum_{\alpha\sqrt{t}\le n\le\sqrt{t}} \psi((t^{k/2} - n^k)^{1/k}) + O(1), \tag{3.1}$$

with $\psi(w) = w - [w] - 1/2$ throughout, and $\alpha := 2^{-1/k}$.

In what follows, let $T$ be sufficiently large and $t \in [T, 2T]$. We split up the domain of summation into subintervals $\mathcal{N}_j(t) = [N_j, N_{j+1}]$, where

$$N_j = \frac{\sqrt{t}}{(1+2^{-jq})^{1/k}}, \qquad j = 0, 1, \ldots, J,$$

with $J$ minimal such that $\sqrt{t}-N_J<1$ for $T\le t\le 2T$. Thus

$$P_k(t)=-8\sum_{j=0}^{J}\sum_{n\in\mathcal{N}_j(t)}\psi((t^{k/2}-n^k)^{1/k})+O(\log T).$$

Further let

$$P_k^*(t)=-8\sum_{j=0}^{J}\sum_{n\in\mathcal{N}_j(t)}\psi_H^*((t^{k/2}-n^k)^{1/k}), \tag{3.2}$$

with $\psi_H^*(w)$ defined as in Lemma 1.

We will prove the following Proposition. By applying Cauchy's inequality and summing over $T=X/2,X/4,\ldots$, our Theorem readily follows.

PROPOSITION. *For sufficiently large $T$ and $H=[T^{1/4+\alpha_k}]$, with $\alpha_k$ defined as in* (1.5), *we have*

(i) $$\int_T^{2T}|P_k(t)-P_k^*(t)|^2\,dt\ll T^{3/2-2\alpha_k},$$

(ii) $$\int_T^{2T}|P_k^*(t)|^2\,dt=\mathcal{H}(T)+O(T^{3/2-\alpha_k+\varepsilon}),$$

*with*

$$\mathcal{H}(T)=\frac{8}{\pi^2(k-1)}\sum_{\substack{a,b,c,d\in\mathbb{N}\\ |(a,b)|_q=|(c,d)|_q}}(abcd)^{-1+q/2}|(a,b)|_q^{-2q+1}\int_T^{2T}t^{1/2}\,dt.$$

Proof. (i) By (3.1), (3.2) and Lemma 1, the left-hand side of (i) is

$$\ll\int_T^{2T}\Bigl(\frac{1}{H^2}\sum_{1\le h\le H}\sum_{j=0}^{J}\sum_{n\in\mathcal{N}_j(t)}\cos(-2\pi h(t^{k/2}-n^k)^{1/k})\Bigr)^2\,dt$$
$$+O(T^{3/2-2\alpha_k}).$$

By Cauchy's inequality, it thus suffices to show that there exists a constant $c_0>1$ such that, for $T$ sufficiently large and $0\le j\le J$,

$$I_j(T):=\int_T^{2T}\Bigl(\frac{1}{H^2}\sum_{1\le h\le H}\sum_{n\in\mathcal{N}_j(t)}\cos(-2\pi h(t^{k/2}-n^k)^{1/k})\Bigr)^2\,dt \ll c_0^{-j}T^{3/2-2\alpha_k}. \tag{3.3}$$

We transform each of the inner trigonometric sums over $n$ by Lemma 2, with $[A,B]=[N_j,N_{j+1}]$, and

$$f(w)=-h(t^{k/2}-w^k)^{1/k}.$$

We note that $f'(N_j)$ is independent of $t$, more precisely

$$f'(N_j) = h2^j \quad \text{and} \quad N_{j+1} - N_j \asymp \sqrt{T}/2^{jq}. \tag{3.4}$$

Calculating derivatives, we get

$$\begin{aligned}
f^{(1)}(w) &= hw^{k-1}(t^{k/2} - w^k)^{-1+1/k},\\
f^{(2)}(w) &= h(k-1)t^{k/2}w^{k-2}(t^{k/2} - w^k)^{-2+1/k}\\
&\asymp hT^{1/2-1/(2k)}(\sqrt{t} - w)^{-2+1/k}\\
&\asymp hT^{1/2-1/(2k)}(\sqrt{T}/2^{jq})^{-2+1/k}\\
&\asymp hT^{-1/2}2^{jq(2-1/k)},\\
f^{(3)}(w) &= h(k-1)t^{k/2}w^{k-3}(t^{k/2} - w^k)^{-3+1/k}((k-2)t^{k/2} + (k+1)w^k)\\
&\asymp hT^{1/2-1/(2k)}(\sqrt{t} - w)^{-3+1/k}\\
&\asymp hT^{-1}2^{jq(3-1/k)},\\
f^{(4)}(w) &= h(k-1)t^{k/2}w^{k-4}(t^{k/2} - w^k)^{-4+1/k}\\
&\quad \times ((k-2)t^{k/2}((k-3)t^{k/2} + 2(k+1)w^k)\\
&\quad + (1+k)w^k((2k-3)t^{k/2} + (k+2)w^k))\\
&\asymp hT^{1/2-1/(2k)}(\sqrt{t} - w)^{-4+1/k}\\
&\asymp hT^{-3/2}2^{jq(-4+1/k)}.
\end{aligned}$$

