On sums of two $k$th powers: an asymptotic formula for the mean square of the error term

by

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1. Introduction. For a fixed natural number $k \geq 2$ we consider the arithmetic function $r_k(n)$ which counts the number of ways to write the positive integer $n$ as a sum of the $k$th powers of two integers taken absolutely:

$$r_k(n) = \#\{(u_1, u_2) \in \mathbb{Z}^2 : |u_1|^k + |u_2|^k = n\}.$$

To study the average order of this arithmetic function, one is interested in the Dirichlet summatory function

$$A_k(t) = \sum_{1 \leq n \leq t^{k/2}} r_k(n),$$

where $t$ is a large real variable. For the special case $k = 2$, Gauß proved that

$$A_2(t) = \pi t + P_2(t),$$

with $P_2(t) \ll t^{1/2}$. Since then the question of the exact order of $P_2(t)$ has been called the circle problem of Gauß. For an exposition of its history, see e.g. the textbook of Krätzel [10]. At present the sharpest upper bound is

$$(1.1) \quad P_2(t) = O(t^{23/73} (\log t)^{315/146}),$$
due to Huxley [7], [8]. In the opposite direction the best results to date are

$$P_2(t) = \Omega_-(t \log t)^{1/4} (\log \log t)^{(\log 2)/4} \exp (-c \sqrt{\log \log \log t}) \quad (c > 0),$$

and

$$P_2(t) = \Omega_+ (t^{1/4} \exp (c' (\log \log t)^{1/4} (\log \log \log t)^{-3/4})) \quad (c' > 0),$$
due to Hafner [5], and Corrádi and Kátai [2], respectively. They refined earlier work of Hardy [6], resp. Gangadharan [3]. It is usually conjectured that

$$\inf \{\theta \in \mathbb{R} : P_2(t) \ll_\theta t^\theta\} = 1/4.$$
This hypothesis is supported by the mean square result
\begin{equation}
\int_0^X (P_2(t))^2 \, dt = CX^{3/2} + O(X(\log X)^2), \quad C = \frac{1}{3\pi^2} \sum_{n=1}^\infty \frac{\left(\frac{r_2(n)}{n}\right)^2}{n^{3/2}},
\end{equation}
which has been established (in this sharp form) by Kátai [9].

For \( k \geq 3 \), the asymptotic formula for \( A_k(t) \) contains a second main term which comes from the points of the boundary curve where the curvature vanishes. It turns out that
\begin{equation}
A_k(t) = \frac{2\Gamma^2(1/k)}{k\Gamma(2/k)} t + B_k \Phi_k(t)t^{1/2 - 1/(2k)} + P_k(t),
\end{equation}
where
\begin{align*}
B_k &= 2^{3-1/k} \pi^{-1/k} k^{1/k} \Gamma \left(1 + \frac{1}{k}\right), \\
\Phi_k(t) &= \sum_{n=1}^\infty n^{-1-1/k} \sin \left(2\pi n \sqrt{t} - \frac{\pi}{2k}\right).
\end{align*}

A thorough account on the history (which goes back to van der Corput [19]) and the diverse aspects of this problem can be found in the textbook of Krätzel [10]. Using Huxley’s deep method in its sharpest form, Kuba [12] proved that the new error term \( P_k(t) \) again satisfies the estimate \((1.1)\). Quite recently Nowak [14] was able to show that this analogy partially extends to the order of the mean square, i.e.
\begin{equation}
\int_0^X (P_k(t))^2 \, dt \ll X^{3/2},
\end{equation}
for a large real parameter \( X \) (the \( \ll \)-constant possibly depending on \( k \)). Concerning lower estimates it is known that
\begin{equation}
P_k(t) = \begin{cases} 
\Omega_{-}(t \log t)^{1/4} & \text{for } k \geq 3, \\
\Omega_{+}(t \log \log t)^{1/4} & \text{for } k = 3,
\end{cases}
\end{equation}
due to Nowak [15] and Nowak, Schoissengeier, Wooley and the author [13], which corresponds to the results of Hardy [6], resp. Gangadharan [3]. See also earlier works of Krätzel [11] and Schnabel [17] where somewhat weaker estimates were obtained.

