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Transcendence measure for η/ω

by

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Dedicated to my parents

1. Introduction. Let $\wp(z)$ be the Weierstrass elliptic function with invariants g_2 and g_3 and fundamental periods ω_1 and ω_2 such that $\mathrm{Im}(\omega_2/\omega_1) > 0$. Let $\zeta(z)$ be the zeta function associated with $\wp(z)$. For any period ω of $\wp(z)$, let $\eta(\omega)$ be the quasi-period of $\wp(z)$. Thus $\zeta(z+\omega) = \zeta(z) + \eta(\omega)$. Let $|\ | = |\ |_{\mathbb{C}}$ denote the ordinary absolute value in $\mathbb{C}$. For any polynomial $B(X) \in \mathbb{Z}[X]$, we denote by $H(B)$ the maximum of the absolute values of the coefficients of B. For any non-zero algebraic number α, we define the degree and height of α as the degree and height of the minimal polynomial of α. In this paper we prove

Theorem. *For $i \in \{1,2,3\}$, let α_i be an algebraic number of height h_i and degree d_i. Suppose*

$$[\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3):\mathbb{Q}] = d^* \quad \text{and} \quad 1 + \sum_{i=1}^{3} \frac{\log h_i}{d_i} = h^*.$$

Then for any period ω of $\wp(z)$, we have

$$\max\left(\left|\frac{\eta(\omega)}{\omega} - \alpha_1\right|, |g_2 - \alpha_2|, |g_3 - \alpha_3|\right) > \exp\{-C_0((h^* d^* \log(h^* d^* + 2))^2 + (d^*)^2 \log^4(d^* + 2))\}$$

where C_0 is an effectively computable number depending only on g_2, g_3 and ω.

Let the invariants g_2 and g_3 be algebraic. In 1937, Schneider [11] showed that $\eta(\omega)/\omega$ is transcendental. In 1980, Reyssat [10, p. 90, inequality (3)] gave an approximation measure for $\eta(\omega)/\omega$. Reyssat proved that for any algebraic number α of degree $\le d$ and height $\le h$ with $h > e^e$,

$$|\omega - \alpha\eta(\omega)| > \exp\{-C_1(d \log h \log\log h + (d \log d)^3)\} \tag{1}$$

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where C_1 is an effectively computable number depending only on g_2, g_3 and ω. Thus it follows that $\eta(\omega)/\omega$ has transcendence type $\le 3+\varepsilon$ for any $\varepsilon > 0$. As a consequence of the Theorem, we deduce

COROLLARY 1. *Let g_2, g_3 and α be algebraic numbers with α having height $\le h$ and degree $\le d$ where $h > e^e$ and $d > e$. Then for any period ω of $\wp(z)$ we have*

$$|\omega - \alpha\eta(\omega)| > \exp\{-C_2(\log^2 h(\log\log h)^2 + d^2\log^4 d)\} \tag{2}$$

where C_2 is an effectively computable number depending only on g_2, g_3 and ω.

For the deduction of the above corollary, we take in the Theorem $\alpha_2 = g_2$, $\alpha_3 = g_3$ and observe that $d^* \le c_1 d$ and $h^* d^* \le c_2(d + \log h)$ where c_1, c_2 are effectively computable numbers depending only on g_2, g_3 and ω. Thus it follows that $\eta(\omega)/\omega$ has transcendence type $\le 2 + \varepsilon$ for any $\varepsilon > 0$. We observe that (2) is better than (1) whenever $\log h \le d^{3/2}(\log d)^{1/2}$. By a straightforward comparison, we combine the two bounds in (1) and (2) to get

COROLLARY 2. *Let g_2, g_3 and α be algebraic numbers with α having height $\le h$ and degree $\le d$ where $h > e^e$ and $d > e$. Then for any period ω of $\wp(z)$ we have*

$$-\log|\omega - \alpha\eta(\omega)| \ll \begin{cases} d^2\log^4 d & \text{if } \log h \le d\log d, \\ \log^2 h(\log\log h)^2 & \text{if } d\log d < \log h \le d^{3/2}(\log d)^{1/2}, \\ d^3\log^3 d & \text{if } d^{3/2}(\log d)^{1/2} < \log h \le (d\log d)^2, \\ d\log h\log\log h & \text{if } \log h > (d\log d)^2, \end{cases}$$

where the constant involved in the symbol $\ll$ is effectively computable depending only on g_2, g_3 and ω.