One easily verifies that the conditions of Lemma 2 are satisfied, with $L = \sqrt{T}\,2^{-jq}$ and $U = h2^j$. We note that in view of (3.4) $f'(N_j)$ and $f'(N_{j+1})$ are integers. We may thus apply the lemma to conclude by a straightforward calculation that, for $T \le t \le 2T$,

$$\begin{aligned}
&\sum_{n\in\mathcal{N}_j(t)} e(-h(t^{k/2} - n^k)^{1/k})\\
&\quad = \frac{e(1/8)}{\sqrt{k-1}} ht^{1/4} \sum_{m\in\mathcal{M}_j(h)}'' (hm)^{-1+q/2}|(h,m)|_q^{-q+1/2} e(-\sqrt{t}\,|(h,m)|_q)\\
&\qquad + O(j + \log h),
\end{aligned} \tag{3.5}$$

where

$$\mathcal{M}_j(h) = [f'(N_j), f'(N_{j+1})].$$

Therefore, using the real part of (3.5), we obtain

$$I_j(T) \ll I_j^*(T) + T(\log T)^4 \tag{3.6}$$

with

$$I_j^*(T) := \int_T^{2T} \frac{t^{1/2}}{H^2}\Big|\sum_{1\le h\le H} S_h(t)\Big|^2\,dt, \tag{3.7}$$

and

$$S_h(t) := \sum_{m\in\mathcal{M}_j(h)}'' h(hm)^{-1+q/2}|(h,m)|_q^{-q+1/2} e(\sqrt{t}\,|(h,m)|_q).$$

To estimate the integral in (3.7) we follow the proof of Nowak [14]. We split up the domain of summation over $h$ into dyadic subintervals: Let

$$\mathcal{H}_i = ]H_{i+1}, H_i], \quad H_i = H/2^i, \quad i = 0,1,\ldots,I,$$

where $I$ is the largest integer for which $2^I < H$. By Cauchy's inequality,

$$\Big|\sum_{h=1}^{H} S_h(t)\Big|^2 \ll \sum_{i=0}^{I} H_i^{\varepsilon}\Big|\sum_{h\in\mathcal{H}_i} S_h(t)\Big|^2,$$

with some fixed $\varepsilon > 0$ sufficiently small.

In what follows, we write $\mathbf{u} = (u_1,u_2)$, $\mathbf{v} = (v_1,v_2)$ for elements of $\mathbb{Z}^2$, and put

$$\mathcal{U}_{i,j} := \{(h,m) : h \in \mathcal{H}_i,\ m \in \mathcal{M}_j(h)\}.$$

By squaring and integrating term by term we get

$$\int_T^{2T} \frac{t^{1/2}}{H^2}\Big|\sum_{h\in\mathcal{H}_i} S_h(t)\Big|^2\,dt \ll \frac{T^{1/2}}{H^2}\sum_{\mathbf{u},\mathbf{v}\in\mathcal{U}_{i,j}} \frac{u_1v_1(|\mathbf{u}|_q|\mathbf{v}|_q)^{-q+1/2}}{(u_1u_2v_1v_2)^{1-q/2}}\Big|\int_T^{2T} e(\sqrt{t}\,(|\mathbf{u}|_q - |\mathbf{v}|_q))\,dt\Big|.$$

Recalling $f'(N_j) = h2^j$ we conclude that $\mathbf{u} = (u_1,u_2) \in \mathcal{U}_{i,j}$ implies that

$$u_1 \asymp H_i, \quad |\mathbf{u}|_q \asymp u_2 \asymp H_i 2^j. \tag{3.8}$$

Consequently,

$$\int_T^{2T} \frac{t^{1/2}}{H^2}\Big|\sum_{h\in\mathcal{H}_i} S_h(t)\Big|^2\,dt \ll \frac{T^{1/2}}{H^2} H_i^{-1}2^{-j(1+q)} \sum_{\mathbf{u}\in\mathcal{U}_{i,j}}\Big(\sum_{\mathbf{v}:|\mathbf{v}|_q\le|\mathbf{u}|_q}\min\Big(T, \frac{\sqrt{T}}{|\mathbf{u}|_q - |\mathbf{v}|_q}\Big)\Big). \tag{3.9}$$