The proof of the estimate \((1.2)\) uses the fact that the generating function \( \sum_{n=1}^\infty r_2(n)/n^s (\Re s > 1) \) of the number of lattice points on the circle satisfies a functional equation. For general Dirichlet series satisfying a functional equation with multiple gamma factors, Redmond [16] proved a mean-square asymptotic formula for the error term of the summatory function. In the case
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\( k \geq 3 \), such a handy functional equation is not available. Using a different method we improve (1.4) to

\[
\int_0^X (P_k(t))^2 \, dt \sim C_k X^{3/2} \quad (C_k > 0)
\]

with an explicitly given error term.

**Notation.** For any fixed natural number \( k \) let \( q \) be defined by \( 1/k + 1/q = 1 \), i.e. \( q = k/(k - 1) \). Further, let \( | \cdot |_q \) denote the \( q \)-norm in \( \mathbb{R}^2 \), i.e.

\[
|(w_1, w_2)|_q = (|w_1|^q + |w_2|^q)^{1/q}.
\]

**Theorem.** For any fixed integer \( k \geq 3 \), the error term \( P_k(t) \) defined in (1.3) satisfies

\[
\int_0^X (P_k(t))^2 \, dt = C_k X^{3/2} + O(X^{3/2 - \alpha_k + \varepsilon})
\]

for any \( \varepsilon > 0 \), where

\[
C_k = \frac{16}{3\pi^2 (k - 1)} \sum_{a,b,c,d \in \mathbb{N}} \frac{(abcd)^{-1+q/2} |(a,b)|_{q}^{-2q+1}}{|(a,b)|_q = |(c,d)|_q}
\]

and

\[
12(q + 1/6 + k(k - 1)^4).
\]

**Remarks.** 1. The convergence of the above series will be a by-result of our proof: see (3.15) ff.

2. The constant \( \alpha_k \) can certainly be improved by a more elaborate analysis. The author did not invest much effort to obtain the optimal \( \alpha_k \) in reach of the present method.

3. It is natural to compare our constant \( C_k \) with the constant \( C \) of (1.2) which may be written in the form

\[
C = \frac{1}{3\pi^2} \sum_{a,b,c,d \in \mathbb{Z}} |(a,b)|_{2}^{-3} \quad 0 < |(a,b)|_{2} = |(c,d)|_{2}
\]

We notice that this latter sum ranges also over the pairs \( (a, b), (c, d) \) with one vanishing component, in contrast to the series for \( C_k, k \geq 3 \). The reason for this is that the Lamé’s curve has curvature 0 at its points of intersection with the coordinate axes: These give rise to the second main term in (1.3).
2. Some lemmas

**Lemma 1** (see Vaaler [18]). For arbitrary $w \in \mathbb{R}$ and $H \in \mathbb{N}$, let

$$
\psi(w) = w - \lfloor w \rfloor - 1/2, \quad \psi^*_H(w) = -\frac{1}{\pi} \sum_{h=1}^{H} \frac{\sin(2\pi hw)}{h} \tau\left(\frac{h}{H+1}\right),
$$

where

$$
\tau(x) = \pi x(1-x) \cot(\pi x) + x \quad \text{for } 0 < x < 1.
$$

Then

$$
|\psi(w) - \psi^*_H(w)| \leq \frac{1}{H+1} \sum_{h=1}^{H} \left( 1 - \frac{h}{H+1} \right) \cos(2\pi hw) + \frac{1}{2H+2}.
$$

**Lemma 2.** Let $f(w)$ be a real-valued function with continuous derivatives up to the fourth order on the interval $[A,B]$. Let $L$ and $U$ be real parameters not less than 1 such that $B - A \asymp L$,

$$
f^{(j)}(w) \ll UL^{1-j} \quad \text{for } w \in [A,B], \quad j = 1, 2, 3, 4,
$$

and, for some $C > 0$,

$$
f''(w) \geq CUL^{-1} \quad \text{for } w \in [A,B].
$$

Suppose that $f'(A)$ and $f'(B)$ are integers, and denote by $\phi$ the inverse function of $f'$. Then

$$
\sum_{A \leq n \leq B} e(f(n)) = e\left(\frac{1}{8}\right) \sum_{f'(A) \leq m \leq f'(B)}'' e(f(\phi(m)) - m\phi(m)) \sqrt{f''(\phi(m))} + O(\log(1+U)),
$$

where $e(u) = e^{2\pi i u}$ for real $u$, and $\sum''$ indicates that the terms corresponding to $m = f'(A)$, resp. $m = f'(B)$ are weighted with the factor 1/2. The implied $O$-constant depends at most on $C$ and on the constants implied in the order symbols in the suppositions.

