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Asymptotic aspects of the Diophantine equation $p^k x^{nk} - z^k = l$

by

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1. Introduction. Let $(k,n) \in \mathbb{N}^2$ with $n(k-1) > 1$ and $p \in \mathbb{N}$. For some $l \in \mathbb{N}$ consider the equation

$$p^k x^{nk} - z^k = l.$$

We are interested in the number $a_{k,n}(\alpha,\gamma,m;l)$ of solutions $(x,z) \in \mathbb{N}\times\mathbb{N}_0$ with the restriction $x \equiv \alpha \bmod m$ and $z \equiv \gamma \bmod m$ for some $m \in \mathbb{N}$, $0 < \alpha \le m$ and $0 \le \gamma < m$.

Given $T > 0$, we are going to derive an asymptotic expansion for

$$A_{k,n}(\alpha,\gamma,m;T) = \sum_{l\le T} a_{k,n}(\alpha,\gamma,m;l).$$

This generalizes results in the case $n = 1$, which are due to Krätzel [6], who takes up the case $m = 1$, and to Kuba [7], who deals with arbitrary m.

The reader will notice that our method for counting the lattice points in question, though it might look different, is fundamentally related to the procedure employed in [6], the technical differences stemming mainly from the fact that we use the hyperbola method (see Section 2) instead of an *ad hoc* argument that, perhaps, would have been more difficult to adapt to the general case.

In order to state our result, we have yet to define the integer $1 \le \beta \le m$ by $\beta \equiv p\alpha^n - \gamma \bmod m$.

Theorem 1. *With the notations introduced above, we have*

$$A_{k,n}(\alpha,\gamma,m;T) = \frac{T^{\frac{n+1}{nk}}}{m^2 p^{\frac{1}{n}}} B\biggl(\frac{(k-1)n-1}{kn}, \frac{1}{k}\biggr) + \frac{T^{\frac{1}{n(k-1)}}}{m^{\frac{1+n(k-1)}{n(k-1)}} k^{\frac{1}{n(k-1)}}} \zeta\biggl(\frac{1}{(k-1)n}, \frac{\beta}{m}\biggr) + \frac{T^{\frac{1}{nk}}}{p^{\frac{1}{n}} m}\biggl(\frac{1}{2} - \frac{\gamma}{m}\biggr)$$

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$$
\begin{aligned}
&+T^{\frac{kn-1}{nk^2}} \frac{p^{\frac{1}{nk}} n^{\frac{1}{k}}}{2^{\frac{1}{k}} \pi^{\frac{k+1}{k}} (mk)^{\frac{k-1}{k}}} \\
&\times \sum_{\nu=1}^{\infty} \frac{\Gamma(1/k)}{\nu^{\frac{k+1}{k}}} \sin\left(\frac{2\pi\nu(T^{\frac{1}{nk}} - p^{\frac{1}{n}}\alpha)}{mp^{\frac{1}{n}}} + \frac{\pi}{2k}\right) \\
&+ \Delta_{k,n}(\alpha,\gamma,m;T)
\end{aligned}
$$

where

$$\Delta_{k,1}(\alpha,\gamma,m;T) \ll \left(\frac{T}{m^k}\right)^{\frac{46}{73k}} \left(\log\left(\frac{T}{m^k}+1\right)\right)^{\frac{315}{146}} + 1 \quad \textit{for } k \ge 3,$$

$$\Delta_{k,2}(\alpha,\gamma,m;T) \ll T^{\frac{1}{2k-1}} + 1 \quad \textit{for } k \ge 2,$$

$$\Delta_{k,n}(\alpha,\gamma,m;T) \ll T^{\frac{n-1}{nk}} + 1 \quad \textit{for } n \ge 3 \textit{ and } k \ge 2,$$

and the implied constants depend on k, n and p.

As usual, we denote by $B(\cdot,\cdot)$ the beta function and by $\zeta(\cdot,\cdot)$ the Hurwitz zeta function.

