A new irrationality measure for $\zeta(3)$

by

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1. Introduction and the result. The aim of this paper is to give a new irrationality measure for the number

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3},$$

as an application of Legendre-type polynomials

$$L(a,b,c,d;x) = \frac{x^a}{d!}(x^b(1-x)^c)^{(d)} \in \mathbb{Z}[x]$$

of degree $a + b + c - d$, where a, b, c, d are integers satisfying $b, c, d \geq 0$, $b + c \geq d$ and $a \geq \min\{0, d - b\}$.

The irrationality of $\zeta(3)$ was first shown by R. Apéry [1] in 1978. F. Beukers [2] reconstructed Apéry's rational approximation to $\zeta(3)$ by introducing the triple improper integral

$$\iiint_B \frac{L_n(x)L_n(y)}{1 - (1 - xy)u} \, dx \, dy \, du,$$

where $B = (0,1)^3$ is the open unit cube and $L_n(x) = L(0, n, n, n; x)$ is the usual Legendre polynomial of degree n.

Various proofs of the irrationality of $\zeta(3)$ are known. V. N. Sorokin [9] constructed a number of Hermite–Padé approximations to certain series which lead to the irrationality of $\zeta(3)$. Yu. V. Nesterenko [7], inspired by L. A. Gutnik's work [4], obtained a new continued fraction expansion of $\zeta(3)$ by using the so-called Meyer functions. M. Prévost [8] recovered Apéry's sequences using Padé approximations to the asymptotic expansion of the partial sum of $\zeta(3)$. We note that all the approximations mentioned above give the same irrationality measure

$$\mu_0 = 1 + \frac{4\log(\sqrt{2} + 1) + 3}{4\log(\sqrt{2} + 1) - 3} = 13.4178202\ldots$$

for $\zeta(3)$. Nothing is known about arithmetical properties of the values of Riemann zeta function at odd points greater than 3.

The above irrationality measure μ_0 was improved by R. Dvornicich and C. Viola [3] to $\mu_1 = 12.74359\ldots$ The author [5] obtained a fairly improved measure $\mu_2 = 8.8302837\ldots$ by introducing the Legendre-type polynomials

$$L(0, n - m, n + m, n; x) = \frac{1}{n!}(x^{n-m}(1 - x)^{n+m})^{(n)}$$

of degree n, where $m \in [1, n]$ is an integral parameter. This polynomial, regarded as a perturbation of the Legendre polynomial, has the advantage of possessing a large common factor of its coefficients and gives in fact the best known irrationality measure, for example, of $\log 2$.

In this paper, as another perturbation of the Legendre polynomial, we consider the following Legendre-type polynomials:

$$L_{n,m}(x) = L(0, n + m, n, n; x) = \frac{1}{n!}(x^{n+m}(1 - x)^n)^{(n)}$$

of degree $n+m$, where m is a positive integral parameter. It then follows from Lemma 2.1 of [5] that $L_{n,m}(x)$ is uniquely determined, up to a multiplicative non-zero constant, by the following three conditions:

(a) $\int_0^1 x^j L_{n,m}(x)\,dx = 0$ for $0 \le j < n$;
(b) $L_{n,m}(x)$ vanishes at the origin with order at least m;
(c) $\deg L_{n,m} = n + m$.

Note that the coefficients of $L_{n,m}(x)$ have no common prime factors greater than $\sqrt{2n + m}$; so it seems to be hopeless to use these polynomials in the study of the irrationality measure for $\log 2$. Nevertheless $L_{n,m}(x)$ is suitable for improving the irrationality measure for $\zeta(3)$. Indeed we have

THEOREM 1. *For any $\varepsilon > 0$ there exists an effective constant $q_0(\varepsilon)$ such that*

$$\left|\zeta(3) - \frac{p}{q}\right| \ge q^{-7.377956\ldots-\varepsilon}$$

for any integers p and q satisfying $q \ge q_0(\varepsilon)$. The exact value of the constant $\mu = 7.377956\ldots$ is given by

$$\mu = 1 + \frac{6\log c_0 + d_0}{6\log c_0 - d_0}$$

where

$$c_0 = \frac{352 + 133\sqrt{7}}{9} \quad and \quad d_0 = 26 + \pi\left\{\sqrt{3} - \cot\frac{\pi}{9} - \cot\frac{2\pi}{9}\right\}.$$

To prove this theorem we need a few lemmas given in the next section.

