On a system of two diophantine inequalities with prime numbers

by

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1. Introduction and results. In 1952 Piatetski-Shapiro [7] considered the following analogue of the Goldbach–Waring problem: Assume that $c > 1$ is not an integer and let $\varepsilon$ be a small positive number. Let $H(c)$ denote the smallest natural number $r$ such that the inequality

$$|p_1^c + \ldots + p_r^c - N| < \varepsilon$$

is solvable in prime numbers $p_1, \ldots, p_r$ for sufficiently large $N$. Then it is proved in [7] that

$$\limsup_{c \to \infty} \frac{H(c)}{c \log c} \leq 4.$$

Piatetski-Shapiro also proved that $H(c) \leq 5$ for $1 < c < 3/2$. In [8] Tolev first improved this result for $c$ close to one. More precisely, he proved that if $1 < c < 15/14$, then the inequality

$$|p_1^c + p_2^c + p_3^c - N| < \varepsilon(N)$$

has prime solutions $p_1, p_2, p_3$ for large $N$, where

$$\varepsilon(N) = N^{-(1/c)(15/14-c)} \log^3 N.$$

This result was improved by several authors (see [1, 4, 5]).

In [9] Tolev first studied the system of two inequalities with primes

$$|p_1^c + \ldots + p_5^c - N_1| < \varepsilon_1(N_1),$$

$$|p_1^d + \ldots + p_5^d - N_2| < \varepsilon_2(N_2),$$

where $1 < d < c < 2$ are different numbers and $\varepsilon_1(N_1)$ and $\varepsilon_2(N_2)$ tend to zero as $N_1$ and $N_2$ tend to infinity. Tolev proved that if $c, d, \alpha, \beta$ are real

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numbers satisfying

\[(1.4) \quad 1 < d < c < 35/34,\]
\[(1.5) \quad 1 < \alpha < \beta < 5^{1-d/c},\]

then there exist numbers \(N_1^{(0)}, N_2^{(0)},\) depending on \(c, d, \alpha, \beta,\) such that for all real numbers \(N_1, N_2\) satisfying \(N_1 > N_1^{(0)},\) \(N_2 > N_2^{(0)}\) and

\[(1.6) \quad \alpha \leq N_2/N_1^{d/c} \leq \beta,\]

the system (1.3) has prime solutions \(p_1, \ldots, p_5\) for

\[
\varepsilon_1(N_1) = N_1^{-(1/c)(35/34-c)} \log^{12} N_1, \quad \varepsilon_2(N_2) = N_2^{-(1/d)(35/34-d)} \log^{12} N_2.
\]

In this paper we shall prove

**Theorem.** Suppose \(c\) and \(d\) are real numbers such that

\[(1.7) \quad 1 < d < c < 25/24,\]

and \(\alpha\) and \(\beta\) are real numbers satisfying (1.5). Then for all real numbers \(N_1, N_2\) satisfying (1.6), the system (1.3) has prime solutions \(p_1, \ldots, p_5\) for

\[
\varepsilon_1(N_1) = N_1^{-(1/c)(25/24-c)} \log^{335} N_1, \quad \varepsilon_2(N_2) = N_2^{-(1/d)(25/24-d)} \log^{335} N_2.
\]

A short proof, which follows the argument of Tolev [9], will be given in Section 2. The main difficulty is to prove the Proposition of Section 2, which improves Lemma 13 of Tolev [9] and is the key to our result. In Section 3, some preliminary lemmas are given. A detailed proof of the Proposition is given in Section 4. The new idea of the proof combines elementary methods and van der Corput’s classical estimates.

**Notations.** Throughout, \(c\) and \(d\) are real numbers satisfying (1.7), \(\alpha\) and \(\beta\) are real numbers satisfying (1.5), and \(\lambda\) denotes a sufficiently small positive number determined precisely by Lemma 1 of Tolev [9], depending on \(c, d, \alpha, \beta,\) \(N_1\) and \(N_2\) are large numbers satisfying (1.6), \(X = N_1^{1/c},\)

\[
\varepsilon_1(N_1) = N_1^{-(1/c)(25/24-c)} \log^{335} N_1, \quad \varepsilon_2(N_2) = N_2^{-(1/d)(25/24-d)} \log^{335} N_2,
\]

\(K_1 = \varepsilon_1^{-1} \log X,\) \(K_2 = \varepsilon_2^{-1} \log X,\) \(\eta\) is a sufficiently small positive number in terms of \(c\) and \(d,\)

\(\tau_1 = X^{3/4-c-\eta},\) \(\tau_2 = X^{3/4-d-\eta},\) \(e(t) = e^{2\pi it},\) \(\varphi(t) = e^{-\pi t},\)

