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Magic p-dimensional cubes of order $n \not\equiv 2 \pmod 4$

by

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A *magic p-dimensional cube* of order n is a p-dimensional matrix

$$\mathbf{M}_n^p = |\mathbf{m}(i_1, \ldots, i_p) : 1 \le i_1, \ldots, i_p \le n|,$$

containing natural numbers $1, \ldots, n^p$ such that the sum of the numbers along every row and every diagonal is the same, i.e. $n(n^p + 1)/2$. (*Note.* A magic 1-dimensional cube $\mathbf{M}_n^1$ of order n is given by an arbitrary permutation of the natural numbers $1, \ldots, n$.)

By a *row* of $\mathbf{M}_n^p$ we mean an n-tuple of elements $\mathbf{m}(i_1, \ldots, i_p)$ which have identical coordinates at $p-1$ places. A magic p-dimensional cube $\mathbf{M}_n^p$ contains pn^{p-1} rows. A *diagonal* of $\mathbf{M}_n^p$ is an n-tuple $\{\mathbf{m}(x, i_2, \ldots, i_p) : x = 1, \ldots, n,\ i_j = x \text{ or } i_j = \overline{x} \text{ for all } 2 \le j \le p\}$. The symbol $\overline{x}$ denotes the number $n+1-x$, and $\lfloor x \rfloor$ denotes the integer part of x. Every p-dimensional cube has exactly 2^{p-1} great diagonals.

Figure 1 depicts a magic cube $\mathbf{M}_3^3$.

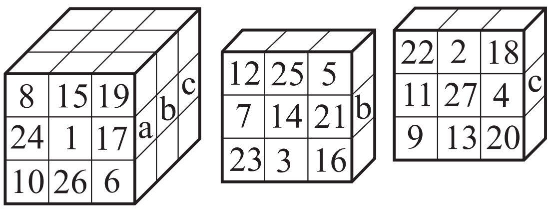

Fig. 1. Magic cube $\mathbf{M}_3^3$

A special case, for $p = 2$, of a magic p-dimensional cube $\mathbf{M}_n^p$ is a magic square. The first references to magic squares can be found in ancient Chinese and Indian literature. They have been the object of study of many mathematicians (e.g. Pierre de Fermat, Leonard Euler), but not only of them (also e.g. Arabian astrologers, Benjamin Franklin). A very famous magic

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square is in the painting *Melancholy* ([2, p. 147]) made by Albrecht Dürer in 1514. The construction of a magic square of order 3 appears in the tragedy *Faust* by J. W. Göthe. Probably, the first magic cube appeared in a letter of P. Fermat from 1640.

There is a lot of information and results about magic squares and cubes in the 1917 book by W. S. Andrews. A revised and enlarged edition [2] was published in 1960. More up-to-date information and references can be found in a paper by Allan Adler [1]. Knowledge of magic p-dimensional cubes can find its use not only in recreational mathematics, but also in many fields of mathematics and physics (see [1]). Although many papers have been published concerning magic squares and cubes, relatively little is known about magic p-dimensional cubes for $p \geq 4$. A universal algorithm for their construction has probably not been published yet. The construction of an $\mathbf{M}_n^3$ for every $n \neq 2$ is in [4]. In [3] there is a construction of "magic p-dimensional cubes" without the constant sum on diagonals.

46	8	69	17	78	28	60	37	26
62	42	19	51	1	71	10	80	33
15	73	35	55	44	24	53	6	64
59	39	25	48	7	68	16	77	30
12	79	32	61	41	21	50	3	70
52	5	66	14	75	34	57	43	23
18	76	29	58	38	27	47	9	67
49	2	72	11	81	31	63	40	20
56	45	22	54	4	65	13	74	36

Fig. 2. Magic cube $\mathbf{M}_3^4$

Figure 2 shows the nine layers of an $\mathbf{M}_3^4$. The element $\mathbf{m}(1,1,1,1) = 46$ is in four rows containing the triplets $\{46, 8, 69\}$, $\{46, 62, 15\}$, $\{46, 17, 60\}$ and $\{46, 59, 18\}$. On the eight diagonals there are the triplets

$$\begin{aligned}
&\{\mathbf{m}(1,1,1,1) = 46, 41, 36\}, \qquad \{\mathbf{m}(1,1,1,3) = 69, 41, 13\},\\
&\{\mathbf{m}(1,1,3,1) = 15, 41, 67\}, \qquad \{\mathbf{m}(1,1,3,3) = 35, 41, 47\},\\
&\{\mathbf{m}(1,3,1,1) = 60, 41, 22\}, \qquad \{\mathbf{m}(1,3,1,3) = 26, 41, 56\},\\
&\{\mathbf{m}(1,3,3,1) = 53, 41, 29\}, \qquad \{\mathbf{m}(1,3,3,3) = 64, 41, 18\}.
\end{aligned}$$