2. Preliminaries. The following lemma is due to Beukers [2, Lemma 1].

LEMMA 2. *For any non-negative integers r and s, we have*

$$\iiint_B \frac{x^r y^s}{1 - (1 - xy)u}\, dx\, dy\, du - 2\delta_{r,s}\zeta(3)$$

$$= \begin{cases} -\displaystyle\sum_{l=1}^{r} \frac{2}{l^3} & \text{if } r = s, \\[2ex] \displaystyle\sum_{l=\min\{r,s\}+1}^{\max\{r,s\}} \frac{1}{|r-s|l^2} & \text{if } r \neq s, \end{cases}$$

where $\delta_{r,s}$ is Kronecker's delta.

It immediately follows from the above lemma that the integral

$$I = \iiint_B \frac{P(x)Q(y)}{1 - (1 - xy)u}\, dx\, dy\, du,$$

can always be written in the form $\alpha\zeta(3) + \beta$ with $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{Q}$ for any polynomials $P(x), Q(x) \in \mathbb{Z}[x]$. We call β the *rational part* of the integral I, which is uniquely determined by the irrationality of $\zeta(3)$. Thus we have the possibility of improving the irrationality measure for $\zeta(3)$ by choosing suitable polynomials $P(x)$ and $Q(y)$. However this is not an easy task because the corresponding measure depends highly both on the asymptotic decay of the remainder term and on the arithmetical properties of the rational part.

The factor $r - s$ in the denominators on the right-hand side when $r \neq s$ in Lemma 2 is very important. For, if one of the denominators is a multiple of p^3 for some prime p satisfying $p^2 > \max\{r, s\}$, then obviously $r \equiv s \pmod{p}$. And this enables us to obtain some arithmetical information on the rational part. The details will be discussed in Section 3.

Let D_n be the least common multiple of $\{1, \ldots, n\}$. Suppose now that $0 \leq \deg P - \deg Q \leq \mathrm{ord}_0 Q$ where $\mathrm{ord}_0 Q$ denotes the order of the zero of $Q(x)$ at the origin. Then it follows from Lemma 2 that the rational part β of I belongs to $\mathbb{Z}/(D_{\deg P}^2 D_{\deg Q})$, since $|r-s| \leq \max\{\deg P - \mathrm{ord}_0 Q, \deg Q\} = \deg Q$. Furthermore this can be sharpened if either $P(x)$ or $Q(y)$ is a specific Legendre-type polynomial, as follows.

LEMMA 3. *Suppose that $P(x), Q(x) \in \mathbb{Z}[x]$ satisfy $0 \leq \deg P - \deg Q \leq \mathrm{ord}_0 Q$. Suppose further that either*

(1) $P(x) = L(a, b, c, d; x)$ *with* $\deg P - \deg Q \leq a \leq d \leq \deg Q$, *or*

(2) $Q(y) = L(a', b', c', d'; y)$

  with $\deg P - \deg Q \leq \min\{a', a' + b' - d'\}$ *and* $a' \leq d' \leq c'$.

Then the rational part β of the integral

$$I = \iiint_B \frac{P(x)Q(y)}{1-(1-xy)u}\,dx\,dy\,du$$

belongs to the set

$$\frac{\Delta}{D_{\deg P}^2 D_{\deg Q}}\mathbb{Z},$$

where Δ is the product of all primes lying in the interval $(\deg Q, \deg P]$.

Proof. Put $P(x) = \sum a_r x^r$ and $Q(y) = \sum b_s y^s$. For any prime $p \in (\deg Q, \deg P]$ one of the denominators on the right-hand side in Lemma 2 is a multiple of p^2 if and only if $r \geq p$, since $\max\{s, |r-s|\} \leq \deg Q$ for any r and s. More precisely, the sum of the coefficients of p^{-2} is equal to

$$J_p = \sum_{r \geq p}\sum_s \frac{a_r b_s}{r-s}.$$

If (1) holds, then clearly

$$a_r = (-1)^{r-a-b+d}\binom{c}{r-a-b+d}\binom{r-a+d}{d},$$

which is a multiple of p for any $r \geq p$ since $d \leq \deg Q < p \leq r \leq r-a+d$ and $r-a \leq \deg P - (\deg P - \deg Q) = \deg Q < p$.