To prove Lemma 2, let us first state the following result.

**Lemma 3.** Let $F(w)$ be a real-valued function with continuous derivatives up to the fourth order on the interval $[A,B]$. Let $L$ and $U$ be real parameters not less than 1 such that $B - A \asymp L$,

$$
F^{(j)}(w) \ll UL^{1-j} \quad \text{for } w \in [A,B], \quad j = 1, 2, 3, 4,
$$

and, for some $C > 0$,

$$
F''(w) \geq CUL^{-1} \quad \text{for } w \in [A,B].
$$
Suppose that there exists a value \( c \in [A, B] \) for which \( F'(c) = 0 \). Then
\[
\int_A^B e(F(w)) \, dw = \begin{cases} 
\frac{1}{2} (F''(A))^{-1/2} e\left(\frac{1}{8} + F(A)\right) + O\left(\frac{1}{F'(B)}\right) + O\left(\frac{1}{U}\right) & \text{if } c = A, \\
\frac{1}{2} (F''(B))^{-1/2} e\left(\frac{1}{8} + F(B)\right) + O\left(\frac{1}{F'(A)}\right) + O\left(\frac{1}{U}\right) & \text{if } c = B, \\
(F''(c))^{-1/2} e\left(\frac{1}{8} + F(c)\right) + O\left(\frac{1}{F'(A)} + \frac{1}{F'(B)}\right) + O\left(\frac{1}{U}\right) & \text{else.}
\end{cases}
\]

Proof. For \( A < c < B \), this is explicitly contained in Lemma 3.4 of Graham and Kolesnik [4]. The case \( c = B \) can be reduced to \( c = A \) by the substitution \( w \mapsto A + B - w \). Finally, to deal with the case \( c = A \), it suffices to have a close look at the proof of Lemma 3.4 in [4]: Here \( F \) is approximated by its quadratic Taylor expansion \( q(w) \) at the stationary point \( c \). If \( c = A \), the integral \( \int_A^B e(q(w)) \, dw \) can be evaluated by an obvious variant of Lemma 3.3 in [4], viz.
\[
\int_0^\infty e(Hw^2) \, dw = \frac{e(1/8)}{2\sqrt{2H}} + O\left(\frac{1}{HX}\right).
\]
The ingenious estimation of the remainder integral then works exactly as in [4].

Proof of Lemma 2. Again by [4], Lemma 3.5 (after taking conjugates),
\[
\sum_{A \leq n \leq B} e(f(n)) = \sum_{m=f'(A)+1}^{f'(B)+1} e(f(w) - mw) \, dw + O(\log(1 + U)).
\]
To each of the integrals on the right-hand side we apply Lemma 3. (The first and last one are estimated as \( O(1) \) by the first derivative test.) The main term produces no difficulties, and the error terms are readily
\[
\ll \frac{1}{f'(B) - f'(A)} + \sum_{m=f'(A)+1}^{f'(B)-1} \left(\frac{1}{m - f''(A)} + \frac{1}{f''(B) - m}\right) + \frac{1}{U} (f'(B) - f'(A) + 1)
\ll \log (1 + U).
\]

Lemma 4. Let \( s, t, u, v \) be natural numbers with \( \|(s, t)_q - (u, v)_q\|_q \neq 0 \).
Then
\[ \|(s, t)|_q - |(u, v)|_q \gtrsim M^{-(q^{-1} + k(k-1)^4)}, \]
where \( M = \max(s, t, u, v) \) and the implied \( \gtrsim \) constant depends at most on \( k \).

**Proof.** By the mean value theorem we have
\[ (s, t)|^q_q - |(u, v)|^q_q \ll M^{q-1}||(s, t)|_q - |(u, v)|_q|. \]
The left-hand side of (2.1) can be written as
\[ L := a^{1/r} + b^{1/r} - c^{1/r} - d^{1/r} \neq 0, \]
with natural numbers \( a = s^k, b = t^k, c = u^k, d = v^k, \) and \( r := k - 1 \).
Consider the field extension
\[ \mathbb{F} = \mathbb{Q}(a^{1/r}, b^{1/r}, c^{1/r}, d^{1/r}, e^{2\pi i/r}). \]
The corresponding Galois group \( G = \text{Gal}(\mathbb{F}/\mathbb{Q}) \) contains at most \( r^5 \) elements \( \chi \). It is clear that
\[ \left| \prod_{\chi \in G} \chi(L) \right| \geq 1, \]
since the left-hand side is the modulus of the norm of a nonzero algebraic integer. Furthermore, for every \( \chi \in G \),
\[ |\chi(L)| \leq |\chi(a^{1/r})| + |\chi(b^{1/r})| + |\chi(c^{1/r})| + |\chi(d^{1/r})| \leq 4M^{k/r}. \]
Consequently,
\[ |L| \geq \prod_{\chi \in G \setminus \{\text{id}\}} |\chi(L)|^{-1} \gtrsim M^{-r^4}, \]
which establishes Lemma 4.