For the inner sum over $\mathbf{v}$ we have the estimate

$$\sum_{\mathbf{v}:|\mathbf{v}|_q\le|\mathbf{u}|_q}\min\Big(T, \frac{\sqrt{T}}{|\mathbf{u}|_q - |\mathbf{v}|_q}\Big) \ll |\mathbf{u}|_q\sqrt{T}\log T + T|\mathbf{u}|_q^{2/3}$$

(see Nowak [14], formula (2.14) and below). Inserting this in (3.9), and recalling (3.8) we obtain

$$\int_T^{2T} \frac{t^{1/2}}{H^2}\Big|\sum_{h\in\mathcal{H}_i} S_h(t)\Big|^2\,dt$$
$$\ll \frac{T^{1/2}}{H^2} H_i^{-1}2^{-j(q+1)}H_i^2 2^j(\sqrt{T}\log T H_i 2^j + T(H_i 2^j)^{2/3})$$
$$\ll T\log T\, 4^{-i}2^{-j(q-1)} + T^{17/12-\alpha_k/3}2^{-5/(3i)}2^{-j(q-2/3)}.$$

Therefore,

$$I_j^*(T) \ll T^{5/4}2^{-j(q-1)} + T^{3/2-2\alpha_k}2^{-j(q-2/3)}.$$

In view of (3.6) this proves (3.3) and therefore part (i) of the Proposition.

(ii) We insert (3.2) and the definition of $\psi_H^*(\cdot)$ into the left-hand side of (ii), transform the inner sums over $n$ by Lemma 2, and take the imaginary part of (3.5) to obtain

$$\int_T^{2T} (P_k^*(t))^2\,dt = \frac{64}{\pi^2(k-1)}\int_T^{2T} t^{1/2}(S_h^*(t))^2\,dt + O(T^{5/4}(\log T)^3), \tag{3.10}$$

where

$$S_h^*(t) := \sideset{}{'}\sum_{(h,m)\in\mathcal{D}(T)} \tau\left(\frac{h}{H+1}\right)(hm)^{-1+q/2}|(h,m)|_q^{-q+1/2} \times \cos(2\pi\sqrt{t}\,|(h,m)|_q + \pi/4),$$

and $H = [T^{1/4+\alpha_k}]$, anticipating that the first term of (3.10) is bounded by $O(T^{3/2})$. The domain of summation is given by

$$\mathcal{D}(T) = \{(h,m)\in\mathbb{N}^2 : 1\le h\le T^{1/4+\alpha_k},\ h\le m\le f'(N_J)\},$$

and $\sum'$ indicates that the terms corresponding to $h=m$ are weighted with the factor $1/2$. For a large real parameter $M$, we define the set

$$\mathcal{B}(M) := \{(h,m)\in\mathbb{N}^2 : h\le m,\ |(h,m)|_q\le M\},$$

such that $\mathcal{B}(M)\subset\mathcal{D}(T)$. We write the sum $S_h^*(t)$ as

$$\Big\{\sideset{}{'}\sum_{(h,m)\in\mathcal{B}(M)} + \sideset{}{'}\sum_{\substack{(h,m)\in\mathcal{D}(T)\\ |(h,m)|_q>M}}\Big\}\tau\left(\frac{h}{H+1}\right)(hm)^{-1+q/2}|(h,m)|_q^{-q+1/2} \times\cos(2\pi\sqrt{t}\,|(h,m)|_q+\pi/4) =: \Sigma_1(t)+\Sigma_2(t).$$

In what follows we choose

$$M = T^{6\alpha_k}. \tag{3.11}$$

Let us first consider

$$\int_T^{2T} t^{1/2}(\Sigma_2(t))^2\,dt. \tag{3.12}$$

Repeating the proof of (3.7) above, with $S_h(T)$ replaced by $\Sigma_2(t)$, we conclude with the notation there that

$$\begin{aligned}\int_T^{2T} t^{1/2}\Big(\sum_{h\in\mathcal{H}_i}\Sigma_2(t)\Big)^2\,dt &\ll 2^{-j(q-1)}T\log T + 2^{-j(q-2/3)}H_i^{-1/3}T^{3/2}\\ &\ll 2^{-j(q-1)}T\log T + M^{-1/3+\varepsilon}H_i^{-\varepsilon}2^{-j/(k-1)}T^{3/2},\end{aligned}$$

with a fixed positive $\varepsilon > 0$. Therefore, (3.12) is $\ll T^{3/2-2\alpha_k+\varepsilon}$.