\(\varphi_\delta(t) = \delta \varphi(\delta t),\) and \(\chi(t)\) is the characteristic function of the interval \([-1, 1]\). We set

\[
B = \sum_{\lambda X < p_1, \ldots, p_5 < X} \log p_1 \ldots \log p_5 \chi \left( \frac{p_1^c + \ldots + p_5^c - N_1}{\varepsilon_1 \log X} \right) \times \chi \left( \frac{p_1^d + \ldots + p_5^d - N_2}{\varepsilon_2 \log X} \right),
\]
\[ S(x, y) = \sum_{\lambda X < p < X} (\log p)e(xp^c + yp^d), \]

\[ D = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S^5(x, y)e(-N_1 x - N_2 y)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y) \, dx \, dy, \]

\[ \Omega_1 = \{(x, y) \mid \max(\left| \frac{x}{\tau_1} \right|, \left| \frac{y}{\tau_2} \right|) < 1\}, \]

\[ \Omega_2 = \{(x, y) \mid \max(\left| \frac{x}{\tau_1} \right|, \left| \frac{y}{\tau_2} \right|) \geq 1, \max(\left| \frac{x}{K_1} \right|, \left| \frac{y}{K_2} \right|) \leq 1\}, \]

\[ \Omega_3 = \{(x, y) \mid \max(\left| \frac{x}{\tau_1} \right|, \left| \frac{y}{\tau_2} \right|) \geq 1, \max(\left| \frac{x}{K_1} \right|, \left| \frac{y}{K_2} \right|) > 1\}. \]

**2. A short proof of the Theorem.** The Theorem follows if we can show that \( B \) tends to infinity as \( X \) tends to infinity. By Lemma 3 of Tolev [9], it is sufficient to show that \( D \) tends to infinity as \( X \) tends to infinity. Write

\[ D = D_1 + D_2 + D_3, \]

where

\[ D_i = \int_{\Omega_i} S^5(x, y)e(-N_1 x - N_2 y)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y) \, dx \, dy. \]

By the same arguments as in Section 4 of Tolev [9], we have

\[ D_1 \gg \varepsilon_1 \varepsilon_2 X^{5-c-d}. \]

By Lemma 4 of Tolev [9], we have

\[ D_3 \ll 1. \]

So now the Theorem follows from (2.1)–(2.4) and the estimate

\[ D_2 \ll \varepsilon_1 \varepsilon_2 X^{5-c-d}(\log X)^{-1}. \]

By Lemma 14 of Tolev [9] we have

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S^4(x, y)|\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y) \, dx \, dy \ll X^2 \log^6 X. \]

It suffices to prove the following

**Proposition.** Uniformly for \((x, y) \in \Omega_2\), we have

\[ S(x, y) \ll X^{11/12} \log^{660} X. \]

**3. Some preliminary lemmas.** In order to prove the Proposition, we need the following lemmas. Lemma 1 is Theorem 2.2 of Min [6]. Lemma 2 is Lemma 2.5 of Graham and Kolesnik [2]. Lemma 3 is contained in Lemma 2.8 of Krätzel [3]. Lemma 4 is well known (see Graham and Kolesnik [2], for example).
Lemma 1. Suppose $f(x)$ and $g(x)$ are algebraic functions in $[a, b]$ and
\[ |f'(x)| \sim 1/R, \quad |f''(x)| \ll 1/(RU), \]
\[ |g(x)| \ll G, \quad |g'(x)| \ll GU_1^{-1}, \quad U, U_1 \geq 1. \]
Then
\[
\sum_{a<n\leq b} g(n)e(f(n)) = \sum_{a<u\leq \beta} b_u \frac{g(n_u)}{\sqrt{f''(n_u)}} e(f(n_u) - un_u + 1/8)
+ O(G\log(\beta - \alpha + 2) + G(b - a + R)(U^{-1} + U_1^{-1}))
+ O(G\min(\sqrt{R}, 1/\langle \alpha \rangle) + G\min(\sqrt{R}, 1/\langle \beta \rangle)),
\]
where $[\alpha, \beta]$ is the image of $[a, b]$ under the mapping $y = f'(x)$, $n_u$ is the solution of the equation $f'(x) = u$, $b_u = \begin{cases} 1 & \text{for } \alpha < u < \beta, \\ 1/2 & \text{for } u = \alpha \in \mathbb{Z} \text{ or } u = \beta \in \mathbb{Z}, \end{cases}$ and the function $\langle t \rangle$ is defined as follows:
\[
\langle t \rangle = \begin{cases} \|t\| & \text{if } t \text{ is not an integer,} \\ \beta - \alpha & \text{otherwise,} \end{cases}
\]
where $\|t\| = \min_{n \in \mathbb{Z}} \{|t - n|\}$.

Lemma 2. Suppose $z(n)$ is any complex number and $1 \leq Q \leq N$. Then
\[
\left| \sum_{N<n\leq CN} z(n) \right|^2 \ll \frac{N}{Q} \sum_{0\leq q\leq Q} \left(1 - \frac{q}{Q}\right) \Re \sum_{N<n\leq CN-q} z(n)\overline{z(n+q)}.
\]

Lemma 3. Suppose $f(x) \ll P$ and $f'(x) \gg \Delta$ for $x \sim N$. Then
\[
\sum_{n\sim N} \min(D, \frac{1}{\|f(n)\|}) \ll (P + 1)(D + \Delta^{-1})\log(2 + \Delta^{-1}).
\]

Lemma 4. Suppose $5 < A < B \leq 2A$ and $f'''(x)$ is continuous on $[A, B]$. If $0 < c_1 \lambda_1 \leq |f'(x)| \leq c_2 \lambda_1 \leq 1/2$, then
\[
\sum_{A<n\leq B} e(f(n)) \ll \lambda_1^{-1}.
\]
If $0 < c_3 \lambda_2 \leq |f''(x)| \leq c_4 \lambda_2$, then
\[
\sum_{A<n\leq B} e(f(n)) \ll A\lambda_2^{1/2} + \lambda_2^{-1/2}.
\]

Now we prove the following two lemmas, which are important in the proof of the Proposition. Let
\[
S = S(M, a, b, \gamma_1, \gamma_2) = \sum_{M<m\leq M_1} e(am^{\gamma_1} + bm^{\gamma_2}),
\]
where $M$ and $M_1$ are positive numbers such that $5 \leq M < M_1 \leq 2M$, $a$ and $b$ are real numbers such that $ab \neq 0$, and $\gamma_1$ and $\gamma_2$ are real numbers such that $1 < \gamma_1, \gamma_2 < 2, \gamma_1 \neq \gamma_2$. Let $R = |a|M^{\gamma_1} + |b|M^{\gamma_2}$.