**3. Proof of the Theorem.** We use \( \varepsilon \) to denote an arbitrary small positive quantity which need not be the same at each occurrence. The constants implied in the symbols \( \ll \) and \( O \) may depend on \( \varepsilon \) and \( k \). We start from formulae (3.57), (3.58) of Krätzel [10], p. 148, and the asymptotic expansion below:
\[ P_k(t) = -8 \sum_{\alpha \sqrt{t} \leq n \leq \sqrt{t}} \psi((t^{k/2} - n^k)^{1/k}) + O(1), \]
with \( \psi(w) = w - [w] - 1/2 \) throughout, and \( \alpha := 2^{-1/k} \).

In what follows, let \( T \) be sufficiently large and \( t \in [T, 2T] \). We split up the domain of summation into subintervals \( N_j(t) = [N_j, N_{j+1}] \), where
\[ N_j = \frac{\sqrt{t}}{(1 + 2^{-jq})^{1/k}}, \quad j = 0, 1, \ldots, J, \]
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with $J$ minimal such that $\sqrt{t} - N_j < 1$ for $T \leq t \leq 2T$. Thus

$$P_k(t) = -8 \sum_{j=0}^{J} \sum_{n \in N_j(t)} \psi((t^{k/2} - n^k)^{1/k}) + O(\log T).$$

Further let

$$P^*_k(t) = -8 \sum_{j=0}^{J} \sum_{n \in N_j(t)} \psi^*_H((t^{k/2} - n^k)^{1/k}),$$

with $\psi^*_H(w)$ defined as in Lemma 1.

We will prove the following Proposition. By applying Cauchy’s inequality and summing over $T = X/2, X/4, \ldots$, our Theorem readily follows.

**Proposition.** For sufficiently large $T$ and $H = [T^{1/4 + \alpha_k}]$, with $\alpha_k$ defined as in (1.5), we have

(i) $$2 \int_T^{2T} |P_k(t) - P^*_k(t)|^2 dt \ll T^{3/2 - 2\alpha_k},$$

(ii) $$\int_T^{2T} |P^*_k(t)|^2 dt = \mathcal{H}(T) + O(T^{3/2 - \alpha_k + \varepsilon}),$$

with

$$\mathcal{H}(T) = \frac{8}{\pi^2(k - 1)} \sum_{a,b,c,d \in \mathbb{N}} (abcd)^{-1+q/2} |(a,b)|_q^{-2q+1} T^{1/2} \int_T^{2T} dt.$$

**Proof.** (i) By (3.1), (3.2) and Lemma 1, the left-hand side of (i) is

$$\ll \int_T^{2T} \left( \frac{1}{H^2} \sum_{1 \leq h \leq H} \sum_{j=0}^{J} \sum_{n \in N_j(t)} \cos(-2\pi h(t^{k/2} - n^k)^{1/k}) \right)^2 dt$$

$$+ O(T^{3/2 - 2\alpha_k}).$$

By Cauchy’s inequality, it thus suffices to show that there exists a constant $c_0 > 1$ such that, for $T$ sufficiently large and $0 \leq j \leq J$,

$$I_j(T) := \int_T^{2T} \left( \frac{1}{H^2} \sum_{1 \leq h \leq H} \sum_{n \in N_j(t)} \cos(-2\pi h(t^{k/2} - n^k)^{1/k}) \right)^2 dt$$

$$\ll c_0^{-1} T^{3/2 - 2\alpha_k}.$$

We transform each of the inner trigonometric sums over $n$ by Lemma 2, with $[A, B] = [N_j, N_{j+1}]$, and

$$f(w) = -h(t^{k/2} - w^k)^{1/k}.$$
We note that \( f'(N_j) \) is independent of \( t \), more precisely

\[
(3.4) \quad f'(N_j) = h 2^j \quad \text{and} \quad N_{j+1} - N_j \asymp \sqrt{T}/2^j.
\]