By (3.10) and the Cauchy–Schwarz inequality, this implies that, again anticipating that the main term on the right-hand side of (3.10) is $O(T^{3/2})$, we have

$$\int_T^{2T}(P_k^*(t))^2\,dt = \frac{64}{\pi^2(k-1)}\int_T^{2T} t^{1/2}(\Sigma_1(t))^2\,dt + O(T^{3/2-\alpha_k+\varepsilon}).$$

The next step is to get rid of Vaaler's smoothing factors $\tau(\cdot)$, i.e. to approximate $\Sigma_1(t)$ by

$$\Sigma(t) := \sideset{}{'}\sum_{(h,m)\in\mathcal{B}(M)} (hm)^{-1+q/2}|(h,m)|_q^{-q+1/2}\cos(2\pi\sqrt{t}\,|(h,m)|_q + \pi/4).$$

In view of the Taylor expansion $\tau(x) = 1+O(x^2)$ for $x\to 0$, and the estimate

$$\sum_{(h,m)\in\mathcal{B}(M)} (hm)^{-1+q/2}|(h,m)|_q^{-q+1/2} \ll \sum_{m\le M}\sum_{h\le m} m^{-1/2-q/2}h^{-1+q/2} \ll M^{1/2} \tag{3.13}$$

it follows that

$$\Sigma(t) - \Sigma_1(t) \ll T^{-1/2+13\alpha_k}.$$

We therefore conclude that

$$\int_T^{2T}(P_k^*(t))^2\,dt = \frac{64}{\pi^2(k-1)}\int_T^{2T} t^{1/2}(\Sigma(t))^2\,dt + O(T^{3/2-\alpha_k+\varepsilon}). \tag{3.14}$$

Write $\mathbf{u} = (u_1,u_2)$, $\mathbf{v} = (v_1,v_2)$ for elements of $\mathbb{N}^2\cap\{(w_1,w_2): w_1\le w_2\}$. Squaring out $(\Sigma(t))^2$ and using the elementary formula

$$\cos A\cos B = \tfrac{1}{2}(\cos(A-B)+\cos(A+B)),$$

we can write

$$(\Sigma(t))^2 := S_0 - S_1(T) + S_2(t,T) + S_3(t,T),$$

where

$$S_0 := \frac{1}{2} \sideset{}{'}\sum_{\substack{|\mathbf{u}|_q=|\mathbf{v}|_q \\ u_1\le u_2, v_1\le v_2}} (u_1u_2v_1v_2)^{-1+q/2}(|\mathbf{u}|_q|\mathbf{v}|_q)^{-q+1/2},$$

$$S_1(T) := \frac{1}{2} \sideset{}{'}\sum_{\substack{\mathbf{u},\mathbf{v}\notin\mathcal{B}(M) \\ |\mathbf{u}|_q=|\mathbf{v}|_q}} (u_1u_2v_1v_2)^{-1+q/2}(|\mathbf{u}|_q|\mathbf{v}|_q)^{-q+1/2},$$

$$S_2(t,T) := \frac{1}{2} \sideset{}{'}\sum_{\substack{\mathbf{u},\mathbf{v}\in\mathcal{B}(M) \\ |\mathbf{u}|_q\ne|\mathbf{v}|_q}} (u_1u_2v_1v_2)^{-1+q/2}(|\mathbf{u}|_q|\mathbf{v}|_q)^{-q+1/2} \times \cos(2\pi\sqrt{t}\,(|\mathbf{u}|_q - |\mathbf{v}|_q)),$$

$$S_3(t,T) := \frac{1}{2} \sideset{}{'}\sum_{\mathbf{u},\mathbf{v}\in\mathcal{B}(M)} (u_1u_2v_1v_2)^{-1+q/2}(|\mathbf{u}|_q|\mathbf{v}|_q)^{-q+1/2} \times \cos(2\pi\sqrt{t}\,(|\mathbf{u}|_q + |\mathbf{v}|_q) + \pi/2).$$