**Lemma 5.** If $RM^{-1} \leq 1/8$, then

$$S \ll MR^{-1/2}.$$ 

**Proof.** Suppose $R > 100$; otherwise Lemma 5 is trivial. Let

$$f(m) = am^{\gamma_1} + bm^{\gamma_2}.$$

Then

$$f'(m) = \gamma_1 am^{\gamma_1 - 1} + \gamma_2 bm^{\gamma_2 - 1}.$$

If $ab > 0$, then $R/M \leq |f'(m)| \leq 4R/M \leq 1/2$, hence the assertion follows from Lemma 4.

Now suppose $ab < 0$. Let

$$I = \{t \in [M, M_1] | |f'(t)| \leq R^{1/2}M^{-1}\},$$

$$J = \{t \in [M, M_1] | |f'(t)| > R^{1/2}M^{-1}\}.$$

By the definition we see that if $m \in J$, then

$$R^{1/2}/M \leq |f'(m)| \leq 4R/M \leq 1/2;$$

thus by Lemma 4,

$$\sum_{m \in J} e(f(m)) \ll MR^{-1/2}. \tag{3.1}$$

We only need to estimate $|I|$. If $t \in I$, then

$$\gamma_1 at^{\gamma_1} = -\gamma_2 bt^{\gamma_2} + O(R^{1/2}) = -\gamma_2 bt^{\gamma_2}(1 + O(R^{-1/2})),

$$

$$t^{\gamma_1 - \gamma_2} = \frac{-\gamma_2 b}{\gamma_1 a} (1 + O(R^{-1/2})),

$$

which implies that

$$t = \left(\frac{-\gamma_2 b}{\gamma_1 a}\right)^{1/(\gamma_1 - \gamma_2)} (1 + O(R^{-1/2}))^{1/(\gamma_1 - \gamma_2)} \tag{3.2}$$

$$= \left(\frac{-\gamma_2 b}{\gamma_1 a}\right)^{1/(\gamma_1 - \gamma_2)} (1 + O(R^{-1/2})),

$$

$$= \left(\frac{-\gamma_2 b}{\gamma_1 a}\right)^{1/(\gamma_1 - \gamma_2)} + O(MR^{-1/2}).$$

So

$$|I| \ll MR^{-1/2}. \tag{3.3}$$

Now the conclusion follows from (3.1) and (3.3).
Lemma 6. If $M \ll R \ll M^2$, then

$$S \ll R^{1/2} + MR^{-1/3}.$$  

Proof. We have

$$f''(m) = \gamma_1(\gamma_1 - 1)am^{\gamma_1 - 2} + \gamma_2(\gamma_2 - 1)bm^{\gamma_2 - 2}.$$  

If $ab > 0$, then $|f''(m)| \sim RM^{-2}$, and by Lemma 4 we get $S \ll R^{1/2} + MR^{-1/2}$. If $ab < 0$, we define $\Delta_0 = R^{2/3}M^{-2}$.

Now suppose $ab < 0$. Let $I_0 = \{ t \in [M, M_1] \mid |f''(t)| \leq \Delta_0 \}$, $I_j = \{ t \in [M, M_1] \mid 2^{j-1}\Delta_0 < |f''(t)| \leq 2^{j}\Delta_0 \leq 2R/M^2 \}$, $1 \leq j \leq \log \left( \frac{R}{2^2\Delta_0} \right) = J_0$. If $I_0$ is not empty, then by the same argument as in Lemma 5 we get $|I_0| \ll MR^{-1/3}$. Thus Lemma 4 yields

$$\sum_{M < m \leq M_1} e(f(m)) = \sum_{m \in I_0} e(f(m)) + \sum_{1 \leq j \leq J_0} \sum_{m \in I_j} e(f(m))$$

$$\ll MR^{-1/3} + \sum_{1 \leq j \leq J_0} \{ M(2^j\Delta_0)^{1/2} + (2^j\Delta_0)^{-1/2} \}$$

$$\ll MR^{-1/3} + R^{1/2}.$$  

This completes the proof.

4. Proof of the Proposition. In this section we shall estimate $S(x, y)$ for $(x, y) \in \Omega_2$. Suppose $1 < d < c < 25/24$ and fix $(x, y) \in \Omega_2$. Let $R = |x|X^c + |y|X^d$. Obviously, $X^{3/4-\eta} \ll R \ll X^{25/24} \log^{-300} X$. Without loss of generality, we suppose $xy \neq 0$. For the case $x = 0$ or $y = 0$, previous methods yield better results (see [1, 5]).