Calculating derivatives, we get

\[
\begin{align*}
f^{(1)}(w) &= h w^{k-1}(t^{k/2} - w^k)^{-1+1/k}, \\
f^{(2)}(w) &= h(k-1)t^{k/2}w^{k-2}(t^{k/2} - w^k)^{-2+1/k} \\
&\quad \times h T^{1/2-1/(2k)}(\sqrt{t} - w)^{-2+1/k} \\
&\quad \times h T^{1/2-1/(2k)}(\sqrt{T}/2^jq)^{-2+1/k} \\
&\quad \times h T^{1/2}2^j q^{(2-1/k)}, \\
f^{(3)}(w) &= h(k-1)t^{k/2}w^{k-3}(t^{k/2} - w^k)^{-3+1/k}((k-2)t^{k/2} + (k+1)w^k) \\
&\quad \times h T^{1/2-1/(2k)}(\sqrt{t} - w)^{-3+1/k} \\
&\quad \times h T^{1/2}2^j q^{(3-1/k)}, \\
f^{(4)}(w) &= h(k-1)t^{k/2}w^{k-4}(t^{k/2} - w^k)^{-4+1/k} \\
&\quad \times ((k-2)t^{k/2}(k-3)t^{k/2} + 2(k+1)w^k) \\
&\quad + (1+k)w^k((2k-3)t^{k/2} + (k+2)w^k) \\
&\quad \times h T^{1/2-1/(2k)}(\sqrt{t} - w)^{-4+1/k} \\
&\quad \times h T^{1/2}2^j q^{(4-1/k)}.
\end{align*}
\]

One easily verifies that the conditions of Lemma 2 are satisfied, with \( L = \sqrt{T}2^{-jq} \) and \( U = h 2^j \). We note that in view of (3.4) \( f'(N_j) \) and \( f'(N_{j+1}) \) are integers. We may thus apply the lemma to conclude by a straightforward calculation that, for \( T \leq t \leq 2T \),

\[
(3.5) \quad \sum_{n \in N_j(t)} e(-h(t^{k/2} - n^k)^{1/k}) = \frac{e(1/8)}{\sqrt{k-1}} h t^{1/4} \sum_{m \in M_j(h)} \left( hm \right)^{-1+q/2} \| (h, m) \|_q^{-q+1/2} e(-\sqrt{t} \| (h, m) \|_q) \\
\quad \quad + O(j + \log h),
\]

where

\[
M_j(h) = [f'(N_j), f'(N_{j+1})].
\]

Therefore, using the real part of (3.5), we obtain

\[
(3.6) \quad I_j(T) \ll I_j^*(T) + T(\log T)^4
\]

with

\[
(3.7) \quad I_j^*(T) := \int_T^{2T} \left| \frac{1}{H^2} \sum_{1 \leq h \leq H} S_h(t) \right|^2 dt,
\]
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$$S_h(t) := \sum_{m \in \mathcal{M}_j(h)} h(m)^{-1+q/2} (h, m)^{-q+1/2} e(\sqrt{T} |(h, m)|_q).$$

To estimate the integral in (3.7) we follow the proof of Nowak [14]. We split up the domain of summation over $h$ into dyadic subintervals: Let

$$\mathcal{H}_i = [H_{i+1}, H_i], \quad H_i = H/2^i, \quad i = 0, 1, \ldots, I,$n

where $I$ is the largest integer for which $2^I < H$. By Cauchy’s inequality,

$$\left| \sum_{h=1}^{H} S_h(t) \right| \leq \sum_{i=0}^{I} H_i^{1/2} \sum_{h \in \mathcal{H}_i} S_h(t),$$

with some fixed $\varepsilon > 0$ sufficiently small.

In what follows, we write $u = (u_1, u_2), \ v = (v_1, v_2)$ for elements of $\mathbb{Z}^2$, and put

$$\mathcal{U}_{i,j} := \{(h, m) : h \in \mathcal{H}_i, \ m \in \mathcal{M}_j(h)\}.$$

By squaring and integrating term by term we get

$$\int_{\mathcal{H}_i} \left| \sum_{h=1}^{H} S_h(t) \right|^2 dt \ll \frac{T}{H^2} \sum_{u, v \in \mathcal{U}_{i,j}} \frac{u_1 v_1 (|u|_q |v|_q)^{-q+1/2}}{(u_1 u_2 v_1 v_2)^{1-q/2}} \left| \int_T^{2T} e\left((|u|_q - |v|_q)\right) dt \right|.$$