Note that the order in which the various terms appear in Theorem 1 reflects only the case $n = 1$. As a matter of fact, we have for $n = 1$ (and, consequently, $k \ge 3$)

$$\frac{46}{73k} < \frac{kn-1}{nk^2} < \frac{1}{nk} < \frac{1}{n(k-1)} < \frac{n+1}{nk},$$

for $n = 2$ and $k = 2$

$$\frac{1}{nk} < \frac{1}{2k-1} < \frac{kn-1}{nk^2} < \frac{1}{n(k-1)} < \frac{n+1}{nk},$$

for $n = 2$ and $k \ge 3$

$$\frac{1}{nk} < \frac{1}{2k-1} < \frac{1}{n(k-1)} < \frac{kn-1}{nk^2} < \frac{n+1}{nk},$$

and in all other cases

$$\frac{1}{nk} < \frac{1}{n(k-1)} \le \frac{n-1}{nk} < \frac{kn-1}{nk^2} < \frac{n+1}{nk}.$$

We note that the case $n = k = 2$ is of particular interest since it is related to elliptic curves (cf. [1]).

For results concerning the arithmetic and quadratic mean of the number of *primitive* lattice points in this case see [4].

2. The principal terms. In what follows, we shall write

$$a = p^{-1/n} T^{1/(nk)} \quad \text{and} \quad f(x) = px^n - (p^k x^{nk} - T)^{1/k}.$$

For $t \geq 1$ and k and n real numbers with $k > 1$, $n \geq 1$ and $n(k-1) > 1$, we consider the function given by

$$g(t) = g_{k,n}(t) = t^n - (t^{kn} - 1)^{1/k},$$

which has the property

$$f(x) = pa^n g(x/a).$$

For later use we define $\widetilde{g} = g_{k/(k-1),n(k-1)}$. Note that $\widetilde{\widetilde{g}} = g$ and

$$g'(t) = -nt^{n-1}(t^{kn} - 1)^{(1-k)/k}\, \widetilde{g}(t). \tag{1}$$

LEMMA 1. 1. *The function g is strictly decreasing*, *and we have the inequality*

$$\frac{t^{n(1-k)}}{k} < g(t) \leq \frac{t^{n(1-k)}}{k} + \frac{k-1}{k} t^{n(1-2k)}. \tag{2}$$

2. *The inverse function g^{-1}*, *defined for* $0 < s \leq 1$, *satisfies the equation*

$$g^{-1}(s) = (ks)^{-1/((k-1)n)}\left(1 + \frac{\vartheta_0(s)}{kn}(ks)^{k/(k-1)}\right) \tag{3}$$

for some $0 < \vartheta_0(s) < 1$.

3. *The function $(g^{-1})'$ satisfies*

$$(g^{-1})'(s) = -\frac{k}{(k-1)n}(ks)^{-1/((k-1)n)-1}(1 + \vartheta_1(s)(ks)^{k/(k-1)}) \tag{4}$$

for some $-1 < \vartheta_1(s) < 1$.

REMARK. This is a slightly more precise version of Hilfssatz 1 in [6]. Though this kind of precision is quite useless for our present purpose, it might be of some interest when investigating the dependencies on k and n of the remainder terms (which is not done in the present paper).

Proof (of Lemma 1). We define the auxiliary function $h(t) = t - (t^k - 1)^{1/k}$, which has the property $g(t) = h(t^n)$. Note that for $0 < x \leq 1$ and $0 < \alpha < 1$ we have

$$\frac{1-\alpha}{2\alpha} < \frac{1}{\alpha x} - \frac{1}{1-(1-x)^\alpha} \leq \frac{1-\alpha}{\alpha} \tag{5}$$

since the function given by the middle expression is strictly increasing. This implies for $x = t^{-k}$ and $\alpha = 1/k$ the inequality

$$\frac{t}{k(t^k - 1/2) + 1/2} < h(t) \leq \frac{t}{k(t^k - 1) + 1}.$$

Now, the left-hand side is $\geq k^{-1}t^{1-k}$ and

$$\frac{t}{k(t^k-1)+1} - \frac{1}{kt^{k-1}} = \frac{k-1}{kt^{2k-1}(k-(k-1)t^{-k})} \leq \frac{k-1}{kt^{2k-1}}.$$

On the other hand, Bernoulli's inequality shows that

$$h(t) < \frac{1}{k(t^k-1)^{(k-1)/k}}.$$

Setting $t = h^{-1}(s)$ and doing an easy calculation, we deduce

(6) $$(ks)^{-1/(k-1)} < h^{-1}(s) < (ks)^{-1/(k-1)}(1+(ks)^{k/(k-1)})^{1/k}.$$

This implies the second statement of the lemma.