Lemma 7. Suppose $a(m)$ are complex numbers such that

$$\sum_{m \sim M} |a(m)|^2 \ll M \log^{2A} M, \quad A > 0.$$  

Then for $M \ll \min(X^{2/3}, X^{19/12}R^{-1})$, $MN \sim X$, we have

$$S_1 = \sum_{m \sim M} a(m) \sum_{n \sim N} e(xmn^c + ymn^d) \ll X^{11/12} \log^{A+1} X.$$  

Proof. If $M \ll X^{11/12}R^{-1/2}$, then by Lemma 6 we get

$$S_1 \ll M(R^{1/2} + NR^{-1/3}) \log A X \ll X^{11/12} \log A X.$$  

From now on we always suppose $M \gg X^{11/12}R^{-1/2}$. Let $Q = [X^{1/6}]$.

By Cauchy’s inequality and Lemma 2 we have

$$(4.3) \quad |S_t|^2 \ll \sum_{m \sim M} |a(m)|^2 \sum_{m \sim M} \left| \sum_{n \sim N} e(x(mn)^c + y(mn)^d) \right|^2$$

$$\ll X^2 Q^{-1} \log^{2A} X + XQ^{-1} \log^{2A} X \sum_{q=1}^{Q} |E_q|,$$

where

$$E_q = \sum_{m \sim M} \sum_{N < n \leq 2N - q} e(xm^c \Delta(n, q; c) + ym^d \Delta(n, q; d)),$$

$$\Delta(n, q; t) = (n + q)^t - n^t.$$

Now the problem is reduced to showing that

$$(4.4) \quad \sum_{q=1}^{Q} |E_q| \ll X \log^2 X.$$  

For each fixed $1 \leq q \leq Q$, let

$$f(m, n) = xm^c \Delta(n, q; c) + ym^d \Delta(n, q; d).$$

We first consider several simple cases.

**Case 0: A special case.** For constants $a, b > 0$, let $N(a, b)$ denote the solution of the inequality

$$(4.5) \quad |ax(mn)^c + by(mn)^d| \leq \frac{R}{Q^{1/2 \log X}}, \quad m \sim M, \ n \sim N.$$  

Suppose $0 < \sigma < 1$ is a positive constant small enough. Then we can prove that uniformly for $a, b \in [\sigma, 1/\sigma]$, we have

$$(4.6) \quad N(a, b) \ll_{\sigma} X^{11/12}.$$  

If $xy > 0$, then $N(a, b) = 0$; so suppose $xy < 0$. If $(m, n)$ satisfies the inequality (4.5), then

$$ax(mn)^c = -by(mn)^d + O \left( \frac{R}{Q^{1/2 \log X}} \right)$$

$$= -by(mn)^d (1 + O(Q^{-1/2 \log^{-1} X})),$$

which implies that
\[ mn = \left( \frac{-by}{ax} \right)^{1/(c-d)} \left( 1 + O(Q^{-1/2} \log^{-1} X) \right)^{1/(c-d)} \]
\[ = \left( \frac{-by}{ax} \right)^{1/(c-d)} \left( 1 + O(Q^{-1/2} \log^{-1} X) \right) \]
\[ = \left( \frac{-by}{ax} \right)^{1/(c-d)} + O(XQ^{-1/2} \log^{-1} X). \]

Thus (4.5) follows from a divisor argument. Why we study this case will be explained later.

**Case 1:** \(|\partial f/\partial m| \leq 500^{-1}\). It is obvious that
\[ |xm^c\Delta(n, q; c)| \sim q|x|m^c n^{c-1} \sim q|x|X^c N^{-1}, \]
\[ |ym^d\Delta(n, q; d)| \sim q|y|m^d n^{d-1} \sim q|y|X^d N^{-1}, \]
thus
\[ |xm^c\Delta(n, q; c)| + |ym^d\Delta(n, q; d)| \sim qRN^{-1}. \]
We use Lemma 5 to estimate the sum over \( m \) and get
\[ E_q \ll NM(qRN^{-1})^{-1/2} \ll MN^{3/2} q^{-1/2} R^{-1/2}. \]
Summing over \( q \) we find that (4.4) holds if noticing \( M \gg X^{11/12} R^{-1/2} \) and \( R \ll X^{25/24} \).

**Case 2:** \(|\partial f/\partial n| \leq 500^{-1}\). For fixed \( m \), we estimate the sum over \( n \).
Since
\[ \partial f/\partial n = cxm^c\Delta(n, q; c-1) + dym^d\Delta(n, q; d-1), \]
\[ \Delta(n, q; c-1) = (c-1)qn^{c-2} + O(q^2 N^{c-3}), \]
\[ \Delta(n, q; d-1) = (d-1)qn^{d-2} + O(q^2 N^{d-3}), \]
we get
\[ \partial f/\partial n = c(c-1)xqm^c n^{c-2} + d(d-1)yqm^d n^{d-2} + O(q^2 RN^{-3}). \]
If \( xy > 0 \), then
\[ c_1 qRN^{-2} < |\partial f/\partial n| \leq c_2 qRN^{-2} < 1/2 \]
for some constants \( c_1, c_2 > 0 \). Thus by Lemma 4 we get
\[ E_q \ll MN^2 q^{-1} R^{-1}. \]
Now suppose \( xy < 0 \), \( 0 < \delta = o(qRN^{-2}) \) is a parameter to be determined. Define
\[ I = \{ t \in [N, 2N-q] \mid |\partial f/\partial t| \leq \delta \}, \]
\[ J = \{ t \in [N, 2N-q] \mid |\partial f/\partial t| > \delta \}. \]
If \( n \in I \), then we have