(*Note*. This picture is a magic square of order 9 with some special proprieties.) This magic 4-dimensional cube was constructed using the following formula (from Theorem 1):

$$\begin{aligned}
\mathbf{m}(i_1, i_2, i_3, i_4) = {} & \left[\left(i_1 - i_2 + i_3 - i_4 + \frac{n+1}{2} - 1\right) \pmod n\right] n^3 \\
& + \left[\left(i_1 - i_2 + i_3 + i_4 - \frac{n+1}{2} - 1\right) \pmod n\right] n^2
\end{aligned}$$

$$+\left[\left(i_1 - i_2 - i_3 - i_4 + 3\frac{n+1}{2} - 1\right) \pmod{n}\right]n$$
$$+\left[\left(i_1 + i_2 + i_3 + i_4 - 3\frac{n+1}{2} - 1\right) \pmod{n}\right] + 1.$$

This paper is concerned with the construction of a magic p-dimensional cube of order n for every $n \not\equiv 2 \pmod 4$ and $p \geq 1$.

THEOREM 1. *A magic p-dimensional cube $\mathbf{M}_n^p$ of order n exists for every odd natural number n and every natural number p.*

Proof. We define a magic p-dimensional cube $\mathbf{M}_n^p = |\mathbf{m}(i_1,\ldots,i_p)|$ of odd order n by

$$\mathbf{m}(i_1,\ldots,i_p) = \sum_{k=0}^{p-1} m_k(i_1,\ldots,i_p)n^k + 1,$$

where

$$m_k(i_1,\ldots,i_p) = \Big[\sum_{x=1}^{k}(-1)^{x-1}i_x + (-1)^k \sum_{x=k+1}^{p} i_x + C_k\Big] \pmod{n}$$

(note $\sum_{x=1}^{0}(-1)^{x-1} = 0$) and

$$C_k = (-1)^{k+1}[p-k-(k+1) \pmod 2]\frac{n+1}{2} - 1.$$

The constant C_k is chosen so that

$$m_k\left(\frac{n+1}{2},\frac{n+1}{2},\ldots,\frac{n+1}{2}\right) = \frac{n-1}{2} \qquad \text{for all } 0 \leq k \leq p-1.$$

The proof consists of four steps. First, we prove that each element of $\mathbf{M}_n^p$ is in $\{1,\ldots,n^p\}$; second, no two elements of $\mathbf{M}_n^p$ with different coordinates are equal; third, the sums of elements in all rows are the same; fourth, the sums of elements on the diagonals are also the same.

1. Because $0 \leq m_k(i_1,\ldots,i_p) \leq n-1$ for all $0 \leq k \leq p-1$ we get $1 \leq \mathbf{m}(i_1,\ldots,i_n) \leq n^p$ for every element of $\mathbf{M}_n^p$.

2. Suppose that $\mathbf{m}(i_1',\ldots,i_p') = \mathbf{m}(i_1,\ldots,i_p)$. The definition of $\mathbf{M}_n^p$ gives

$$\sum_{k=0}^{p-1} m_k(i_1',\ldots,i_p')n^k + 1 = \sum_{k=0}^{p-1} m_k(i_1,\ldots,i_p)n^k + 1.$$

Hence

$$\sum_{k=0}^{p-1}[m_k(i_1',\ldots,i_p') - m_k(i_1,\ldots,i_p)]n^k = 0.$$

Because the differences in brackets are less than n we get p equations

$$m_k(i'_1,\dots,i'_p)=m_k(i_1,\dots,i_p)\qquad\text{for all } 0\le k\le p-1.$$

By rearranging them according to the definition of $\mathbf{M}_n^p$ we get

$$\begin{aligned}
&(\mathrm{E}_0)\qquad (i'_1+i'_2+i'_3+\dots+i'_p+C_0)\pmod n\\
&\qquad\qquad\qquad =(i_1+i_2+i_3+\dots+i_p+C_0)\pmod n,\\
&(\mathrm{E}_1)\qquad (i'_1-i'_2-i'_3-\dots-i'_p+C_1)\pmod n\\
&\qquad\qquad\qquad =(i_1-i_2-i_3-\dots-i_p+C_1)\pmod n,\\
&(\mathrm{E}_2)\qquad (i'_1-i'_2+i'_3-\dots+i'_p+C_2)\pmod n\\
&\qquad\qquad\qquad =(i_1-i_2+i_3-\dots+i_p+C_2)\pmod n,\\
&(\mathrm{E}_3)\qquad (i'_1-i'_2+i'_3-\dots-i'_p+C_3)\pmod n\\
&\qquad\qquad\qquad =(i_1-i_2+i_3-\dots-i_p+C_3)\pmod n,\\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(\mathrm{E}_{p-1})\qquad (i'_1-i'_2+\dots+(-1)^{p-1}i'_p+C_{p-1})\pmod n\\
&\qquad\qquad\qquad =(i_1-i_2+\dots+(-1)^{p-1}i_p+C_{p-1})\pmod n.
\end{aligned}$$

By adding (E_0) and (E_1) we get either $2i'_1=2i_1$ or $2i'_1=2i_1+n$ or $2i'_1=2i_1-n$. Because $i'_1\le n$ and n is odd we get $i'_1=i_1$. Replace i'_1 by i_1 in (E_1), $(\mathrm{E}_2),\dots,(\mathrm{E}_{p-1})$. From the relations rearranged in this way, by adding (E_1) and (E_2) we get $i'_2=i_2$. Continuing in this manner, we get $i'_3=i_3$, $i'_4=i_4,\dots,i'_p=i_p$.

