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On the hybrid mean value of Dedekind sums and Hurwitz zeta-function

by

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1. Introduction. For a positive integer k and an arbitrary integer h, the Dedekind sum $S(h,k)$ is defined by

$$S(h,k)=\sum_{a=1}^{k}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-1/2 & \text{if } x \text{ is not an integer,}\\ 0 & \text{if } x \text{ is an integer.}\end{cases}$$

The various properties of $S(h,k)$ were investigated by many authors. Maybe the most famous property of the Dedekind sums is the reciprocity formula (see [2], [3], [5] and [6])

$$S(h,k)+S(k,h)=\frac{h^2+k^2+1}{12hk}-\frac{1}{4} \tag{1}$$

for all $(h,k)=1$, $h>0$, $k>0$. A three-term version of (1) was discovered by Rademacher [7]. Walum [8] has shown that for prime $p\ge 3$,

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^4=\frac{\pi^4(p-1)}{p^2}\sum_{h=1}^{p}|S(h,p)|^2 \tag{2}$$

and

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2=\frac{\pi^2(p-1)^2(p-2)}{12p^2}. \tag{3}$$

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Recently, J. B. Conrey *et al.* [4] studied the mean value distribution of $S(h,k)$, and proved the following important asymptotic formula:

$$\sum_{h=1}^{k}{}' |S(h,k)|^{2m} = f_m(k)\left(\frac{k}{12}\right)^{2m} + O((k^{9/5} + k^{2m-1+1/(m+1)})\log^3 k), \tag{4}$$

where $\sum_h'$ denotes summation over all h such that $(k,h)=1$, and

$$\sum_{n=1}^{\infty} \frac{f_m(n)}{n^s} = 2\frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\zeta(s).$$

In the spirit of [4] and [8], the author [9] obtained a sharper asymptotic formula for (4) with $m=1$ and $k=p^n$, where p is a prime:

$$\sum_{h=1}^{k}{}' |S(h,k)|^2 = \frac{5}{144}k^2\frac{(p^2-1)^2}{p(p^3-1)} + O\left(k\exp\left(\frac{3\log k}{\log\log k}\right)\right). \tag{5}$$

In this paper, as a note of [4] and [9], we shall give a hybrid mean value formula involving Dedekind sums and Hurwitz zeta-function. The constants implied by the O-symbols and the symbols $\ll$ used in this paper do not depend on any parameter, unless otherwise indicated. By using the estimates for character sums and the mean value theorem for Dirichlet L-functions, we shall prove the following main result:

Theorem. *Let $q \ge 3$ be an integer. Then for any fixed positive integer m, we have the asymptotic formula*

$$\sum_{a=1}^{q}{}' \zeta^2\left(\frac{1}{2},\frac{a}{q}\right)S^{2m}(a,q) = \frac{q^{2m+1}}{(12)^{2m}}\zeta(2m+1)\prod_{p\mid q}\left(1-\frac{1}{p^{2m+1}}\right) + O\left(q^{2m+1/2}\exp\left(\frac{3\log q}{\log\log q}\right)\right),$$

where $\zeta(s,\alpha)$ is the Hurwitz zeta-function, $\zeta(s)$ is the Riemann zeta-function, and $\exp(y)=e^y$.

From this Theorem we may immediately deduce the following

Corollary. *Let p be an odd prime. Then for any fixed positive integer m, we have the asymptotic formula*

$$\sum_{a=1}^{p-1} \zeta^2\left(\frac{1}{2},\frac{a}{p}\right)S^{2m}(a,p) = \frac{\zeta(2m+1)}{(12)^{2m}}p^{2m+1} + O\left(p^{2m+1/2}\exp\left(\frac{3\log p}{\log\log p}\right)\right).$$

2. Some lemmas. To prove the Theorem, we need the following lemmas:

LEMMA 1. *Let $q \geq 3$ be an integer with $(a, q) = 1$. Then*

$$S(a,q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2,$$

where $\phi(d)$ is the Euler function, χ denotes a Dirichlet character modulo d with $\chi(-1) = -1$, and $L(s, \chi)$ denotes the Dirichlet L-function corresponding to χ.