\[
c(c-1)xqm^cn^{c-2} = -d(d-1)yqm^dn^{d-2} + O(\delta + q^2RN^{-3})
\]

\[
= -d(d-1)yqm^dn^{d-2}(1 + O(\delta N^2(qR)^{-1} + qN^{-1})),
\]

which gives

\[
n = \left(\frac{-d(d-1)yqm^d}{c(c-1)xm^c}\right)^{1/(c-d)} (1 + O(\delta N^2(qR)^{-1} + qN^{-1}))^{1/(c-d)}
\]

\[
= \left(\frac{-d(d-1)yqm^d}{c(c-1)xm^c}\right)^{1/(c-d)} (1 + O(\delta N^2(qR)^{-1} + qN^{-1}))
\]

\[
= \left(\frac{-d(d-1)yqm^d}{c(c-1)xm^c}\right)^{1/(c-d)} + (q + \delta N^3q^{-1}R^{-1}).
\]

Thus

\[ (4.7) \quad |I| \ll q + \delta N^3q^{-1}R^{-1}. \]

By Lemma 4 we get

\[ (4.8) \quad \sum_{n \in I, |\partial f/\partial n| \leq 500^{-1}} e(f(m,n)) \ll \delta^{-1}. \]

Thus we get

\[ (4.9) \quad \sum_{n \sim N, |\partial f/\partial n| \leq 500^{-1}} e(f(m,n)) \ll q + N^{3/2}(qR)^{-1/2}, \]

by choosing \( \delta = (qR)^{1/2}N^{-3/2} \).

Combining the above, we get

\[ (4.10) \quad \sum_{|\partial f/\partial n| \leq 500^{-1}} e(f(m,n)) \ll Mq + MN^{3/2}(qR)^{-1/2} + MN^2(qR)^{-1}. \]

Summing over \( q \) we find

\[ (4.11) \quad \sum_{q} \sum_{\langle m,n \rangle} e(f(m,n)) \ll MQ^2 + MN^{3/2}Q^{1/2}R^{-1/2} + MN^2R^{-1} \log Q \ll X \log X, \]

if we recall \( X^{11/12}R^{-1/2} \ll M \ll X^{2/3} \).

**Case 3:** For some \( i \) and \( j \), \( 2 \leq i + j \leq 3 \),

\[
(*) \quad \left| \frac{\partial^{i+j} f}{\partial m^i \partial n^j} \right| \leq \frac{qR \log X}{QM^iN^j+1}.
\]
Let \(c(\gamma, 0) = 1, c(\gamma, n) = \gamma(\gamma - 1) \ldots (\gamma - n + 1)\) for \(n \neq 0\). Then

\[
\frac{\partial^{i+j} f}{\partial m^i \partial n^j} = c(c, i)c(c, j)x^m c^{i-j} \Delta(n, q; c - j)
+ c(d, i)c(d, j)y^m d^{i-j} \Delta(n, q; d - j).
\]

Since \(c(c, i)c(c, j)\) and \(c(d, i)c(d, j)\) always have the same sign, we may suppose \(xy < 0\); otherwise there is no \((m, n)\) satisfying \((\ast)\).

If \((m, n)\) satisfies \((\ast)\), then

\[
c(c, i)c(c, j)x^m c^{i-j} \Delta(n, q; c - j)
= -c(d, i)c(d, j)y^m d^{i-j} \Delta(n, q; d - j)
+ O\left(\frac{qR \log X}{QM^2 N^{j+1}}\right).
\]

which implies that

\[
m = \left(\frac{-c(d, i)c(d, j)y \Delta(n, q; d - j)}{c(c, i)c(c, j)x \Delta(n, q; c - j)}\right)^{1/(c-d)}
\left(1 + O\left(\frac{\log X}{Q}\right)\right)^{1/(c-d)}
= \left(\frac{-c(d, i)c(d, j)y \Delta(n, q; d - j)}{c(c, i)c(c, j)x \Delta(n, q; c - j)}\right)^{1/(c-d)}
+ O\left(\frac{M \log X}{Q}\right).
\]

Thus

\[
\sum_{(m, n), (\ast)} e(f(m, n)) \ll \frac{X \log X}{Q}
\]

and

\[
(4.12) \sum_q \sum_{(m, n), (\ast)} e(f(m, n)) \ll X \log X.
\]

Now we turn to the most difficult part. We suppose that none of the conditions from Cases 0 to 3 holds. Without loss of generality, we suppose \(\partial f / \partial n > 0\). For any fixed \(0 \leq j \leq \log 10Q / \log 2\), let \(I_j\) denote the subinterval of \([N, 2N - q]\) in which

\[
2^j qR < \left|\frac{\partial^2 f}{\partial n^2}\right| \leq 2^{j+1} qR.
\]