Further, setting $t = g^{-1}(s)$, we have

$$(g^{-1})'(s) = \frac{1}{nt^{n-1}}\left(1 - \frac{1}{1-(1-t^{-kn})^{(k-1)/k}}\right).$$

Applying inequality (5) for $x = t^{-kn}$ and $\alpha = (k-1)/k$, we find

$$\frac{1}{2n}\left(1+\frac{k}{k-1}\right)\frac{1}{t^{n-1}} < (g^{-1})'(s) + \frac{k}{(k-1)n}t^{(k-1)n+1} < \frac{k}{(k-1)n}\cdot\frac{1}{t^{n-1}}.$$

Note that, as far as the leftmost expression is concerned, we use only the fact that it is positive. Substituting (6), we arrive at the third statement of the lemma. ∎

With $\psi(x) = x - [x] - 1/2$ as usual, we will use the notation

$$\psi_\alpha(t) = \psi\left(\frac{t-\alpha}{m}\right).$$

Proposition 1. *Let $c > 1$, $u > cp^{-1/n}T^{1/(nk)}$ and $v = f(u)$. Let $0 < b < \beta$. Then*

$$\Delta_{k,n}(\alpha,\gamma,m;T) = -\sum_{a<x\le u,\, x\equiv\alpha}\psi_\beta(f(x)) - \sum_{b<y\le v,\, y\equiv\beta}\psi_\alpha(f^{-1}(y)) + O(p^{\frac{1}{n}}T^{\frac{n-1}{nk}} + T^{\frac{1}{n(k-1)}}p^{\frac{1-k}{n(k-1)}}v^{-\frac{n(k-1)+1}{n(k-1)}} + 1),$$

the implied constants depending on k, n and c.

Proof. It being understood that x and z run over their respective residue classes, we can express $A_{k,n}(\alpha,\gamma,m;T)$ as the sum of

(7) $$\sum_{x\le a}\sum_{z<px^n} 1$$

and

(8) $$\sum_{a<x}\sum_{(p^kx^{nk}-T)^{1/k}\le z<px^n} 1.$$

We can express the inner sum of (8) as

(9) $$\sum_{0<px^n-z\le f(x)} 1 = \sum_{0<y\le f(x)} 1,$$

where $y \equiv \beta$.

Euler's summation formula shows that (7) is equal to the sum of

$$\frac{pa^{n+1}}{m^2(n+1)} - \frac{pa^n\psi_\alpha(a)}{m} - \frac{a}{m}\left(\frac{\beta+\gamma}{m} - 1\right)$$

and the expression

$$\frac{np}{m}\int_0^a t^{n-1}\psi_\alpha(t)\,dt - \left(-\frac{\alpha}{m} - \psi_\alpha(a) + \frac{1}{2}\right)\left(\frac{\beta+\gamma}{m} - 1\right),$$

which is $O(pa^{n-1}+1)$.

The hyperbola method (cf. [5], Theorem 1.5) shows in view of (9) that (8) equals

$$\frac{1}{m^2}\Big(\int_a^u f(x)\,dx + \int_b^v f^{-1}(y)\,dy - uv + ab\Big) \tag{10}$$

$$\begin{aligned} &+ \frac{1}{m}\Big(\int_a^u \psi_\alpha(x)f'(x)\,dx + \int_b^v \psi_\beta(y)(f^{-1})'(y)\,dy\Big) \qquad (11)\\ &+ \frac{1}{m}(f(a)\psi_\alpha(a) - b\psi_\alpha(a) + f^{-1}(b)\psi_\beta(b) - a\psi_\beta(b))\\ &- \sum_{a<x\le u,\, x\equiv\alpha}\psi_\beta(f(x)) - \sum_{b<y\le v,\, y\equiv\beta}\psi_\alpha(f^{-1}(y))\\ &+ \psi_\alpha(a)\psi_\beta(b) - \psi_\alpha(u)\psi_\beta(v). \end{aligned}$$