Proof. See [9].

LEMMA 2. *Let $q \geq 3$ and m be positive integers and let χ be any Dirichlet character modulo q. Then*

$$\begin{aligned}\sum_{a=1}^{q} \chi(a)\zeta(s, a/q) S^m(a,q) = \frac{q^{s-m}}{\pi^{2m}} \sum_{d_1|q} \dots \sum_{d_m|q} \frac{d_1^2 \dots d_m^2}{\phi(d_1)\dots\phi(d_m)} \\ \times \sum_{\substack{\chi_1 \bmod d_1 \\ \chi_1(-1)=-1}} \dots \sum_{\substack{\chi_m \bmod d_m \\ \chi_m(-1)=-1}} L(s, \chi\chi_1 \dots \chi_m) \\ \times |L(1,\chi_1)|^2 \dots |L(1,\chi_m)|^2,\end{aligned}$$

where $s = \sigma + it$, $1/2 \leq \sigma < 1$.

Proof. For any complex number $s = \sigma + it$ with $1/2 \leq \sigma < 1$, from [1] we know that

$$L(s,\chi) = \frac{1}{q^s} \sum_{a=1}^{q} \chi(a)\zeta\left(s, \frac{a}{q}\right).$$

Applying this identity and Lemma 1 we immediately get

$$\begin{aligned}&\sum_{a=1}^{q} \chi(a)\zeta(s, a/q) S^m(a,q) \\ &\quad = \frac{1}{\pi^{2m} q^m} \sum_{d_1|q} \dots \sum_{d_m|q} \frac{d_1^2 \dots d_m^2}{\phi(d_1)\dots\phi(d_m)} \\ &\qquad \times \sum_{\substack{\chi_1 \bmod d_1 \\ \chi_1(-1)=-1}} \dots \sum_{\substack{\chi_m \bmod d_m \\ \chi_m(-1)=-1}} \left(\sum_{a=1}^{q} \chi(a)\chi_1(a)\dots\chi_m(a)\zeta\left(s, \frac{a}{q}\right)\right) \\ &\qquad \times |L(1,\chi_1)|^2 \dots |L(1,\chi_m)|^2\end{aligned}$$

$$= \frac{q^{s-m}}{\pi^{2m}} \sum_{d_1 \mid q} \dots \sum_{d_m \mid q} \frac{d_1^2 \dots d_m^2}{\phi(d_1) \dots \phi(d_m)}$$

$$\times \sum_{\substack{\chi_1 \bmod d_1 \\ \chi_1(-1)=-1}} \dots \sum_{\substack{\chi_m \bmod d_m \\ \chi_m(-1)=-1}} L(s, \chi\chi_1 \dots \chi_m) |L(1, \chi_1)|^2 \dots |L(1, \chi_m)|^2.$$

This proves Lemma 2.

Lemma 3. *Let $q \ge 3$ be an integer, let χ denote an odd Dirichlet character modulo d with $d \mid q$, and χ_1 be any Dirichlet character modulo q. Then for any fixed positive integer m, we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} L(m - 1/2, \chi_1\chi) |L(1, \chi)|^2$$

$$= \frac{\pi^2}{12} \phi(d) L(m + 1/2, \chi_1) \prod_{p \mid d} \left(1 - \frac{1}{p^2}\right) + O\left(\frac{q}{\sqrt{d}} \exp\left(\frac{2 \log d}{\log \log d}\right)\right).$$

Proof. For simplicity we only prove the statement for $m = 1$. Other cases are similar. Let $A(y, \chi) = \sum_{d < a \le y} \chi(a)$, χ be any odd character modulo d, and let χ_q^0 denote the principal character modulo q. If $\chi\chi_1 \ne \chi_q^0$, then

$$L(1/2, \chi\chi_1) = \sum_{1 \le n \le d} \frac{\chi\chi_1(n)}{n^{1/2}} + \frac{1}{2} \int_d^\infty \frac{A(y, \chi\chi_1)}{y^{3/2}}\, dy,$$