We suppose \(I_j = [A_j, B_j]\), say; \(A_j\) and \(B_j\) may depend on \(m\), but this does not affect our final result.
By Lemma 1 we get

$$\sum_{n \in I_j} e(f(m, n)) = e(1/8) \sum_{v_1(m) < v \leq v_2(m)} b_v e(s(m, v)) \frac{1}{\sqrt{|G(m, v)|}} + O(R(m, q, j)),$$

where

$$f_n(m, g(m, v)) = v, \quad s(m, v) = f(m, g(m, v)) - vg(m, v), \quad G(m, v) = f_{nn}(m, g(m, v)), \quad R(m, q, j) = \log X + \frac{QN^2}{2^j qR} + \min \left( \frac{1}{2^{j/2}q^{1/2}R^{1/2}}, \frac{1}{\|v_1(m)\|} \right)$$

$$+ \min \left( \frac{Q^{1/2}N^{3/2}}{2^{j/2}q^{1/2}R^{1/2}}, \frac{1}{\|v_2(m)\|} \right),$$

$$\frac{qR}{QN^2} \ll v_1(m), v_2(m) \ll \frac{qR}{N^2}.$$

Since

$$qRN^{-2} \gg 1,$$

$$v'_1(m) = \frac{\partial^2 f}{\partial n \partial m}(m, B_j) \gg qRq^{-1}M^{-1}N^{-2},$$

$$v'_2(m) = \frac{\partial^2 f}{\partial n \partial m}(m, A_j) \gg qRq^{-1}M^{-1}N^{-2},$$

by Lemma 3 we get

$$\sum_{1 \leq q \leq Q} \sum_{j \geq 0} \sum_m R(m, q, j)$$

$$\ll \sum_{1 \leq q \leq Q} \sum_{j \geq 0} \left( M \log X + \frac{QM^2}{2^j qR} + \frac{Q^{1/2}N^{3/2}}{2^{j/2}q^{1/2}R^{1/2}} + \frac{qR}{N^2} \cdot \frac{QM^2}{qR} \right)$$

$$\ll MQ^2 \log^2 X + QMN^2R^{-1} \log X + Q^2 R^{1/2}N^{-1/2}$$

$$\ll X \log^2 X.$$

Let $v_1 = \min v_1(m), v_2 = \max v_2(m)$. Then

$$\sum_{M < m \leq 2M} \sum_{v_1(m) < v \leq v_2(m)} \frac{b_v e(s(m, v))}{\sqrt{|G(m, v)|}} \ll \sum_{v_1 \leq v \leq v_2} \sum_{m \in I_v} \frac{e(s(m, v))}{\sqrt{|G(m, v)|}}.$$

where $I_v$ is a subinterval of $[M, 2M]$.

Now the problem is reduced to estimating the sum over $m$. We first prove that $|G(m, v)|^{-1/2}$ is monotonic. Let $g = g(m, v)$. Differentiating the
equation \( f_n(m, g(m, v)) = v \) over \( m \) we get
\[
g_m(m, v) = -\frac{f_{nm}(m, g)}{f_{nn}(m, g)}.
\]
Thus
\[
(4.16) \quad G_m(m, v) = f_{mnn} + f_{nmm}g_m = \frac{f_{mnm}f_{nn} - f_{nmm}f_{nn}}{f_{nn}}.
\]
We only need to consider \( f_{mnm}f_{nn} - f_{nmm}f_{nn} \), since \( f_{nn} \) always has the same sign. Here we remark that we actually consider subintervals of \([M, 2M]\) such that \( f_{nn} \) is always positive or negative. This is so for other derivatives.

We now compute the corresponding derivatives. We have
\[
f_{nm} = c^2 x m^{c-1} \Delta(g, q; c - 1) + d^2 y m^{d-1} \Delta(g, q; d - 1)
\]
\[
= c^2 (c - 1) x m^{c-1} g^{c-2} + d^2 (d - 1) y m^{d-1} g^{d-2} + O\left(\frac{q^2 R}{M N^3}\right).
\]
Since \( |f_{nm}| > (qR \log X)/(QM N^2) \), we have
\[
f_{nm} = (c^2 (c - 1) x m^{c-1} g^{c-2} + d^2 (d - 1) y m^{d-1} g^{d-2}) \left(1 + O\left(\frac{Q^2}{N \log X}\right)\right).
\]
Similarly,
\[
f_{nn} = (c(c - 1)(c - 2) x m^{c-1} g^{c-3} + d(d - 1)(d - 2) y m^{d-1} g^{d-3})
\]
\[
\times \left(1 + O\left(\frac{Q^2}{N \log X}\right)\right),
\]
\[
f_{mnm} = (c^2 (c - 1)(c - 2) x m^{c-1} g^{c-3} + d^2 (d - 1)(d - 2) y m^{d-1} g^{d-3})
\]
\[
\times \left(1 + O\left(\frac{Q^2}{N \log X}\right)\right),
\]
\[
f_{nmm} = (D(c) x m^{c-1} g^{c-4} + D(d) y m^{d-1} g^{d-4}) \left(1 + O\left(\frac{Q^2}{N \log X}\right)\right),
\]
where \( D(\gamma) = \gamma(\gamma - 1)(\gamma - 2)(\gamma - 3) \).