On the other hand, if (2) holds, then

$$\sum_s \frac{b_s}{r-s} = \int_0^1 x^{r-1}Q\left(\frac{1}{x}\right)dx = \int_1^\infty t^{-r-1}Q(t)\,dt$$

$$= \frac{1}{d'!}\int_1^\infty t^{-r+a'-1}(t^{b'}(1-t)^{c'})^{(d')}\,dt$$

$$= \binom{r-a'+d'}{d'}\int_1^\infty t^{-r+a'+b'-d'-1}(1-t)^{c'}\,dt$$

$$= \binom{r-a'+d'}{d'}\int_0^1 x^{r-a'-b'-c'+d'-1}(x-1)^{c'}\,dx.$$

Since $r-a'-b'+d' \leq \deg P - a'-b'+d' \leq \deg Q$, the denominator of J_p is a divisor of $D_{\deg Q}$. The numerator of J_p is a multiple of p, since $d' \leq a'+b'-\deg P+\deg Q \leq a'+b' = \deg Q - c'+d' \leq \deg Q < p \leq r \leq r-a'+d'$ and since $r-a' \leq \deg P - (\deg P - \deg Q) = \deg Q < p$.

In either case it follows that the integer $D_{\deg Q}J_p$ is a multiple of p; therefore the integer $D_{\deg P}^2 D_{\deg Q}\beta$ is a multiple of Δ. This completes the proof. ∎

3. Construction of the approximations. We consider the integral

$$I_{n,m} = \iiint\limits_B \frac{x^m L_{n,m}(x) L_{n,m}(y)}{1 - (1 - xy)u} \, dx \, dy \, du,$$

where $m = [\lambda n]$ and $\lambda \in [0, 1)$ is a real parameter. This integral can be written as $\alpha_{n,m} \zeta(3) + \beta_{n,m}$ by Lemma 2. Putting $P(x) = x^m L_{n,m}(x)$ and $Q(y) = L_{n,m}(y)$, we have $\deg P - \deg Q = m = \mathrm{ord}_0 Q$. Moreover since $P(x) = L(m, n+m, n, n; x)$ satisfies condition (1) in Lemma 3, the rational part $\beta_{n,m}$ of $I_{n,m}$ belongs to the set

$$\frac{\Delta_n}{D_{n+2m}^2 D_{n+m}} \mathbb{Z},$$

where Δ_n is the product of all primes lying in $(n + m, n + 2m]$.

We put $P(x) = \sum a_r x^r$ and $Q(y) = \sum b_s y^s$; that is,

$$a_r = (-1)^r \binom{n}{r - 2m} \binom{n - m + r}{n} \quad \text{and} \quad b_s = (-1)^{s+m} \binom{n}{s - m} \binom{n + s}{n}.$$

Then we have $\beta_{n,m} = (-1)^m (-2\beta + \beta')$, where

$$(3) \qquad \beta = \sum_{r=2m}^{n+m} \binom{n}{r - 2m} \binom{n - m + r}{n} \binom{n}{r - m} \binom{n + r}{n} \sum_{l=1}^{r} \frac{1}{l^3}$$

and

$$(4) \qquad \beta' = \sum_{\substack{r \neq s \\ 2m \leq r \leq n + 2m \\ m \leq s \leq n + m}} (-1)^{r+s} \binom{n}{r - 2m} \binom{n - m + r}{n} \binom{n}{s - m} \binom{n + s}{n}$$

$$\times \sum_{l=\min\{r,s\}+1}^{\max\{r,s\}} \frac{1}{|r - s| l^2}.$$

Let $\omega = \{n/p\}, \eta = \{m/p\}, \theta_r = \{r/p\}$ and $\theta_s = \{s/p\}$ for brevity, where $\{x\}$ denotes the fractional part of x. Suppose now that one of the denominators on the right-hand side in (4) is a multiple of p^3 for some prime $p \in [\sqrt{3n}, n + m]$. Then clearly $r \equiv s \pmod{p}$; hence $\theta_r = \theta_s$. Put

$$S_1 = \{0 \leq \theta < 1 : \omega \geq \{\theta - 2\eta\}\},$$
$$S_2 = \{0 \leq \theta < 1 : \omega + \{\theta - \eta\} < 1\},$$
$$S_3 = \{0 \leq \theta < 1 : \omega \geq \{\theta - \eta\}\},$$
$$S_4 = [0, 1 - \omega).$$