$$L(1, \chi) = \sum_{1 \le n \le d} \frac{\chi(n)}{n} + \int_d^\infty \frac{A(y, \chi)}{y^2}\, dy,$$

so that

$$\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1 \\ \chi\chi_1 \ne \chi_q^0}} L(1/2, \chi\chi_1) |L(1, \chi)|^2 \tag{6}$$

$$= \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1 \\ \chi\chi_1 \ne \chi_q^0}} \left(\sum_{1 \le l \le d} \frac{\chi\chi_1(l)}{l^{1/2}} + \frac{1}{2} \int_d^\infty \frac{A(y, \chi\chi_1)}{y^{3/2}}\, dy \right)$$

$$\times \left(\sum_{1 \le m \le d} \frac{\overline{\chi}(m)}{m} + \int_d^\infty \frac{A(y, \overline{\chi})}{y^2}\, dy \right)$$

$$\times \left(\sum_{1 \le n \le d} \frac{\chi(n)}{n} + \int_d^\infty \frac{A(y, \chi)}{y^2}\, dy \right).$$

Note that for $(lmn, d) = 1$, from the orthogonality relations for character sums modulo d, we have

$$\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(l)\chi(n)\overline{\chi}(m) = \begin{cases} \frac{1}{2}\phi(d) & \text{if } ln \equiv m \bmod d, \\ -\frac{1}{2}\phi(d) & \text{if } ln \equiv -m \bmod d, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Now we can use (7) to estimate each term on the right side of (6). First we have

$$\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \bigg(\sum_{1\le l\le d} \frac{\chi\chi_1(l)}{l^{1/2}}\bigg)\bigg(\sum_{1\le m\le d} \frac{\overline{\chi}(m)}{m}\bigg)\bigg(\sum_{1\le n\le d} \frac{\chi(n)}{n}\bigg) \tag{8}$$

$$= \frac{\phi(d)}{2} \sum_{1\le l<d} \sideset{}{'}\sum_{\substack{1\le m<d \\ ln\equiv m \,(\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1\le n<d} \frac{\chi_1(l)}{\sqrt{l}mn} - \frac{\phi(d)}{2} \sum_{1\le l<d} \sideset{}{'}\sum_{\substack{1\le m<d \\ ln\equiv -m \,(\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1\le n<d} \frac{\chi_1(l)}{\sqrt{l}mn}$$

$$= \frac{\phi(d)}{2} \sum_{1\le l<d} \sideset{}{'}\sum_{\substack{1\le m<d \\ ln=m}} \sideset{}{'}\sum_{1\le n<d} \frac{\sqrt{l}\chi_1(l)}{lmn} + \frac{\phi(d)}{2} \sum_{1\le l<d} \sideset{}{'}\sum_{\substack{1\le m<d \\ ln\equiv m \,(\mathrm{mod}\, d) \\ ln>d}} \sideset{}{'}\sum_{1\le n<d} \frac{\sqrt{l}\chi_1(l)}{lmn}$$

$$- \frac{\phi(d)}{2} \sum_{1\le l<d} \sideset{}{'}\sum_{\substack{1\le m<d \\ ln=d-m}} \sideset{}{'}\sum_{1\le n<d} \frac{\sqrt{l}\chi_1(l)}{lmn} - \frac{\phi(d)}{2} \sum_{1\le l<d} \sideset{}{'}\sum_{\substack{1\le m<d \\ ln\equiv -m \,(\mathrm{mod}\, d) \\ ln>d}} \sideset{}{'}\sum_{1\le n<d} \frac{\sqrt{l}\chi_1(l)}{lmn}.$$