For simplicity, we write \( s = x m^c g^c \), \( t = y m^d g^d \). Then we get
\[
(4.18) \quad f_{nn}f_{mnm} - f_{nmm}f_{nn} = m^{-1} g^{-6} \left(As^2 + 2Bst + Ct^2\right) \left(1 + O\left(\frac{Q^2}{N \log X}\right)\right),
\]
where
\[
A = c^3 (c - 2)^2 (c - 2) < 0,
\]
\[
B = c(c - 1)d(d - 1)(3cd - c^2 - d^2 - c - d) < 0,
\]
\[
C = d^3 (d - 2)^2 (d - 2) < 0.
\]
We only need to show that
\[(4.19) \quad A s^2 + 2 B s t + C t^2 \neq 0.\]

If \(xy > 0\), (4.19) is obvious. Now suppose \(xy < 0\). It is easy to show that
\[B^2 - AC = c^2(c - 1)^2 d^2(d - 1)^2(c - d)^2(2c + 2d + 1 + c^2 + d^2 - 4cd) > 0.\]
Thus there exist constants \(a_1, a_2, b_1, b_2\) such that
\[A s^2 + 2 B s t + C t^2 = (a_1 s + b_1 t)(a_2 s + b_2 t).\]

Since \(A < 0, B < 0, C < 0\), it can be easily seen that \(a_1 b_1 > 0, a_2 b_2 > 0\). Now we recall that \(s\) and \(t\) do not satisfy the condition of Case 0. Taking
\[\sigma = \frac{1}{2} \min(|a_1|, |a_2|, |b_1|^{-1}, |b_2|^{-1})\]
in Case 0, we obtain
\[|a_1 s + b_1 t| > \frac{R}{Q^{1/2} \log X}, \quad |a_2 s + b_2 t| > \frac{R}{Q^{1/2} \log X}.\]

Thus
\[|A s^2 + 2 B s t + C t^2| \geq \frac{R^2}{Q \log^2 X}.\]

This is the reason why we consider Case 0.

By the above discussion we know that \(|G(m, v)|\) is monotonic in \(m\). So is \(|G(m, v)|^{-1/2}\).

Now we compute \(s_{mm}(m, v)\). We have
\[(4.20) \quad s_{mm}(m, v) = f_{mm}(m, g) + f_{mn}(m, g) g_m - v g_m = f_{mm}(m, g),\]
\[s_{mm}(m, v) = f_{mm}(m, g) + f_{mn}(m, g) g_m = (f_{mm} f_{nn} - f_{mn}^2) / f_{nn}.\]

Similar to \(G_m\), we have
\[f_{mm} f_{nn} - f_{mn}^2 = -\frac{2 N^2}{M^2} (A_1 s^2 + B_1 s t + C_1 t^2) \left(1 + O\left(\frac{Q^2}{N \log X}\right)\right),\]

where \(A_1 = c^3(c - 1)^2, B_1 = c(c - 1) d(d - 1) (c + d), C_1 = d^3(d - 1)^2, B_1^2 - 4 A_1 C_1 > 0\). Now if \(xy > 0\), we immediately get
\[|f_{mm} f_{nn} - f_{mn}^2| \geq \frac{q^2 R^2}{M^2 N^4};\]
if \(xy < 0\), then similar to \(G_m\), we have
\[|A_1 s^2 + B_1 s t + C_1 t^2| \geq \frac{R^2}{Q \log^2 X},\]

which implies
\[|f_{mm} f_{nn} - f_{mn}^2| \geq \frac{q^2 R^2}{Q M^2 N^4 \log^2 X}.\]

Combining the above, we get
\[(4.21) \quad |s_{mm}| \geq \frac{q R}{Q M^2 N \log^2 X}.\]
On the other hand, we trivially have
\[
(4.22) \quad |s_{mm}| \ll |f_{mm}| + |f_{mn}g_m| \ll \frac{qR}{M^2N} + \frac{qR}{N^2M} \cdot \frac{N}{M} \ll \frac{qR}{M^2N}.
\]

Now let
\[
I_{v,l} = \left\{ m \in I_v \left| \frac{2^l qR}{QM^2N \log^2 X} \leq |s_{mm}| \leq \frac{2^{l+1} qR}{QM^2N \log^2 X} \right. \right\},
0 \leq l \leq \log(Q \log X) / \log 2.
\]

Then by partial summation and Lemma 4 we get
\[
(4.23) \quad \sum_{q=1}^{Q} \sum_{j \geq 0} \sum_{v \in v_1} \sum_{m \in I_{v,l}} \left| \sum_{m \in I_v} e(s(m, v)) \right| \leq \sum_{q=1}^{Q} \sum_{j \geq 0} \sum_{v \in v_1} \sum_{l \geq 0} \left| \sum_{m \in I_{v,l}} e(s(m, v)) \right| \leq \sum_{q=1}^{Q} \sum_{j \geq 0} \sum_{v \in v_1} \sum_{l \geq 0} \left( \frac{QN^3}{qR} \right)^{1/2} \times \left( M \left( \frac{2^l qR}{QM^2N \log^2 X} \right)^{1/2} + \left( \frac{QM^2N \log^2 X}{2^l qR} \right)^{1/2} \right) \leq \sum_{q=1}^{Q} \sum_{j \geq 0} \sum_{v \in v_1} \sum_{l \geq 0} \left( \frac{QN^3}{qR} \right)^{1/2} \left( \frac{(qR)^{1/2}}{N^{1/2}} + \frac{M(QN \log^2 X)^{1/2}}{(qR)^{1/2}} \right) \leq \sum_{q=1}^{Q} \sum_{j \geq 0} \sum_{v \in v_1} \sum_{l \geq 0} \frac{qR}{N^2} \left( \frac{QN^3}{qR} \right)^{1/2} \left( \frac{(qR)^{1/2}}{N^{1/2}} + \frac{M(QN \log^2 X)^{1/2}}{(qR)^{1/2}} \right) \leq Q^{5/2} R N^{-1} \log^2 X + MQ^2 \log^2 X \leq X \log^2 X,
\]
if we recall the condition \( M \ll \min(x^{2/3}, x^{19/12} R^{-1}) \). This completes the proof of Lemma 7.