We find that m^2 times (10) equals

$$pa^n\int_a^{f^{-1}(b)} g\left(\frac{x}{a}\right)dx - (f^{-1}(b)-a)b.$$

Consider the first term of this expression. A change of variables and splitting the resulting integral in an appropriate way gives the sum of

$$pa^{n+1}\int_1^\infty g(t)\,dt \tag{12}$$

and

$$-pa^{n+1}\int_{f^{-1}(b)/a}^\infty g(t)\,dt. \tag{13}$$

The expression (12) gives after integration by parts and a change of variables

$$\frac{pa^{n+1}}{k(n+1)}\int_0^1 (1-t)^{1/k-1}t^{-(n+1)/(nk)}\,dt - \frac{pa^{n+1}}{n+1},$$

which is
$$\frac{pa^{n+1}}{k(n+1)}B\biggl(\frac{(k-1)n-1}{kn},\frac{1}{k}\biggr)-\frac{pa^{n+1}}{n+1}.$$
Next, (3) applied to $s=bp^{-1}a^{-n}$, combined with the upper bound of the inequality (2), shows that (13) equals
$$-pa^{n+1}\int\limits_{(kb)^{-\frac{1}{n(k-1)}}p^{\frac{1}{n(k-1)}}a^{\frac{1}{k-1}}}^{\infty} g(t)\,dt+O(p^{\frac{1-kn}{n(k-1)}}a^{\frac{k(1-n)}{k-1}}).$$
Note that
$$p^{\frac{1-kn}{n(k-1)}}a^{\frac{k(1-n)}{k-1}}=O(1).$$
Using once more (2) for the integrand g, we find that (13) equals
$$-\frac{a^{\frac{k}{k-1}}b^{\frac{(k-1)n-1}{(k-1)n}}p^{\frac{1}{(k-1)n}}}{k^{\frac{1}{(k-1)n}}(n(k-1)-1)}+O(p^{\frac{1-kn}{n(k-1)}}a^{\frac{k(1-n)}{k-1}}).$$
Another expression of this shape is given by
$$-f^{-1}(b)b=-\frac{a^{\frac{k}{k-1}}b^{\frac{(k-1)n-1}{(k-1)n}}p^{\frac{1}{(k-1)n}}}{k^{\frac{1}{(k-1)n}}}+O(p^{\frac{1-kn}{n(k-1)}}a^{\frac{k(1-n)}{k-1}}).$$

The first integral in (11) is
$$pa^{n-1}\int\limits_{a}^{u}g'\biggl(\frac{x}{a}\biggr)\psi_\alpha(x)\,dx.$$
We split it into a main term
$$pa^{n-1}\int\limits_{a}^{\infty}g'\biggl(\frac{x}{a}\biggr)\psi_\alpha(x)\,dx$$
and a remainder term
$$-a^{n-1}p\int\limits_{u}^{\infty}g'\biggl(\frac{x}{a}\biggr)\psi_\alpha(x)\,dx,$$
which is
$$O(mpa^{n-1}). \tag{14}$$
After a change of variables the main term becomes
$$pa^{n}\int\limits_{1}^{\infty}g'(t)\psi_\alpha(at)\,dt.$$
Substituting (1), we find that this is the same as
$$-npa^{n}\int\limits_{1}^{\infty}t^{n-1}(t^{kn}-1)^{(1-k)/k}\,\widetilde{g}(t)\psi_\alpha(at)\,dt.$$

We split this into

$$-npa^n \int_1^\infty t^{n-1}(t^{kn}-1)^{(1-k)/k}\left(\widetilde{g}(t)-\frac{1}{t^n}\right)\psi_\alpha(at)\,dt,$$

which is again estimated by (14) since the singularity of the integrand at 1 is cancelled, and

$$-npa^n \int_1^\infty (t^{kn}-1)^{(1-k)/k}\psi_\alpha(at)\frac{dt}{t}.$$

We split this last integral into

$$-npa^n \int_1^\infty ((t^{kn}-1)^{(1-k)/k}-(kn(t-1))^{(1-k)/k})\psi_\alpha(at)\frac{dt}{t}$$

and

$$-npa^n \int_1^\infty (kn(t-1))^{(1-k)/k}\psi_\alpha(at)\frac{dt}{t}.$$

The first integral again has no singularity left and is bounded by (14). After substituting for ψ_α its Fourier series, the last integral is of a well known type (cf. [2], 2.8) and has the expansion

$$a^{\frac{kn-1}{k}}\frac{p(nm)^{\frac{1}{k}}}{2^{\frac{1}{k}}\pi^{\frac{k+1}{k}}k^{\frac{k-1}{k}}}\sum_{\nu=1}^\infty \frac{\Gamma(1/k)}{\nu^{\frac{k+1}{k}}}\sin\left(\frac{2\pi\nu(a-\alpha)}{m}+\frac{\pi}{2k}\right)+O(pa^{n-1}).$$