Note that

$$\frac{\phi(d)}{2} \sideset{}{'}\sum_{1\le l<d} \sideset{}{'}\sum_{\substack{1\le m<d \\ ln=m}} \sideset{}{'}\sum_{1\le n<d} \frac{\sqrt{l}\chi_1(l)}{lmn} \tag{9}$$

$$= \frac{\phi(d)}{2} \sideset{}{'}\sum_{t=1}^{d} \frac{\sum_{n|t} \sqrt{n}\chi_1(n)}{t^2}$$

$$= \frac{\phi(d)}{2} \sideset{}{'}\sum_{t=1}^{\infty} \frac{\sum_{n|t} \sqrt{n}\chi_1(n)}{t^2} + O\bigg(\phi(d) \sum_{t=d}^{\infty} \frac{\sum_{n|t} 1}{t^{3/2}}\bigg)$$

$$= \frac{\phi(d)}{2} \bigg(\sum_{t=1}^{\infty} \frac{\chi_1(t)}{t^{3/2}}\bigg)\bigg(\sideset{}{'}\sum_{n=1}^{\infty} \frac{1}{n^2}\bigg) + O\bigg(\frac{\phi(d)}{\sqrt{d}}\bigg)$$

$$= \frac{\pi^2}{12}\phi(d) L\bigg(\frac{3}{2}, \chi_1\bigg) \prod_{p|d} \bigg(1 - \frac{1}{p^2}\bigg) + O\bigg(\frac{\phi(d)}{\sqrt{d}}\bigg),$$

$$\frac{\phi(d)}{2}\sum_{\substack{1\le l<d\\ ln\equiv m\,(\mathrm{mod}\,d)\\ ln>d}}\ \sideset{}{'}\sum_{1\le m<d}\ \sideset{}{'}\sum_{1\le n<d}\frac{\sqrt{l}\chi_1(l)}{lmn} \tag{10}$$
$$= \frac{\phi(d)}{2}\sum_{1\le u<d}\sum_{\substack{1\le l<d\\ ln=ud+m}}\sideset{}{'}\sum_{1\le m<d}\sideset{}{'}\sum_{1\le n<d}\frac{\sqrt{l}\chi_1(l)}{lmn}$$
$$= O\biggl(\phi(d)\sum_{1\le u<d}\sum_{1\le m<d}\frac{\sqrt{d}}{(ud+m)m}\biggr)$$
$$= O\biggl(\frac{\phi(d)}{\sqrt{d}}\sum_{1\le u<d}\sum_{1\le m<d}\frac{1}{mu}\biggr) = O\biggl(\frac{\phi(d)}{\sqrt{d}}\log^2 d\biggr)$$

and

$$\frac{\phi(d)}{2}\sum_{\substack{1\le l<d\\ ln=d-m}}\ \sideset{}{'}\sum_{1\le m<d}\ \sideset{}{'}\sum_{1\le n<d}\frac{\sqrt{l}\chi_1(l)}{lmn} \tag{11}$$
$$= \frac{\phi(d)}{2}\sum_{\substack{1\le l<d\\ 1\le ln<d/2}}\sideset{}{'}\sum_{1\le n<d}\frac{\chi_1(l)}{(d-ln)n\sqrt{l}} + \frac{\phi(d)}{2}\sum_{\substack{1\le l<d\\ d/2\le ln<d}}\sideset{}{'}\sum_{1\le n<d}\frac{\sqrt{l}\chi_1(l)}{ln(d-ln)}$$
$$= O\biggl(\frac{\phi(d)}{d}\sum_{1\le l<d}\sum_{1\le n<d}\frac{1}{n\sqrt{l}}\biggr) + O\biggl(\frac{\phi(d)}{d}\sum_{1\le u<d/2}\frac{\sqrt{d}\tau(d-u)}{u}\biggr)$$
$$= O\biggl(\frac{\phi(d)}{\sqrt{d}}\log^2 d\biggr) + O\biggl(\frac{\phi(d)}{\sqrt{d}}\exp\biggl(\frac{2\log d}{\log\log d}\biggr)\biggr)$$
$$= O\biggl(\frac{\phi(d)}{\sqrt{d}}\exp\biggl(\frac{2\log d}{\log\log d}\biggr)\biggr),$$

where $\tau(d)$ is the divisor function and $\tau(d) \ll \exp\bigl(\frac{\log d}{\log\log d}\bigr)$. Similarly,

$$\frac{\phi(d)}{2}\sum_{\substack{1\le l<d\\ ln\equiv -m\,(\mathrm{mod}\,d)\\ ln>d}}\ \sideset{}{'}\sum_{1\le m<d}\ \sideset{}{'}\sum_{1\le n<d}\frac{\sqrt{l}\chi_1(l)}{lmn} \tag{12}$$
$$= \frac{\phi(d)}{2}\sum_{2\le u<d}\sideset{}{'}\sum_{1\le m<d}\frac{\sqrt{l}\chi_1(l)}{(ud-m)m}$$
$$= O\biggl(\phi(d)\sum_{1\le u<d}\sum_{1\le m<d}\frac{\sqrt{d}}{umd}\biggr) = O\biggl(\frac{\phi(d)}{\sqrt{d}}\log^2 d\biggr).$$