**Lemma 8.** Suppose \( a_m \) and \( b_n \) are complex numbers such that
\[
\sum_{m \sim M} |a_m|^2 \ll M \log^{2A} M, \quad \sum_{n \sim N} |b_n|^2 \ll N \log^{2A} N, \quad A > 0, B > 0.
\]

Then for \( X^{1/6} \ll N \ll \min(X^{3/2} R^{-1}, RX^{-1/3}) \), we have
\[
(4.24) \quad S_{II} = \sum_{m \sim M} \sum_{n \sim N} a_m b_n e(x(mn)^c + y(mn)^d) \ll X^{11/12} \log^{A+B+1} X.
\]
Proof. Take \( Q = \lfloor X^{1/6} \log^{-1} X \rfloor = o(N) \). Then by Cauchy’s inequality and Lemma 2 again we get

\[
|S_II|^2 \ll \frac{X^2 \log^{2A+2B} X}{Q} + \frac{X \log^{2A} X}{Q} \sum_{q=1}^{Q} \sum_{n} |b_n b_{n+q}| \sum_{m \sim M} e(f(m, n)),
\]

where \( f(m, n) \) is defined as in the proof of Lemma 7.

By Lemma 6 we get

\[
\sum_{m \sim M} e(f(m, n)) \ll q^{1/2} R^{1/2} N^{-1/2} + MN^{1/3} q^{-1/3} R^{-1/3}.
\]

Notice that for fixed \( q \), we have

\[
\sum_{n} |b_n b_{n+q}| \ll \sum_{n} |b_n|^2 + \sum_{n} |b_{n+q}|^2 \ll N \log^{2B} N.
\]

The conclusion follows from the above three estimates.

Now we prove our Proposition. Let

\[
D = \min(X^{2/3}, X^{19/12} R^{-1}), \quad E = \min(X^{3/2} R^{-1}, RX^{-1/3}), \quad F = X^{1/6}.
\]

Then it is easy to check that under our assumptions we have

\[
DE > X, \quad X/D > (2X)^{1/13}, \quad F^2 < E.
\]

Using Heath-Brown’s identity \( (k = 13) \) we know that \( S(x, y) \) can be written as \( O(\log^{26} X) \) exponential sums of the form

\[
T = \sum_{n_1 \sim N_1} \ldots \sum_{n_{26} \sim N_{26}} a_1(n_1) \ldots a_{26}(n_{26}) e(x(n_1 \ldots n_{26})^c + y(n_1 \ldots n_{26})^d),
\]

where

\[
N_i < n_i \leq 2N_i (i = 1, \ldots, 26), \quad X \ll N_1 \ldots N_{26} \ll X,
\]

\[
N_i \leq (2X)^{1/13} (i = 14, \ldots, 26),
\]

\[
a_1(n_1) = \log n_1, \quad a_i(n_i) = 1 (i = 2, \ldots, 13),
\]

\[
a_i(n_i) = \mu(n_i) (i = 14, \ldots, 26).
\]

Some \( n_i \) may only take value 1. It suffices to show that for each \( T \) we have

\[
T \ll X^{11/12} \log^{630} X.
\]

We consider three cases.

Case 1: There is an \( N_j \) such that \( N_j \geq X/D \). Since \( X/D > X^{1/13} \), it follows that \( 1 \leq j \leq 13 \). Without loss of generality, suppose \( j = 1 \). Let \( m = n_2 n_3 \ldots n_{26}, a_m = \sum_{m=n_2 n_3 \ldots n_{26}} \mu(n_{14}) \ldots \mu(n_{26}) \ll d_{25}(m), n = n_1 \).
Then $T$ is a sum of type I. By partial summation, Lemma 7 and a divisor argument we get

$$T \ll X^{11/12} \log^{630} X.$$

**Case 2:** There is an $N_j$ such that $F \leq N_j < X/D \leq E$. In this case we take $n = n_j$, $m = \prod_{i \neq j} n_i$. Then $T$ forms a sum of type II and (4.28) follows from Lemma 8.

**Case 3:** $N_j < F$ ($j = 1, \ldots, 26$). Without loss of generality, we suppose $N_1 \geq \ldots \geq N_{26}$. Let $1 \leq l \leq 26$ be an integer such that

$$N_1 \ldots N_{l-1} \leq F, \quad N_1 \ldots N_l > F.
$$

It is easy to check that $3 \leq l \leq 23$. We have

$$F < N_1 \ldots N_l = (N_1 \ldots N_{l-1})N_l < F^2 < E.$$

Let $n = n_1 \ldots n_l$, $m = n_{l+1} \ldots n_{26}$, $a_n = \prod_{i=1}^{l} a_i(n_i)$, $b_m = \prod_{i=l+1}^{26} a_i(n_i)$. Then $T$ forms a sum of type II and (4.28) follows from Lemma 8.

Now the Proposition follows from the above three cases.

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