Recalling $f'(N_j) = h 2^j$ we conclude that $u = (u_1, u_2) \in \mathcal{U}_{i,j}$ implies that

(3.8) \hspace{1cm} u_1 \asymp H_i, \quad |u|_q \asymp u_2 \asymp H_i 2^j.$$

Consequently,

(3.9) \hspace{1cm} \int_{\mathcal{H}_i} \left| \sum_{h=1}^{H} S_h(t) \right|^2 dt \ll \frac{T^{1/2}}{H^2} H_i^{1/2} \sum_{u, v \in \mathcal{U}_{i,j}} \left( \sum_{v : |v|_q \leq |u|_q} \min \left( T, \frac{\sqrt{T}}{|u|_q - |v|_q} \right) \right).$$

For the inner sum over $v$ we have the estimate

$$\sum_{v : |v|_q \leq |u|_q} \min \left( T, \frac{\sqrt{T}}{|u|_q - |v|_q} \right) \ll |u|_q \sqrt{T} \log T + T |u|_q^{2/3}$$

(see Nowak [14], formula (2.14) and below). Inserting this in (3.9), and recalling (3.8) we obtain
\[
\int_{\mathcal{H}} T^{-1/2} \sum_{h \in \mathcal{H}} S_h(t)^2 dt \\
\ll \frac{T^{1/2}}{H^2} H^{-j(q+1)} H^2 T \sqrt{T \log T} H^2 \left( \sqrt{T \log T} T H^2 \right)^{2/3} \\
\ll T \log T 4^{-1} 2^{-j(q-1)} + T^{17/12} \alpha_k^2 2^{-j(q-2/3)}.
\]

Therefore,
\[
I^*_j(T) \ll T^{5/4} 2^{-j(q-1)} + T^{-3/2} 2^{-j(q-2/3)}.
\]

In view of (3.6) this proves (3.3) and therefore part (i) of the Proposition.

(ii) We insert (3.2) and the definition of \( \psi^*_H(\cdot) \) into the left-hand side of (ii), transform the inner sums over \( n \) by Lemma 2, and take the imaginary part of (3.5) to obtain
\[
(3.10) \quad \int_{\mathcal{H}} \left( P^*_k(t) \right)^2 dt = \frac{64}{\pi^2 (k - 1)} \int_{\mathcal{H}} t^{1/2} (S^*_h(t))^2 dt + O(T^{5/4} (\log T)^3),
\]

where
\[
S^*_h(t) := \sum'_{(h,m) \in \mathcal{D}(T)} \tau \left( \frac{h}{H+1} \right) (hm)^{-1+q/2} |(h,m)|_q^{-q+1/2} \\
\times \cos(2\pi \sqrt{t} |(h,m)|_q + \pi/4),
\]

and \( H = [T^{1/4+\alpha_k}] \), anticipating that the first term of (3.10) is bounded by \( O(T^{3/2}) \). The domain of summation is given by
\[
\mathcal{D}(T) = \left\{ (h, m) \in \mathbb{N}^2 : 1 \leq h \leq T^{1/4+\alpha_k}, h \leq m \leq f'(N_J) \right\},
\]

and \( \sum' \) indicates that the terms corresponding to \( h = m \) are weighted with the factor 1/2. For a large real parameter \( M \), we define the set
\[
\mathcal{B}(M) := \{(h, m) \in \mathbb{N}^2 : h \leq m, \ |(h,m)|_q \leq M \},
\]

such that \( \mathcal{B}(M) \subset \mathcal{D}(T) \). We write the sum \( S^*_h(t) \) as
\[
\left\{ \sum'_{(h,m) \in \mathcal{B}(M)} + \sum'_{(h,m) \in \mathcal{D}(T) \ |(h,m)|_q > M} \right\} \tau \left( \frac{h}{H+1} \right) (hm)^{-1+q/2} |(h,m)|_q^{-q+1/2} \\
\times \cos(2\pi \sqrt{t} |(h,m)|_q + \pi/4) \\
=: \Sigma_1(t) + \Sigma_2(t).
\]

In what follows we choose
\[
(3.11) \quad M = T^{6\alpha_k}.
\]
Let us first consider

\[(3.12) \int_T^{2T} t^{1/2} (\Sigma_2(t))^2 \, dt.\]

Repeating the proof of (3.7) above, with \( S_h(T) \) replaced by \( \Sigma_2(t) \), we conclude with the notation there that

\[
\int_T^{2T} t^{1/2} \left( \sum_{h \in H} \Sigma_2(t) \right)^2 \, dt \ll 2^{-j(q-1)} T \log T + 2^{-j(q-2/3)} H_i^{-1/3} T^{3/2}
\]

\[
\ll 2^{-j(q-1)} T \log T + M^{-1/3+\varepsilon} H_i^{-\varepsilon} 2^{-j/(k-1)} T^{3/2},
\]

with a fixed positive \( \varepsilon > 0 \). Therefore, (3.12) is \( \ll T^{3/2-2\alpha_k+\varepsilon} \).