3. For every $k=0,1,\dots,p-1$ the set $\{m_k(i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_p): i_j=1,\dots,n\}$ is equal to $\{0,1,\dots,n-1\}$ and therefore

$$\sum_{i_j=1}^{n} m_k(i_1,\dots,i_p)=\frac{n(n-1)}{2}\qquad\text{for all } 1\le j\le p.$$

This implies that every row sum is

$$\begin{aligned}
\sum_{i_j=1}^{n}\mathbf{m}(i_1,\dots,i_p)&=\sum_{i_j=1}^{n}\sum_{k=0}^{p-1}[m_k(i_1,\dots,i_p)n^k+1]\\
&=\sum_{k=0}^{p-1}\frac{n(n-1)}{2}n^k+n=\frac{n^{p+1}-n}{2}+n=\frac{n(n^p+1)}{2}.
\end{aligned}$$

4. From the definition of $\mathbf{M}_n^p$ it follows that for every p-tuple $(i_1,\dots,i_p)$,

$$m_k(i_1,\dots,i_p)+m_k(\bar{i}_1,\dots,\bar{i}_p)=n-1;$$

hence

$$\mathbf{m}(i_1,\dots,i_p)+\mathbf{m}(\bar{i}_1,\dots,\bar{i}_p)=\sum_{k=0}^{p-1}(n-1)n^k+2=n^p+1.$$

There are $(n-1)/2$ pairs of elements on each diagonal whose sum is n^p+1, and in the center of $\mathbf{M}_n^p$ there is the element

$$\mathbf{m}\left(\frac{n+1}{2},\frac{n+1}{2},\dots,\frac{n+1}{2}\right)=\frac{n^p+1}{2}.$$

Each diagonal sum is

$$\frac{n-1}{2}(n^p+1)+\frac{n^p+1}{2}=\frac{n(n^p+1)}{2}.$$

This completes the proof. ∎

THEOREM 2. *A magic p-dimensional cube* $\mathbf{M}_n^p$ *of order n exists for every natural number* $n\equiv 0\pmod 4$ *and for every natural number p.*

We define a magic p-dimensional cube $\mathbf{M}_n^p=|\mathbf{m}(i_1,\dots,i_p)|$ of order $n\equiv 0\pmod 4$ by

$$\mathbf{m}(i_1,\dots,i_p)=\begin{cases}\displaystyle\sum_{k=1}^{p}(i_k-1)n^{k-1}+1 & \text{if } \displaystyle\sum_{j=1}^{n}\left(i_j+\left\lfloor\frac{2(i_j-1)}{n}\right\rfloor\right) \text{ is odd,}\\ \displaystyle\sum_{k=1}^{p}(\bar{i}_k-1)n^{k-1}+1 & \text{in the opposite case.}\end{cases}$$

The assertion of Theorem 2 follows from the following three facts:

1. No two elements with different coordinates are equal because $\sum_{j=1}^{n}(i_j+\lfloor 2(i_j-1)/n\rfloor)$ is odd if and only if $\sum_{j=1}^{n}(\bar{i}_j+\lfloor 2(\bar{i}_j-1)/n\rfloor)$ is odd.

2. The row sums are equal because for every odd coordinate i_j,

$$\begin{aligned}\mathbf{m}(i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_p)+\mathbf{m}(i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_p)\\ =n^p-n^{j-1}+1\quad\text{or}\quad n^p+n^{j-1}+1.\end{aligned}$$

In every row there are $n/4$ pairs of elements with sum $n^p-n^{j-1}+1$ and the same number of pairs with sum $n^p+n^{j-1}+1$.

3. The diagonal sums are the same because for every p-tuple $(i_1,\dots,i_p)$,

$$\mathbf{m}(i_1,\dots,i_p)+\mathbf{m}(\bar{i}_1,\dots,\bar{i}_p)=n^p+1.$$

References

[1] A. Adler, *Magic N-cubes form a free matroid*, Electron. J. Combin. 4 (1997) (it will appear in European J. Combin.).

[2] W. S. Andrews, *Magic Squares and Cubes*, Dover, New York, 1960.
[3] M. Trenkler, *Magic cubes*, Math. Gazette 82 (1998), 56–61.
[4] —, *A construction of magic cubes*, ibid., to appear.

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