An approximation measure for $\eta(\omega)/\omega$ as in Corollary 1 leads to a transcendence measure for $\eta(\omega)/\omega$. See Lang [5, p. 61] and Waldschmidt [13]. In Section 4 we shall use the result of Diaz and Mignotte [1] to deduce from Corollary 1 the following result.

COROLLARY 3. *Let g_2 and g_3 be algebraic and ω any period of $\wp(z)$. Let $B(X) \in \mathbb{Z}[X]$ be any non-zero polynomial with $H(B) \le H$ and $\deg B \le d$ where $H > e^e$ and $d > e$. Then*

$$|B(\eta(\omega)/\omega)| > \exp\{-C_3(\log^2 H\,(\log\log H)^2 + d^2\log^4 d)\}$$

where C_3 is an effectively computable number depending only on g_2, g_3 and ω.

As remarked by Reyssat in [9], if $\wp$ has complex multiplication with fundamental periods ω_1, ω_2, then for any algebraic number α of height $\le h$

and degree $\leq d$, the numbers

$$|\eta(\omega_1) - \alpha\eta(\omega_2)| \text{ with } g_2g_3 \neq 0 \quad \text{and} \quad |\omega_2 - \alpha\eta(\omega_1)|$$

will also have the same estimate as in Corollary 1. Hence for $i, j \in \{1, 2\}$, the numbers

$$\eta(\omega_i)/\eta(\omega_j) \text{ with } i \neq j,\ g_2g_3 \neq 0 \quad \text{and} \quad \eta(\omega_i)/\omega_j$$

have transcendence type $\leq 2 + \varepsilon$ for any $\varepsilon > 0$.

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2. Main Proposition and proof of the Theorem. In this section, we construct an auxiliary function and use it to prove the Theorem. Reyssat uses $\wp(z)$ and the corresponding zeta function $\zeta(z)$ for the construction of the auxiliary function. The main method of his proof is the Schneider–Gelfond method together with the knowledge of the number of zeros of certain meromorphic functions involving $\wp(z)$ and $\zeta(z)$. For proving our theorem, we use the Ramanujan functions which are defined for any $z \in \mathbb{C}$ with $|z| < 1$ as follows:

$$P(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)z^n, \qquad Q(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)z^n,$$
$$R(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)z^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$. Here the sum is taken over positive divisors of n. These functions satisfy the differential equations

$$(3) \qquad \theta P = \tfrac{1}{12}(P^2 - Q), \qquad \theta Q = \tfrac{1}{3}(PQ - R), \qquad \theta R = \tfrac{1}{2}(PR - Q^2)$$

where θ is the differential operator zd/dz. The Ramanujan functions are closely connected to the Weierstrass elliptic function as follows. Let $q = e^{2\pi i\omega_2/\omega_1}$ where ω_1 and ω_2 are fundamental periods with $\mathrm{Im}(\omega_2/\omega_1) > 0$. Then from Lang [6, Ch. 4] it is known that

$$(4) \quad P(q) = 3\frac{\omega_1}{\pi}\cdot\frac{\eta(\omega_1)}{\pi}, \qquad Q(q) = \frac{3}{4}\left(\frac{\omega_1}{\pi}\right)^4 g_2, \qquad R(q) = \frac{27}{8}\left(\frac{\omega_1}{\pi}\right)^6 g_3.$$

We use the properties (3) and (4) of the Ramanujan functions for the construction of a sequence of isobaric polynomials (see Section 3 for the definition).

Proposition. *Let* ω *be any period of* $\wp(z)$. *Let* $C_4, \ldots, C_8$ *be effectively computable numbers depending only on* ω. *For every integer* $N > C_4$, *there exists an isobaric polynomial* $B_N(X_1, X_2, X_3) \in \mathbb{Z}[X_1, X_2, X_3]$ *such that*