By (3.10) and the Cauchy–Schwarz inequality, this implies that, again anticipating that the main term on the right-hand side of (3.10) is \( O(T^{3/2}) \), we have

\[
\int_T^{2T} (P_k^*(t))^2 \, dt = \frac{64}{\pi^2 (k-1)} \int_T^{2T} t^{1/2} (\Sigma_1(t))^2 \, dt + O(T^{3/2-\alpha_k+\varepsilon}).
\]

The next step is to get rid of Vaaler’s smoothing factors \( \tau(\cdot) \), i.e. to approximate \( \Sigma_1(t) \) by

\[
\Sigma(t) := \sum_{(h,m) \in B(M)} (hm)^{-1+q/2} |(h,m)|_q^{-q+1/2} \cos(2\pi \sqrt{t} |(h,m)|_q + \pi/4).
\]

In view of the Taylor expansion \( \tau(x) = 1 + O(x^2) \) for \( x \to 0 \), and the estimate

\[(3.13) \sum_{(h,m) \in B(M)} (hm)^{-1+q/2} |(h,m)|_q^{-q+1/2} \ll \sum_{m \leq M} \sum_{h \leq m} m^{-1/2-q/2} h^{-1+q/2} \ll M^{1/2},\]

it follows that

\[
\Sigma(t) - \Sigma_1(t) \ll T^{-1/2+13\alpha_k}.
\]

We therefore conclude that

\[(3.14) \int_T^{2T} (P_k^*(t))^2 \, dt = \frac{64}{\pi^2 (k-1)} \int_T^{2T} t^{1/2} (\Sigma(t))^2 \, dt + O(T^{3/2-\alpha_k+\varepsilon}).\]

Write \( u = (u_1, u_2) \), \( v = (v_1, v_2) \) for elements of \( \mathbb{N}^2 \cap \{(w_1, w_2) : w_1 \leq w_2\} \).

Squaring out \( (\Sigma(t))^2 \) and using the elementary formula

\[
\cos A \cos B = \frac{1}{2} (\cos(A - B) + \cos(A + B)),
\]

we can write

\[
(\Sigma(t))^2 := S_0 - S_1(T) + S_2(t, T) + S_3(t, T),
\]
We will show that the main term on the left-hand side of (3.14) comes from $a, b, c, d$ if either $(u, v)$ or $S$ satisfies $u, v \leq v_2$ with $u_1 \leq u_2, v_1 \leq v_2$, the condition $|u_q| = |v_q|$ is satisfied if and only if either $(u_1, u_2) = (v_1, v_2)$ or $u_1, u_2, v_1, v_2$ all have the same maximal $(k - 1)$-free divisor $r$, say, i.e.,

$$u_1 = a^{k-1}r, \quad u_2 = b^{k-1}r, \quad v_1 = c^{k-1}r, \quad v_2 = d^{k-1}r,$$

with $a, b, c, d \in \mathbb{N}$ satisfying $a^k + b^k = c^k + d^k$. This follows from the fact that the $(k - 1)$th roots of distinct $(k - 1)$-free positive integers are linearly independent over $\mathbb{Q}$ (see Besicovitch [1]). Therefore,

$$S_1(T) \ll \sum_{u_1, u_2 \geq M} (u_1u_2)^{-2+q} (u_1u_2)^{-q+1/2} + R(T),$$

where

$$S_0 := \frac{1}{2} \sum'_{u_1 \leq u_2, v_1 \leq v_2} (u_1u_2v_1v_2)^{-1+q/2} (|u_q| |v_q|)^{-q+1/2},$$

$$S_1(T) := \frac{1}{2} \sum'_{u, v \in B(M)} (u_1u_2v_1v_2)^{-1+q/2} (|u_q| |v_q|)^{-q+1/2},$$

$$S_2(t, T) := \frac{1}{2} \sum'_{u, v \in B(M)} (u_1u_2v_1v_2)^{-1+q/2} (|u_q| |v_q|)^{-q+1/2} \times \cos(2\pi \sqrt{t} (|u_q| - |v_q|)),$$

$$S_3(t, T) := \frac{1}{2} \sum'_{u, v \in B(M)} (u_1u_2v_1v_2)^{-1+q/2} (|u_q| |v_q|)^{-q+1/2} \times \cos(2\pi \sqrt{t} (|u_q| + |v_q|) + \pi/2).$$