For any prime $p > \sqrt{n}$, the exponent of p in the factorization of $\binom{n}{r - 2m}$ into prime powers is $[n/p] - [(r - 2m)/p] - [(n + 2m - r)/p] = \{\theta_r - 2\eta\} + \{\omega + 2\eta - \theta_r\} - \omega$, and therefore is 0 if $\omega \geq \{\theta_r - 2\eta\}$, and 1 if $\omega <$

$\{\theta_r - 2\eta\}$. Thus $p \mid \binom{n}{r-2m}$ if and only if $\theta_r \notin S_1$. The other binomial co-efficients $\binom{n-m+r}{n}$, $\binom{n}{s-m}$ and $\binom{n+s}{n}$ satisfy the similar property for S_2, S_3 and S_4 respectively. Note that $\sum_{j=1}^4 |S_j| = 2$, where $|\cdot|$ denotes the one-dimensional Lebesgue measure.

For any subset $S \subset [0,1)$ we define

$$\sigma_\eta(S) = \{0 \le \theta < 1 : \{\theta - \eta\} \in S\};$$

that is, $\sigma_\eta(S) \equiv \eta + S \pmod 1$. Clearly $S_3 = \sigma_\eta([0,\omega])$ and $S_j = \sigma_\eta(S_{j+2})$ for $j = 1, 2$. We need the following simple lemma.

LEMMA 4. *For any interval $K \subset [0,1)$ satisfying $|K| < \min\{\eta, 1-\eta\}$, we have*

$$K \cap \sigma_\eta(K) = \emptyset.$$

If, in addition, K is not closed, then the condition $|K| < \min\{\eta, 1-\eta\}$ can be replaced by $|K| \le \min\{\eta, 1-\eta\}$.

Proof. Suppose that $K \cap \sigma_\eta(K) \ne \emptyset$. Then take a point $\xi \in K$ with $\{\xi - \eta\} \in K$. If $\xi \ge \eta$, then $|K| \ge \eta$ since $[\xi - \eta, \xi] \subset K$. Otherwise we have $|K| \ge 1 - \eta$ since $[\xi, \xi - \eta + 1] \subset K$. Hence $|K| \ge \min\{\eta, 1 - \eta\}$. It is clear that $|K| > \min\{\eta, 1 - \eta\}$ if K is not closed. ∎

We now distinguish four cases, as follows.

CASE I: $2\omega + \eta < 1$. Obviously $S_3 = [\eta, \omega + \eta] = S_3 \cap S_4$; hence $|S_3 \cap S_4| = \omega$. Therefore it follows from Lemma 4 that

$$\bigcap_{j=1}^4 S_j = (S_3 \cap S_4) \cap \sigma_\eta(S_3 \cap S_4) = \emptyset$$

if $\omega < \eta$, since $\omega < 1 - \eta$.

CASE II: $\omega + \eta < 1$ and $2\omega + \eta \ge 1$. We also have $S_3 = [\eta, \omega + \eta]$; hence $S_3 \cap S_4 = [\eta, 1 - \omega)$ and so $|S_3 \cap S_4| = 1 - \omega - \eta$. Therefore $\bigcap_{j=1}^4 S_j = \emptyset$ if $\omega + 2\eta \ge 1$, since $1 - \omega - \eta \le 1 - \eta$.

CASE III: $\omega + \eta \ge 1$ and $2\omega + \eta < 2$. We then have $S_3 = [0, \omega + \eta - 1] \cup [\eta, 1)$; hence $S_3 \cap S_4 = [0, \omega + \eta - 1]$ and so $|S_3 \cap S_4| = \omega + \eta - 1$. Thus $\bigcap_{j=1}^4 S_j = \emptyset$ if $\omega + 2\eta < 2$, since $\omega + \eta - 1 < \eta$.

CASE IV: $2\omega + \eta \ge 2$. We have $S_3 \cap S_4 = [0, 1 - \omega)$; hence $|S_3 \cap S_4| = 1 - \omega$. Therefore $\bigcap_{j=1}^4 S_j = \emptyset$ if $\omega \ge \eta$, since $1 - \omega \le \eta$.