$$\deg B_N \le C_5 N \log N, \qquad \log H(B_N) \le C_6 N \log^2 N$$

and

$$\exp(-C_7 N^2) < |B_N(\eta(\omega)/\omega, g_2, g_3)| < \exp(-C_8 N^2).$$

Remark 1. Let $0 < |q| < 1$ and $C_9, \ldots, C_{13}$ be effectively computable numbers depending on q. Following the proofs of Lemmas 2.1 to 2.4 of [8] and using Theorem 3 of [8], it is possible to construct, for every integer $N > C_9$, a polynomial $B'_N(X_1, X_2, X_3) \in \mathbb{Z}[X_1, X_2, X_3]$ such that

$$\deg B'_N \le C_{10} N \log N, \qquad \log H(B'_N) \le C_{11} N \log^2 N$$

and

$$\exp(-C_{12} N^3) < |B'_N(P(q), Q(q), R(q))| < \exp(-C_{13} N^3).$$

We note here that the above construction of B'_N depends on the algebraic techniques of Nesterenko [8]. Our method of proving the Proposition is based on his work but does not depend on his algebraic techniques.

Remark 2. Let $\{\omega_1^*, \omega_2^*\}$ be a pair of fundamental periods of $\wp(z)$. Any period ω of $\wp(z)$ is of the form $\omega = m\omega_1^* + n\omega_2^*$. Let $r = \gcd(m, n)$. Then $\omega = r\omega_1$ where $\omega_1 = a\omega_1^* + b\omega_2^*$ with $a = m/r$, $b = n/r$ and $\gcd(a, b) = 1$. Hence there exist integers a^* and b^* such that $aa^* - bb^* = 1$. Let $\omega_2 = b^*\omega_1^* + a^*\omega_2^*$. Then $\{\omega_1, \omega_2\}$ forms a pair of fundamental periods of $\wp(z)$ and we may assume that $\mathrm{Im}(\omega_2/\omega_1) > 0$. Since $\eta(r\omega_1) = r\eta(\omega_1)$, we have $\eta(\omega)/\omega = \eta(\omega_1)/\omega_1$ and hence it is enough to prove the Proposition and Corollary 3 for $\omega = \omega_1$.

In the sequel, we denote by $c_3, c_4, \ldots$ effectively computable numbers depending on g_2, g_3, ω and q. We now deduce the Theorem from the above Proposition.

Proof of the Theorem. Suppose

$$\eta(\omega)/\omega - \alpha_1 = \varepsilon_1, \qquad g_2 - \alpha_2 = \varepsilon_2, \qquad g_3 - \alpha_3 = \varepsilon_3.$$

Let $\varepsilon_0 = \max(|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3|)$. We may assume that $\varepsilon_0 < 1$. We set $t = h^* d^* + d^* \log(d^* + 2)$. We choose N as the smallest integer such that

$$t \le \delta \frac{N+1}{\log(N+1)} \tag{5}$$

where $\delta > 0$ satisfies the inequality $3\delta C_4 / \log C_4 < 1$. Since $t \ge 1$, we see that $N > C_4$. Hence there exists a polynomial $B_N(X_1, X_2, X_3) \in \mathbb{Z}[X_1, X_2, X_3]$ as in the Proposition. Now

$$(6)\quad B_N(\eta(\omega)/\omega, g_2, g_3) = B_N(\alpha_1+\varepsilon_1, \alpha_2+\varepsilon_2, \alpha_3+\varepsilon_3) = B_N(\alpha_1,\alpha_2,\alpha_3) + B_N^{(1)}(\alpha_1,\alpha_2,\alpha_3,\varepsilon_1,\varepsilon_2,\varepsilon_3)$$

for some polynomial $B_N^{(1)}$. It is easy to see that

$$(7)\quad |B_N^{(1)}(\alpha_1,\alpha_2,\alpha_3,\varepsilon_1,\varepsilon_2,\varepsilon_3)| \le \varepsilon_0 \exp\{c_3N^2\}.$$

If $B_N(\alpha_1,\alpha_2,\alpha_3) = 0$, then it follows from (6), (7) and the Proposition that

$$\varepsilon_0 > \exp\{-c_4N^2\}.$$

If $B_N(\alpha_1,\alpha_2,\alpha_3) \ne 0$, then we apply Theorem 1 of [5, p. 58] to conclude that

$$|B_N(\alpha_1,\alpha_2,\alpha_3)| > \exp\{-c_5(d^*N\log^2 N + h^*d^*N\log N)\} > \exp\{-c_6\delta N^2\}.$$

Now if $\varepsilon_0 < \exp\{-c_7N^2\}$ where $c_7 > c_3$ say, then by (6), (7) and the Proposition, we get

$$|B_N(\alpha_1,\alpha_2,\alpha_3)| < \exp\{-c_8N^2\}.$$

Now we choose $\delta < c_8/c_6$ to get a contradiction. Thus

$$\varepsilon_0 > \exp\{-c_9N^2\}.$$

Now the result follows by the choice of N in (5). ■

In Section 3 we prove several lemmas which lead to the proof of the Proposition.