From (8)–(12) we obtain

$$\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1\\ \chi\chi_1\neq\chi_q^0}} \Big(\sum_{1\le l\le d}\frac{\chi\chi_1(l)}{l^{1/2}}\Big)\Big(\sum_{1\le m\le d}\frac{\overline{\chi}(m)}{m}\Big)\Big(\sum_{1\le n\le d}\frac{\chi(n)}{n}\Big) \tag{13}$$

$$= \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \Big(\sum_{1\le l\le d}\frac{\chi\chi_1(l)}{l^{1/2}}\Big)\Big(\sum_{1\le m\le d}\frac{\overline{\chi}(m)}{m}\Big)\Big(\sum_{1\le n\le d}\frac{\chi(n)}{n}\Big) + O(\sqrt{d}\log^2 d)$$

$$= \frac{\pi^2}{12}\phi(d)L\Big(\frac{3}{2},\chi_1\Big)\prod_{p\mid d}\Big(1-\frac{1}{p^2}\Big) + O\Big(\frac{\phi(d)}{\sqrt{d}}\exp\Big(\frac{2\log d}{\log\log d}\Big)\Big).$$

It is clear that $\chi\chi_1$ is also a character modulo q. So in the following, we can assume $d<y<q$. Then from (7) and the properties of characters we have

$$\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1\\ \chi\chi_1\neq\chi_q^0}} \Big(\sum_{1\le n<d}\frac{\chi(n)}{n}\Big)\Big(\sum_{1\le m<d}\frac{\overline{\chi}(m)}{m}\Big)\Big(\sum_{d<l\le y}\chi\chi_1(l)\Big) \tag{14}$$

$$= {\sum_{1\le n<d}}' {\sum_{1\le m<d}}' {\sum_{d<l\le y}}' \frac{\chi_1(l)}{mn} \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1\\ \chi\chi_1\neq\chi_q^0}} \chi(ln)\overline{\chi}(m)$$

$$\ll \phi(d)\sum_{\substack{1\le l<q\\ ln\equiv m \,(\bmod d)}}\sum_{1\le n<d}\sum_{1\le m<d}\frac{1}{mn} + \sum_{1\le l<q}\sum_{1\le n<d}\sum_{1\le m<d}\frac{\chi_d^0(l)}{mn}$$

$$\ll \frac{q}{d}\phi(d)\sum_{1\le n<d}\sum_{1\le m<d}\frac{1}{mn} + \phi(d)\log^2 d$$

$$\ll \frac{q\phi(d)}{d}\log^2 d + \phi(q)\log^2 d \ll q\log^2 d.$$

Similarly, we have

$$\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1\\ \chi\chi_1\neq\chi_q^0}} \Big(\sum_{1\le n<d}\frac{\chi(n)}{n}\Big)\Big(\sum_{d<a\le y_1}\chi\chi_1(a)\Big)\Big(\sum_{d<b\le y_2}\overline{\chi}(b)\Big) \tag{15}$$

$$= {\sum_{1\le n<d}}' {\sum_{d<a\le y_1}}' {\sum_{d<b\le y_2}}' \frac{\chi_1(a)}{n} \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1\\ \chi\chi_1\neq\chi_q^0}} \chi(an)\overline{\chi}(b)$$

$$\ll \phi(d) \sum_{1\le n<d} \sum_{1\le a<q} \sum_{\substack{1\le b<d \\ an\equiv b \,(\mathrm{mod}\, d)}} \frac{\chi_q^0(a)}{n} + \sum_{1\le n<d}{}' \sum_{1\le a<q} \sum_{1\le b<d} \frac{1}{n}$$