We next define

$$E = \{(\eta, \omega) \in [0,1)^2 : \omega < \min\{\eta, 2 - 2\eta\} \text{ or } \omega \ge \max\{\eta, 1 - 2\eta\}\},$$

which is illustrated in Figure 1. For any $(\eta, \omega) \in E$ we thus have $\bigcap_{j=1}^4 S_j = \emptyset$; hence there exists at least one $j = j(\theta)$ satisfying $\theta \notin S_{j(\theta)}$ for every
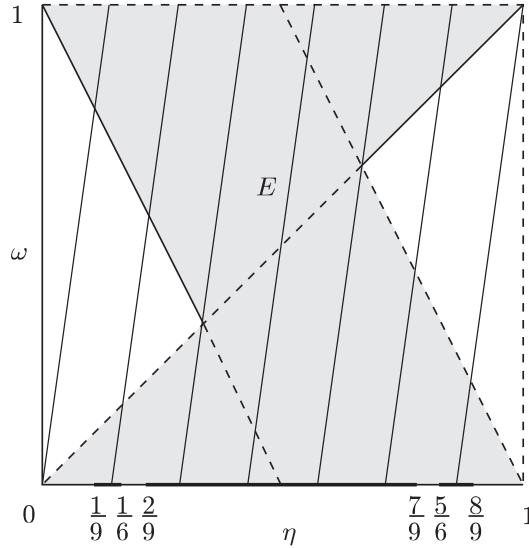
Fig. 1

$\theta \in [0, 1)$. This implies that

$$p \left| \binom{n}{r-2m} \binom{n-m+r}{n} \binom{n}{s-m} \binom{n+s}{n} \right.$$

for any $r \neq s$ satisfying $r \equiv s \pmod{p}$; namely, $p \mid D_{n+2m}^2 D_{n+m} \beta'$. Of course, the same property holds when $r = s$; hence $p \mid D_{n+m}^3 \beta$ from (3). Therefore $M_{n,m} \beta_{n,m}$ becomes an integer, where

$$M_{n,m} = \frac{D_{n+2m}^2 D_{n+m}}{\Delta_n \Delta_{n,m}} \in \mathbb{Z}$$

and $\Delta_{n,m}$ is the product of all primes $p \in [\sqrt{3n}, n+m]$ satisfying $(\eta, \omega) \in E$. Thus

(5) $$M_{n,m} \alpha_{n,m} \zeta(3) + M_{n,m} \beta_{n,m} = M_{n,m} I_{n,m},$$

our rational approximations to $\zeta(3)$. Note that $\Omega_{n,m} \equiv \Delta_n \Delta_{n,m}$ is equal to the product of all primes $p \geq \sqrt{3n}$ satisfying $(\eta, \omega) \in E$.

4. Proof of Theorem 1. It follows from (5) and Lemma 3.1 of [5] that $\zeta(3)$ has an irrationality measure

(6) $$\mu \equiv \mu(\lambda) = 1 + \frac{\sigma(\lambda) + \kappa(\lambda)}{\tau(\lambda) - \kappa(\lambda)}$$

if $\tau(\lambda) > \kappa(\lambda)$, where

$$\kappa(\lambda) = \lim_{n \to \infty} \frac{1}{n} \log M_{n,m},$$

$$\sigma(\lambda) \geq \limsup_{n\to\infty} \frac{1}{n} \log |\alpha_{n,m}| \quad \text{and} \quad \tau(\lambda) = -\lim_{n\to\infty} \frac{1}{n} \log |I_{n,m}|.$$

We first consider the asymptotic behavior of $M_{n,m}$ as $n \to \infty$. The prime number theorem implies, analogously to [5], that

$$\chi(\lambda) \equiv \lim_{n\to\infty} \frac{1}{n} \log \Omega_{n,m} = \lambda \int_W \frac{dt}{t^2},$$

where $W = \{0 < t < \infty : (\{t\}, \{t/\lambda\}) \in E\}$; hence $\kappa(\lambda) = 3 + 5\lambda - \chi(\lambda)$. The value of $\chi(\lambda)$ can be easily calculated when $\lambda = 1/k$ for any integer $k \geq 2$. Indeed we have

$$(7) \qquad k\chi\left(\frac{1}{k}\right) = \sum_{l=0}^{\infty} \int_{W_0} \frac{dx}{(x+l)^2} = \int_{W_0} d\psi(x),$$