The second integral in (11) is

$$\frac{a^{1-n}}{p}\int_b^v (g^{-1})'\left(\frac{y}{a^n p}\right)\psi_\beta(y)\,dy.$$

After a change of variables we get

$$a\int_{b/(a^n p)}^{v/(a^n p)} (g^{-1})'(s)\psi_\beta(pa^n s)\,ds.$$

Substituting the asymptotic expansion (4) for $(g^{-1})'$, we get the sum of a main term

$$-\frac{a}{(k-1)nk^{\frac{1}{(k-1)n}}}\int_{b/(a^n p)}^{v/(a^n p)} \psi_\beta(pa^n s)s^{-\frac{(k-1)n+1}{(k-1)n}}\,ds$$

and a remainder term

$$-\frac{ak^{\frac{kn-1}{(k-1)n}}}{(k-1)n}\int_{b/(a^n p)}^{v/(a^n p)} s^{\frac{n-1}{(k-1)n}}\vartheta_1(s)\psi_\beta(pa^n s)\,ds,$$

which is $O((m/p)a^{n-1})$. The main term is

$$-\frac{a^{\frac{k}{k-1}}p^{\frac{1}{(k-1)n}}}{(k-1)nk^{\frac{1}{(k-1)n}}}\int_b^v \psi_\beta(t)t^{-\frac{(k-1)n+1}{(k-1)n}}\,dt.$$

We split this into

$$-\frac{a^{\frac{k}{k-1}}p^{\frac{1}{(k-1)n}}}{(k-1)nk^{\frac{1}{(k-1)n}}}\int_b^\infty \psi_\beta(t)t^{-\frac{(k-1)n+1}{(k-1)n}}\,dt \tag{15}$$

and

$$\frac{a^{\frac{k}{k-1}}p^{\frac{1}{(k-1)n}}}{(k-1)nk^{\frac{1}{(k-1)n}}}\int_v^\infty \psi_\beta(t)t^{-\frac{(k-1)n+1}{(k-1)n}}\,dt,$$

which is

$$O(a^{\frac{k}{k-1}}p^{\frac{1}{n(k-1)}}v^{-\frac{n(k-1)+1}{n(k-1)}}).$$

After analytic continuation, Euler's summation formula shows that for $\sigma>0$,

$$\frac{1}{m^\sigma}\zeta\left(\sigma,\frac{\beta}{m}\right)=-\sigma\int_b^\infty\frac{\psi_\beta(t)}{t^{\sigma+1}}\,dt+\frac{b^{1-\sigma}}{m(\sigma-1)}+\frac{\psi_\beta(b)}{b^\sigma}.$$

This implies that (15) is the sum of

$$\frac{a^{\frac{k}{k-1}}p^{\frac{1}{n(k-1)}}}{(mk)^{\frac{1}{n(k-1)}}}\zeta\left(\frac{1}{n(k-1)},\frac{\beta}{m}\right)$$

and the terms

$$\frac{a^{\frac{k}{k-1}}b^{\frac{(k-1)n-1}{(k-1)n}}p^{\frac{1}{(k-1)n}}(k-1)n}{k^{\frac{1}{n(k-1)}}m(n(k-1)-1)}$$

and

$$-\frac{a^{\frac{k}{k-1}}p^{\frac{1}{n(k-1)}}\psi_\beta(b)}{b^{\frac{1}{n(k-1)}}k^{\frac{1}{n(k-1)}}}.$$

The proof is finished by collecting the relevant terms. ■

3. Estimation of the remainder term $\Delta_{k,n}$. In view of Proposition 1, the proof of Theorem 1 will be finished by estimating the ψ-sums in the remainder term $\Delta_{k,n}$.