$$\ll \phi(d) \sum_{\substack{1\le a<q \\ (a,q)=1}} \sum_{1\le n<d} \frac{1}{n} + \phi(q)\phi(d)\log d \ll \phi(q)\phi(d)\log d \ll qd\log d$$

and

$$(16)\qquad \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1 \\ \chi\chi_1\ne\chi_q^0}} \Big(\sum_{d<a\le y_1} \chi\chi_1(a)\Big)\Big(\sum_{d<b\le y_2} \chi(b)\Big)\Big(\sum_{d<c\le y_3} \overline{\chi}(c)\Big)$$

$$= \sum_{d<a\le y_1}{}' \sum_{d<b\le y_2}{}' \sum_{d<c\le y_3}{}' \chi_1(a) \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1 \\ \chi\chi_1\ne\chi_q^0}} \chi(ab)\overline{\chi}(c)$$

$$\ll \phi(d) \sum_{1\le a<q}{}' \sum_{1\le b<d}{}' \sum_{\substack{1\le c<d \\ ab\equiv c\,(\mathrm{mod}\, d)}}{}' \chi_q^0(a) + \sum_{\substack{1\le a<q \\ (a,q)=1}} \sum_{\substack{1\le b<d \\ (b,d)=1}} \sum_{\substack{1\le c<d \\ (c,d)=1}} 1$$

$$\ll \phi(d) \sum_{\substack{1\le a<q \\ (a,q)=1}} \sum_{\substack{1\le b<d \\ (b,d)=1}} 1 + \phi(q)\phi^2(d) \ll \phi(q)\phi^2(d) \ll qd^2.$$

Thus from (14)–(16) we get

$$(17)\qquad \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1 \\ \chi\chi_1\ne\chi_q^0}} \Big(\sum_{1\le n<d} \frac{\chi(n)}{n}\Big)\Big(\sum_{1\le m<d} \frac{\chi(m)}{m}\Big)\Big(\int_d^\infty \frac{A(y,\chi\chi_1)}{y^{3/2}}\,dy\Big)$$

$$= \int_d^\infty \frac{1}{y^{3/2}}\Big(\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1 \\ \chi\chi_1\ne\chi_q^0}} \sum_{1\le n<d} \frac{\chi(n)}{n} \sum_{1\le m<d} \frac{\overline{\chi}(m)}{m} \sum_{d<l\le y} \chi\chi_1(l)\Big)dy$$

$$= O\Big(\int_d^\infty \frac{q\log^2 d}{y^{3/2}}\,dy\Big) = O\Big(\frac{q}{\sqrt{d}}\log^2 d\Big),$$

$$(18)\qquad \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1 \\ \chi\chi_1\ne\chi_q^0}} \Big(\sum_{1\le n<d} \frac{\chi(n)}{n}\Big)\Big(\int_d^\infty \frac{A(y,\chi\chi_1)}{y^{3/2}}\,dy\Big)\Big(\int_d^\infty \frac{A(y,\overline{\chi})}{y^2}\,dy\Big)$$

$$= O\Big(\frac{q}{\sqrt{d}}\log d\Big),$$

$$\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1\\ \chi\chi_1\neq\chi_q^0}} \bigg(\int_d^\infty \frac{A(y,\chi\chi_1)}{y^{3/2}}\,dy\bigg)\bigg(\int_d^\infty \frac{A(y,\chi)}{y^2}\,dy\bigg)\bigg(\int_d^\infty \frac{A(y,\overline{\chi})}{y^2}\,dy\bigg) = O\bigg(\frac{q}{\sqrt{d}}\log d\bigg). \tag{19}$$