where $W_0 = W \cap (0,1)$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. Let $H \subset [0,1)^2$ be the union of k segments defined by $\omega = \{k\eta\}$. Since the set $\overline{E}$, as well as the set $\overline{H}$, is symmetric with respect to the point $(\eta^*, \omega^*) = (1/2, 1/2)$, the set $\pi_\eta(\overline{E} \cap \overline{H})$ is also symmetric with respect to $\eta^* = 1/2$ where $\pi_\eta(\cdot)$ denotes the orthogonal projection to the η-axis. This implies that the Stieltjes integral on the right-hand side of (7) can be written as a finite sum

$$\sum_j \varepsilon_j \{\psi(1 - \gamma_j) - \psi(\gamma_j)\},$$

where $\{\gamma_j\}$ are rational numbers which are endpoints lying in $(0, 1/2)$ of the intervals in $\pi_\eta(\overline{E} \cap \overline{H})$ and the sign ε_j is $+1$ or -1 according as γ_j is the left or right endpoint. Then using the well-known formula $\psi(1 - x) - \psi(x) = \pi \cot(\pi x)$, it follows that

$$\chi\left(\frac{1}{k}\right) = \frac{\pi}{k} \sum_j \varepsilon_j \cot(\pi \gamma_j).$$

For example, in the case $k = 7$, we have $\pi_\eta(\overline{E} \cap \overline{H}) = [1/9, 1/6] \cup [2/9, 7/9] \cup [5/6, 8/9]$ (see Figure 1); so $\{\gamma_1, \gamma_2, \gamma_3\} = \{1/9, 1/6, 2/9\}$, $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{+1, -1, +1\}$ and hence

$$\chi\left(\frac{1}{7}\right) = \frac{\pi}{7}\left\{\cot\frac{\pi}{9} + \cot\frac{2\pi}{9} - \sqrt{3}\right\}.$$

Therefore we have $\kappa(1/7) = d_0/7$, where d_0 is the constant defined in Theorem 1.

We next consider the asymptotic behavior of $I_{n,m}$ as $n \to \infty$. After an n-fold partial integration with respect to y, we get

$$I_{n,m} = \iiint_B \frac{x^{n+m} L_{n,m}(x) y^{n+m}(1-y)^n u^n}{(1 - (1-xy)u)^{n+1}} \, dx \, dy \, du.$$

We now need the following convenient lemma. We call

$$\tau_d(u) = (1-u)/(1-du)$$

a *nice transformation* for any $d < 1$; it is a homeomorphism on $[0,1]$ and satisfies $\tau_d \equiv \tau_d^{-1}$. The nice transformation $\tau_{1-xy}(u)$ was used in [2].

LEMMA 5. *By substituting $v = \tau_{1-xy}(u)$ we have*

$$\int_0^1 \frac{u^a(1-u)^b}{(1-(1-xy)u)^{c+1}}\,du = (xy)^{b-c}\int_0^1 \frac{v^b(1-v)^a}{(1-(1-xy)v)^{a+b-c+1}}\,dv$$

for any non-negative integers a, b and c.

The proof is straightforward. It follows from this lemma that

$$I_{n,m} = \iiint_B \frac{x^m L_{n,m}(x)y^m(1-y)^n(1-v)^n}{1-(1-xy)v}\,dx\,dy\,dv.$$

Then, after an n-fold partial integration with respect to x,

$$I_{n,m} = \frac{(-1)^n}{n!}\iiint_B x^{n+m}(1-x)^n \frac{\partial^n}{\partial x^n}\left(\frac{x^m}{1-(1-xy)v}\right)$$

$$\times y^m(1-y)^n(1-v)^n\,dx\,dy\,dv$$

$$= (-1)^m \iiint_B \frac{x^{n+m}(1-x)^n y^n(1-y)^n v^{n-m}(1-v)^{n+m}}{(1-(1-xy)v)^{n+1}}\,dx\,dy\,dv;$$

hence we have

$$\lim_{n\to\infty}\frac{1}{n}\log|I_{n,m}| = \max_{0<x,y,v<1}\log F(x,y,v),$$

where

$$F(x,y,v) = \frac{x^{1+\lambda}(1-x)y(1-y)v^{1-\lambda}(1-v)^{1+\lambda}}{1-(1-xy)v}.$$