3. Lemmas and proof of the Proposition. Before beginning our series of lemmas we fix some notation. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two power series with $a_n \in \mathbb{C}$ for $n \ge 0$ and $b_n \in \mathbb{R}^+$ for $n \ge 0$. We say that g *dominates* f if $|a_n| \le b_n$ for $n \ge 0$ and we write $f \ll g$. As set in the introduction, for any non-zero polynomial $B(X) \in \mathbb{Z}[X]$, $H(B)$ is the maximum of the absolute values of the coefficients of B. Suppose $B_1,\ldots,B_s$ are in $\mathbb{Z}[X]$ and $B = B_1\ldots B_s$. Then by Gelfond [3, p. 135], we have

$$H(B_1)\ldots H(B_s)e^{-\deg B} \le H(B).$$

Thus

$$(8)\quad \log H(B_1) + \ldots + \log H(B_s) \le \log H(B) + \deg B.$$

For any non-zero polynomial $E(z_1,\ldots,z_n) \in \mathbb{Q}[z_1,\ldots,z_n]$, we define the *weight* $w(E)$ as

$$(9)\quad w(E) = \deg_t E(tz_1, t^2z_2, \ldots, t^nz_n).$$

Further we say that E is *isobaric* of weight $w(E)$ if for any monomial

$z_1^{i_1}\dots z_n^{i_n}$ of $E(z_1,\dots,z_n)$, we have

$$w(E)=\sum_{r=1}^{n} r i_r.$$

In the following lemmas, we take q as any complex number with $0<|q|<1$.

LEMMA 1. *For all integers $N\ge 4$ there exists a polynomial $A\in\mathbb{Z}[X_1,X_2,X_3]$, $A\not\equiv 0$, such that A is isobaric in X_1,X_2 and X_3 of weight N and*

$$\log H(A)\le(6N+2)\log N \tag{10}$$

and if $F(z)=A(P(z),Q(z),R(z))$, then

$$F^{(k)}(0)=0 \quad \textit{for } 0\le k<[N^2/24]. \tag{11}$$

Proof. It is known (see [8, p. 1323]) that

$$P(z)\ll\frac{24\cdot 2!}{(1-z)^3},\qquad Q(z)\ll\frac{240\cdot 4!}{(1-z)^5},\qquad R(z)\ll\frac{504\cdot 6!}{(1-z)^7}. \tag{12}$$

For any triple $\overline{k}=(k_1,k_2,k_3)$ with $k_1+2k_2+3k_3=N$, we write

$$(P(z))^{k_1}(Q(z))^{k_2}(R(z))^{k_3}=\sum_{n=0}^{\infty}d(\overline{k},n)z^n. \tag{13}$$

Here and everywhere in the paper we take k_1, k_2 and k_3 as non-negative integers. We note that $d(\overline{k},n)\in\mathbb{Z}$. Using (12) we see that

$$(P(z))^{k_1}(Q(z))^{k_2}(R(z))^{k_3}$$
$$\ll\frac{(504\cdot 6!)^{k_1/(2\cdot 9)+k_2/(1\cdot 45)+k_3}}{(1-z)^{3k_1+5k_2+7k_3}}\ll\frac{(504\cdot 6!)^{(k_1+2k_2+3k_3)/(2\cdot 9)}}{(1-z)^{3N}}\ll\frac{83^N}{(1-z)^{3N}}.$$

Writing $1/(1-z)^{3N}=\sum_{n=0}^{\infty}b_nz^n$, we find that $b_0=1$ and for $n\ge 1$

$$\begin{aligned} b_n&=\frac{3N(3N+1)\dots(3N+n-1)}{n!}=\frac{(n+1)\dots(n+3N-1)}{(3N-1)!}\\ &\le n^{3N-1}\left(1+\frac{1}{n}\right)\left(\frac{1}{2}+\frac{1}{n}\right)\dots\left(\frac{1}{3N-1}+\frac{1}{n}\right)\\ &<n^{3N-1}\left(1+\frac{1}{n}\right)^{3N-1}=(n+1)^{3N-1}. \end{aligned} \tag{14}$$