Using the same method of proving (17) we also have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1\\ \chi\chi_1\neq\chi_q^0}} \bigg(\sum_{1\le l<d} \frac{\chi\chi_1(l)}{l^{1/2}}\bigg)\bigg(\sum_{1\le n<d} \frac{\chi(n)}{n}\bigg)\bigg(\int_d^\infty \frac{A(y,\overline{\chi})}{y^2}\,dy\bigg) \qquad (20)\\
&= \int_d^\infty \frac{1}{y^2}\bigg(\sum_{1\le l<d}\ \sideset{}{'}\sum_{1\le n<d}\ \sideset{}{'}\sum_{d<m\le y} \frac{\chi_1(l)}{nl^{1/2}} \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1\\ \chi\chi_1\neq\chi_q^0}} \chi(ln)\overline{\chi}(m)\bigg)dy\\
&= O\bigg(\frac{\phi(d)}{d}\sum_{1\le l<d}\ \sum_{\substack{1\le n<d\\ ln\equiv m}}\ \sum_{1\le m<d} \frac{1}{nl^{1/2}}\bigg) + O\bigg(\frac{1}{d}\sum_{1\le l<d}\ \sum_{1\le n<d}\ \sum_{1<m\le d} \frac{\chi_d^0(lmn)}{nl^{1/2}}\bigg)\\
&= O\bigg(\frac{\phi(d)}{d}\sum_{1\le l<d}\ \sum_{1\le n<d} \frac{1}{nl^{1/2}}\bigg) + O\bigg(\frac{\phi(d)}{\sqrt{d}}\log d\bigg) = O\bigg(\frac{\phi(d)}{\sqrt{d}}\log d\bigg),
\end{aligned}$$

$$\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1\\ \chi\chi_1\neq\chi_q^0}} \bigg(\sum_{1\le l<d} \frac{\chi\chi_1(l)}{l^{1/2}}\bigg)\bigg(\int_d^\infty \frac{A(y,\chi)}{y^2}\,dy\bigg)\bigg(\int_d^\infty \frac{A(y,\overline{\chi})}{y^2}\,dy\bigg) = O(\sqrt{d}). \tag{21}$$

Combining (6), (13) and (17)–(21) we obtain

$$\begin{aligned}
&\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} L(1/2,\chi_1\chi)|L(1,\chi)|^2\\
&\qquad = \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1\\ \chi\chi_1\neq\chi_q^0}} L(1/2,\chi_1\chi)|L(1,\chi)|^2 + O(q^{1/4}\log^2 d)\\
&\qquad = \frac{\pi^2}{12}\phi(d)L(3/2,\chi_1)\prod_{p\mid d}\bigg(1-\frac{1}{p^2}\bigg) + O\bigg(\frac{q}{\sqrt{d}}\exp\bigg(\frac{2\log d}{\log\log d}\bigg)\bigg).
\end{aligned}$$

This completes the proof of Lemma 3.

LEMMA 4. *Let $q\ge 3$ and m be positive integers, and let χ be any Dirichlet character modulo q. Then*

$$\sum_{a=1}^{q}\chi(a)\zeta(1/2,a/q)S^m(a,q) = \frac{q^{m+1/2}}{(12)^m}L(m+1/2,\chi)+O\biggl(q^m\exp\biggl(\frac{3\log q}{\log\log q}\biggr)\biggr).$$

P r o o f. Note the estimates

$$\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}|L(1,\chi)|^2\le\frac{\pi^2}{12}\phi(d),\qquad q\sum_{d\mid q}\frac{d^{3/2}}{\phi(d)}\ll q^{3/2}\exp\biggl(\frac{\log q}{\log\log q}\biggr)$$