We will show that the main term on the left-hand side of (3.14) comes from $S_0$. Indeed, the contribution of $S_2(t,T)$ is

$$\ll \sum_{\substack{|\mathbf{u}|_q,|\mathbf{v}|_q\le M \\ |\mathbf{u}|_q\ne|\mathbf{v}|_q}} (u_1u_2v_1v_2)^{-1+q/2}(|\mathbf{u}|_q|\mathbf{v}|_q)^{-q+1/2} \frac{T}{||\mathbf{u}|_q - |\mathbf{v}|_q|} \ll T^{3/2-\alpha_k}$$

by (3.13), since

$$\int_T^{2T} t^{1/2}\cos\left(2\pi\sqrt{t}\,(|\mathbf{u}|_q - |\mathbf{v}|_q)\right)dt \ll \frac{T}{||\mathbf{u}|_q - |\mathbf{v}|_q|} \ll TM^{q-1+k(k-1)^4}$$

by Lemma 4. The contribution of $S_3(t,T)$ is clearly not more than this. Finally consider the contribution of $S_1(T)$. For positive integers $u_1$, $u_2$, $v_1$, $v_2$ with $u_1 \le u_2$, $v_1 \le v_2$, the condition $|\mathbf{u}|_q = |\mathbf{v}|_q$ is satisfied if and only if either $(u_1,u_2) = (v_1,v_2)$ or $u_1$, $u_2$, $v_1$, $v_2$ all have the same maximal $(k-1)$-free divisor $r$, say, i.e.,

$$u_1 = a^{k-1}r, \quad u_2 = b^{k-1}r, \quad v_1 = c^{k-1}r, \quad v_2 = d^{k-1}r,$$

with $a,b,c,d \in \mathbb{N}$ satisfying $a^k + b^k = c^k + d^k$. This follows from the fact that the $(k-1)$th roots of distinct $(k-1)$-free positive integers are linearly independent over $\mathbb{Q}$ (see Besicovitch [1]). Therefore,

$$S_1(T) \ll \sum_{\substack{u_1=1 \\ u_2\gg M}}^{\infty} (u_1u_2)^{-2+q}(u_1u_2)^{-q+1/2} + \mathcal{R}(T), \tag{3.15}$$

with

$$\mathcal{R}(T) = \sum_{\substack{a\le b,c\le d\\ b^{k-1}r,d^{k-1}r\gg M}} (abcd)^{(k-1)(-1+q/2)}r^{-3} \times (|(a^{k-1},b^{k-1})|_q|(c^{k-1},d^{k-1})|_q)^{-q+1/2},$$

since $|(u_1,u_2)|_q = r|(a^{k-1},b^{k-1})|_q$. The first term in (3.15) is $\ll M^{-1/2}$. We estimate $\mathcal{R}(T)$ in the cases $k=3,4$, resp. $k\ge 5$ in two different ways. In the first case we use

$$\frac{1}{|(x^{k-1},y^{k-1})|_q^{q-1/2}} \ll (xy)^{-\frac{1}{2}(k-1)(q-1/2)},$$

to get

$$\begin{aligned}\mathcal{R}(T) &\ll \sum_{b^{k-1}r,d^{k-1}r\gg M}\ \sum_{a,c=1}^{\infty}(abcd)^{(k-1)(-1+q/2)}r^{-3}(abcd)^{-\frac{1}{2}(k-1)(q-1/2)}\\ &\ll \Big(\sum_{b^{k-1}r\gg M} r^{-3/2}b^{-\frac{3}{4}(k-1)}\Big)^2 \ll \Big(\sum_{r=1}^{\infty} r^{-3/2}\sum_{b^{k-1}\gg M/r} b^{-\frac{3}{4}(k-1)}\Big)^2\\ &\ll M^{-1/2}.\end{aligned}$$

In the case $k\ge 5$ we use the fact that

$$\sum_{a,c=1}^{\infty}(ac)^{(k-1)(-1+q/2)} \ll 1$$

to get

$$\begin{aligned}\mathcal{R}(T) &\ll \sum_{b^{k-1}r,d^{k-1}r\gg M}(bd)^{(k-1)(-1+q/2-q+1/2)}r^{-3}\\ &\ll \Big(\sum_{b=1}^{\infty} b^{-k+1/2}\sum_{r\gg M/b^{k-1}} r^{-3/2}\Big)^2 \ll M^{-1}.\end{aligned}$$

By (3.11), we therefore conclude that

$$S_1(T)\int_T^{2T} t^{1/2}\,dt \ll T^{3/2-\alpha_k},$$

which completes the proof of (ii).

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