On the one hand, if $n\ge 2$, we choose in Proposition 1

$$u\asymp v\asymp T^{1/(1+n(k-1))},$$

and estimate the sums trivially. This gives the desired estimate for $\Delta_{k,n}$.

On the other hand, in the case $n=1$, we use an exponential sum estimation of Huxley in order to update Kuba's result [7], which relies on another result due to Huxley (cf. [8]). We have to verify a different set of conditions, though.

Proposition 2. *Given an integer $k \ge 3$, for $T > 0$ we have*

$$\Delta_{k,1}(\alpha,\gamma,m;T) \ll \left(\frac{T}{m^k}\right)^{\frac{46}{73k}} \left(\log\left(\frac{T}{m^k}+1\right)\right)^{\frac{315}{146}} + 1,$$

the implied constants depending on k and p.

Proof. We apply Theorem 18.2.3 of [3]. Since there is no point in repeating the details of the proof in [7], we will just check the new condition (18.2.21) of [3] in Huxley's theorem. For

$$h(x) = x - (x^k - a^k)^{1/k},$$

we have

$$\begin{aligned} &h''(x)h^{(4)}(x) - 3h'''(x)^2 \\ &\qquad = -a^{2k}(k-1)^2 x^{2k-6}(x^k-a^k)^{2/k-6}((k+1)(2k+1)x^{2k} \\ &\qquad\quad + a^k(k+1)(2k-5)x^k + a^{2k}(k-2)(2k-3)), \end{aligned}$$

which is clearly < 0 for $x > a$. Note that the last factor of the above is $\asymp x^{2k}$.

Similarly, the growth conditions required by Huxley's theorem for the derivatives of h are easily checked.

On the other hand, writing $x = h^{-1}(y)$, we find

$$\begin{aligned} &(h^{-1})''(y)(h^{-1})^{(4)}(y) - 3((h^{-1})'''(y))^2 \\ &\qquad = \frac{a^{2k}(k-1)^2 x^{6k-6}(x^k-a^k)^{(2(k-4))/k}}{(x^{k-1}-(x^k-a^k)^{1-1/k})^{10}} P\left(\frac{a}{x}\right), \end{aligned}$$

where $P(u)$ equals

$$\begin{aligned} &-(k-2)(2k-3)u^{4k} + (2k^2-11k+17)u^{3k} \\ &-(2k+17)u^{2k} + (k+1)(2k+7)u^k - (k+1)(2k+1) \\ &+(1-u^k)^{1/k}(2(k-2)(2k-1)u^{3k} + 2(8k-1)u^{2k} \\ &-4(k+1)(2k+1)u^k + 2(k+1)(2k+1)) \\ &+(1-u^k)^{2/k}(-(2k-1)(3k-2)u^{2k} \\ &+3(k+1)(2k-1)u^k - (k+1)(2k+1)). \end{aligned}$$

Substituting $u^k = 1 - v^k$, we find for $0 < v \le 1$ that $P(u)$ equals v^{2k+1} times

$$\begin{aligned} &(-2k^2+7k-6)(v^{2k-1}+v^{1-2k}) + (-4k^2+10k-4)(v^k+v^{-k}) \\ &\quad + (6k^2-17k+7)(v^{k-1}+v^{1-k}) + (-6k^2+7k-2)(v+v^{-1}) \\ &\qquad\qquad + 12k^2 - 14k + 10. \end{aligned}$$

Now, using

$$v^{2k-1}+v^{1-2k} \ge v^{k-1}+v^{1-k}, \qquad v^k+v^{-k} \ge v^{k-1}+v^{1-k}$$

and then

$$v^{k-1} + v^{1-k} \ge 2, \qquad v + v^{-1} \ge 2,$$

where equality holds only for $v = 1$, we see that $P(u) < 0$ for $0 < u < 1$. Note that

$$\lim_{u\to 0} \frac{P(u)}{u^{2k}} = -\frac{(k-1)^2(2k-1)(3k-1)}{k^2} < 0,$$

which implies that for $0 < u \le 1 - \varepsilon$,

$$-P(u) \asymp u^{2k},$$

the implied constants depending on $0 < \varepsilon < 1$.

Combined with (3), the same method shows that the derivatives of h^{-1} satisfy the required growth conditions. ■

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