We will show that the main term on the left-hand side of (3.14) comes from $S_0$. Indeed, the contribution of $S_2(t, T)$ is

$$\ll \sum_{|u_q|, |v_q| \leq M} (u_1u_2v_1v_2)^{-1+q/2} (|u_q| |v_q|)^{-q+1/2} \frac{T}{|u_q| - |v_q|} \ll T^{3/2-\alpha_k}$$

by (3.13), since

$$\int_T^{2T} t^{1/2} \cos(2\pi \sqrt{t} (|u_q| - |v_q|)) dt \ll \frac{T}{|u_q| - |v_q|} \ll TM^{-1+k(k-1)/2}$$

by Lemma 4. The contribution of $S_3(t, T)$ is clearly not more than this. Finally consider the contribution of $S_1(T)$. For positive integers $u_1, u_2, v_1, v_2$ with $u_1 \leq u_2, v_1 \leq v_2$, the condition $|u_q| = |v_q|$ is satisfied if and only if either $(u_1, u_2) = (v_1, v_2)$ or $u_1, u_2, v_1, v_2$ all have the same maximal $(k - 1)$-free divisor $r$, say, i.e.,

$$u_1 = a^{k-1}r, \quad u_2 = b^{k-1}r, \quad v_1 = c^{k-1}r, \quad v_2 = d^{k-1}r,$$

with $a, b, c, d \in \mathbb{N}$ satisfying $a^k + b^k = c^k + d^k$. This follows from the fact that the $(k - 1)$th roots of distinct $(k - 1)$-free positive integers are linearly independent over $\mathbb{Q}$ (see Besicovitch [1]). Therefore,
with
\[ R(T) = \sum_{a,b,c \leq d, b^{k-1}r, d^{k-1}r \gg M} (abcd)^{(k-1)(-1+q/2)} r^{-3} \]
\[ \times \left( |(a^{k-1}, b^{k-1})|_q |(c^{k-1}, d^{k-1})|_q \right)^{-q+1/2}, \]

since \(|(u_1, u_2)|_q = r|(a^{k-1}, b^{k-1})|_q\). The first term in (3.15) is \( \ll M^{-1/2} \). We estimate \( R(T) \) in the cases \( k = 3, 4 \), resp. \( k \geq 5 \) in two different ways. In the first case we use
\[ \frac{1}{|(a^{k-1}, y^{k-1})|_q} \ll (xy)^{-\frac{1}{2}(k-1)(q-1/2)}, \]
to get
\[ R(T) \ll \sum_{b^{k-1}r, d^{k-1}r \gg M} \sum_{a,c=1}^{\infty} (abcd)^{(k-1)(-1+q/2)} r^{-3} (abcd)^{-\frac{1}{2}(k-1)(q-1/2)} \]
\[ \ll \left( \sum_{b^{k-1}r \gg M} r^{-3/2} b^{-\frac{3}{2}(k-1)} \right)^2 \ll \left( \sum_{r=1}^{\infty} r^{-3/2} \sum_{b^{k-1}r \gg M/r} b^{-\frac{3}{2}(k-1)} \right)^2 \ll M^{-1/2}. \]

In the case \( k \geq 5 \) we use the fact that
\[ \sum_{a,c=1}^{\infty} (ac)^{(k-1)(-1+q/2)} \ll 1 \]
to get
\[ R(T) \ll \sum_{b^{k-1}r, d^{k-1}r \gg M} (bd)^{(k-1)(-1+q/2-q+1/2)} r^{-3} \]
\[ \ll \left( \sum_{b=1}^{\infty} b^{-k+1/2} \sum_{r \gg M/b^{k-1}} r^{-3/2} \right)^2 \ll M^{-1}. \]

By (3.11), we therefore conclude that
\[ S_1(T) \int_T^{2T} t^{1/2} dt \ll T^{3/2-\alpha_k}, \]
which completes the proof of (ii).

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References


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