and the identity

$$\sum_{d\mid q}d^2\prod_{p\mid d}\biggl(1-\frac{1}{p^2}\biggr)=q^2.$$

Applying Lemmas 2 and 3 repeatedly we have

$$\begin{aligned}
&\sideset{}{'}\sum_{a=1}^{q}\chi(a)\zeta(1/2,a/q)S^m(a,q)\\
&=\frac{q^{1/2-m}}{\pi^{2m}}\sum_{d_1\mid q}\dots\sum_{d_{m-1}\mid q}\frac{d_1^2\dots d_{m-1}^2}{\phi(d_1)\dots\phi(d_{m-1})}\\
&\quad\times\sum_{\substack{\chi_1 \bmod d_1\\ \chi_1(-1)=-1}}\dots\sum_{\substack{\chi_{m-1} \bmod d_{m-1}\\ \chi_{m-1}(-1)=-1}}\sum_{d_m\mid q}\frac{d_m^2}{\phi(d_m)}\\
&\quad\times\Bigl(\sum_{\substack{\chi_m \bmod d_m\\ \chi_m(-1)=-1}}L(1/2,\chi\chi_1\dots\chi_m)|L(1,\chi_m)|^2\Bigr)|L(1,\chi_1)|^2\dots|L(1,\chi_{m-1})|^2\\
&=\frac{q^{1/2-m}}{\pi^{2m}}\sum_{d_1\mid q}\dots\sum_{d_{m-1}\mid q}\frac{d_1^2\dots d_{m-1}^2}{\phi(d_1)\dots\phi(d_{m-1})}\\
&\quad\times\sum_{\substack{\chi_1 \bmod d_1\\ \chi_1(-1)=-1}}\dots\sum_{\substack{\chi_{m-1} \bmod d_{m-1}\\ \chi_{m-1}(-1)=-1}}|L(1,\chi_1)|^2\dots|L(1,\chi_{m-1})|^2\\
&\quad\times\biggl[\frac{\pi^2}{12}q^2L(3/2,\chi\chi_1\dots\chi_{m-1})+O\biggl(q^{3/2}\exp\biggl(\frac{3\log q}{\log\log q}\biggr)\biggr)\biggr]\\
&=\dots
\end{aligned}$$

$$= \frac{q^{m+1/2}}{(12)^m} L(m+1/2,\chi) + O\bigg(q^m \exp\bigg(\frac{3\log q}{\log\log q}\bigg)\bigg).$$

This proves Lemma 4.

3. Proof of the Theorem. In this section, we complete the proof of the Theorem. Let $q \geq 3$ be a positive integer. Then from the orthogonality relation for character sums and Lemma 4 we have

$$\begin{aligned}(22)\quad & {\sum_{a=1}^{q}}' \zeta^2(1/2, a/q) S^{2m}(a,q) \\ &= \frac{1}{\phi(q)} \sum_{\chi \bmod q} \Big| {\sum_{a=1}^{q}}' \chi(a)\zeta(1/2,a/q)S^m(a,q)\Big|^2 \\ &= \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bigg| \frac{q^{m+1/2}}{(12)^m} L(m+1/2,\chi) + O\bigg(q^m \exp\bigg(\frac{3\log q}{\log\log q}\bigg)\bigg)\bigg|^2 \\ &= \frac{1}{(12)^{2m}} \cdot \frac{q^{2m+1}}{\phi(q)} \sum_{\chi \bmod q} |L(m+1/2,\chi)|^2 \\ &\quad + O\bigg(\frac{q^{2m+1/2}}{\phi(q)} \sum_{\chi \bmod q} |L(m+1/2,\chi)| \exp\bigg(\frac{3\log q}{\log\log q}\bigg)\bigg).\end{aligned}$$

Using the method of proof of Lemma 3 we easily get the asymptotic formula

$$(23)\quad \sum_{\chi \bmod q} |L(m+1/2,\chi)|^2 = \zeta(2m+1)\phi(q) \prod_{p\mid q}\bigg(1 - \frac{1}{p^{2m+1}}\bigg) + O(1)$$

and the estimate

$$(24)\quad \sum_{\chi \bmod q} |L(m+1/2,\chi)| \ll \phi(q).$$

Finally, from (22)–(24) we obtain the formula of the Theorem.

Note. It is clear that using the method of proof of the Theorem we can also get the following more general conclusion: For any $1/2 \leq \sigma < 1$, we have

$$\begin{aligned}{\sum_{a=1}^{q}}' \zeta^2(\sigma, a/q) S^{2m}(a,q) &= \frac{q^{2m+2\sigma}}{(12)^{2m}} \zeta(2m+2\sigma) \prod_{p\mid q}\bigg(1 - \frac{1}{p^{2m+2\sigma}}\bigg) \\ &\quad + O\bigg(q^{2m+\sigma} \exp\bigg(\frac{3\log q}{\log\log q}\bigg)\bigg).\end{aligned}$$

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