The above maximum is actually attained at (x^*, y^*, v^*) where

$$x^* = \frac{\lambda+\lambda'}{\lambda+\lambda'+1}, \quad y^* = \frac{\lambda'}{\lambda'+1} \quad \text{and} \quad v^* = \lambda' - \lambda$$

with $\lambda' = \sqrt{(1+\lambda)/2}$, which is a unique solution of the equations

$$\frac{\partial F}{\partial x}(x,y,v) = \frac{\partial F}{\partial y}(x,y,v) = \frac{\partial F}{\partial v}(x,y,v) = 0$$

in $(0,1)^3$. For $\lambda = 1/7$ it can be seen that $F(x^*, y^*, v^*) = c_0^{-6/7}$, where the constant c_0 is defined in Theorem 1.

We finally consider the asymptotic behavior of

$$\alpha_{n,m} = \frac{2}{2\pi i} \int\limits_{C_0} z^{m-1} L_{n,m}(z) L_{n,m}\left(\frac{1}{z}\right) dz$$

$$= \frac{2}{(2\pi i)^3} \int\limits_{C_0} \int\limits_{C_1} \int\limits_{C_2} z^{m-1} \frac{w^{n+m}(1-w)^n}{(w-z)^{n+1}} \cdot \frac{\zeta^{n+m}(1-\zeta)^n}{(\zeta-1/z)^{n+1}} \, d\zeta \, dw \, dz$$

as $n \to \infty$. Taking the contours C_0, C_1, C_2 to be the circles centered at $z = 0$, $w = z$, $\zeta = 1/z$ with radii r, R, ϱ respectively, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log|\alpha_{n,m}| \leq \min_{r,R,\varrho>0} \log G(r, R, \varrho),$$

where

$$G(r, R, \varrho) = \frac{(r+R)^{1+\lambda}(1+r+R)(1+r\varrho)^{1+\lambda}(1+r+r\varrho)}{r^2 R \varrho}.$$

The above minimum is attained at (r^*, R^*, ϱ^*) where

$$r^* = \frac{\lambda' - \lambda}{\lambda' + \lambda}, \quad R^* = \frac{2}{\lambda' + \lambda} - \frac{1}{\lambda'} \quad \text{and} \quad \varrho^* = \frac{1}{\lambda' - \lambda},$$

which is a unique solution of the equations

$$\frac{\partial G}{\partial r}(r, R, \varrho) = \frac{\partial G}{\partial R}(r, R, \varrho) = \frac{\partial G}{\partial \varrho}(r, R, \varrho) = 0$$

in $(0, \infty)^3$. Then it can be seen that $F(x^*, y^*, v^*)G(r^*, R^*, \varrho^*) = 1$; hence we can take $\sigma(\lambda) = \tau(\lambda)$ for any λ. Therefore $\zeta(3)$ has an irrationality measure

$$\mu = \mu\left(\frac{1}{7}\right) = 1 + \frac{6\log c_0 + d_0}{6\log c_0 - d_0},$$

which completes the proof of Theorem 1.

5. Concluding remarks. Theorem 1 is thus proved by taking $\lambda = 1/7$. However we do not know whether the irrationality measure $\mu(\lambda)$ in (6) attains its minimum at $\lambda = 1/7$ as the real parameter $\lambda \in (0, 1)$ varies, although numerical calculations seem to support this.

We also note that the integral

$$\iiint\limits_{B} \frac{x^m L_n(x) L_{n+m}(y)}{1 - (1-xy)u} \, dx \, dy \, du$$

gives almost the same irrationality measure μ as in Theorem 1 numerically. In this case we need to calculate the corresponding Stieltjes integral with respect to the digamma function $\psi(x)$ over twelve intervals, which is more complicated than W_0. Moreover the diffeomorphism

$$T(x, y, u) = \left(1 - (1-xy)u, \; \frac{xy}{1 - (1-xy)u}, \; \frac{1-x}{1-xy}\right)$$

acting on the unit cube B and satisfying $T \equiv T^{-1}$, converts the integral $I_{n,m}$ constructed in Section 3 into

$$\iiint_B \frac{x^m L_{n+m}(x) L_{n-m,2m}(y)}{1 - (1 - xy)u} \, dx \, dy \, du,$$

although this integral representation does not seem to give any further arithmetical information on the rational part $\beta_{n,m}$.

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