From the definition of $d(\overline{k},n)$ and (14) it follows that

$$|d(\overline{k},n)|\le 83^Nb_n\le(83(n+1)^3)^N \quad \text{for } n\ge 0. \tag{15}$$

We solve the system of equations

$$\sum_{\overline{k}}a_{\overline{k}}d(\overline{k},n)=0 \quad \text{for } 0\le n<[N^2/24] \tag{16}$$

in the unknowns $a_{\overline{k}}$. The number of equations in (16) is $[N^2/24]$. The number of unknowns $a_{\overline{k}}$ is equal to the number of non-negative integral solutions in (k_1, k_2, k_3) of $k_1 + 2k_2 + 3k_3 = N$, which is equal to the number of ways N can be partitioned into parts equalling 1, 2 or 3, denoted by $p_3(N)$, say. This is known to be equal to

$$\frac{(N+3)^2}{12} - \frac{7}{72} + \frac{(-1)^N}{8} + \frac{2}{9}\cos\left(\frac{2N\pi}{3}\right)$$

(see [2, p. 112 or p. 115]). In fact, this can be easily derived from the generating function $1/((1-x)(1-x^2)(1-x^3))$ of $p_3(N)$ using partial fractions. Thus the number of unknowns is $\le (N+3)^2/12 + 1$ and exceeds $N^2/12$. We apply Siegel's lemma (see [12]) to the system of equations in (16) to conclude that there exist integers $a_{\overline{k}}$, not all zero, satisfying (16) such that

$$|a_{\overline{k}}| \le \left(\frac{(N+3)^2}{12} + 1\right) \max(|d(\overline{k}, n)|)$$

where the maximum is taken over all $\overline{k}$ with $k_1 + 2k_2 + 3k_3 = N$ and $0 \le n < [N^2/24]$. Now we use (15), $n < N^2/24$ and $N \ge 4$ to get

$$|a_{\overline{k}}| \le N^{6N+2}. \tag{17}$$

Now we set

$$A(X_1, X_2, X_3) = \sum_{\overline{k}} a_{\overline{k}} X_1^{k_1} X_2^{k_2} X_3^{k_3}$$

where $a_{\overline{k}}$ satisfies (16) with (17). Thus (10) holds. Since

$$F(z) = \sum_{n=0}^{\infty} \Big(\sum_{\overline{k}} a_{\overline{k}} d(\overline{k}, n)\Big) z^n,$$

we see that (11) follows from (16). ∎

Since $A(X_1, X_2, X_3) \not\equiv 0$, we observe that $F(z) \not\equiv 0$. Otherwise we have $A(P(z), Q(z), R(z)) \equiv 0$. But this contradicts the fact that the functions $P(z), Q(z), R(z)$ are algebraically independent over $\mathbb{C}(z)$ and hence over $\mathbb{Q}$ in particular. This fact is a consequence of a result of Mahler [7]. Now let $M = \operatorname{ord}_{z=0} F(z)$. Then by Lemma 1,

$$M \ge N^2/24. \tag{18}$$

LEMMA 2. *Let $q \in \mathbb{C}$ with $0 < |q| < 1$. For $N \ge c_{10}$ we have*

$$|F(q)| \le |q|^M M^{3N} N^{11N}. \tag{19}$$

Proof. By Lemma 1, we see that

$$F(z) = \sum_{n=M}^{\infty} f_n z^n \quad \text{where} \quad f_n = \sum_{\overline{k}} a_{\overline{k}} d(\overline{k}, n)$$

with $k_1 + 2k_2 + 3k_3 = N$. Hence by (17), (15), $n \ge M$ and (18), for N sufficiently large ($N \ge 84$ suffices) we obtain

$$|f_n| \le \sum_{\bar{k}} |a_{\bar{k}}\ d(\bar{k}, n)| \le N^{6N+4}(83(n+1)^3)^N \le n^{3N} N^{7N}.$$

Using the above estimate for $|f_n|$ we get

$$\begin{aligned}|F(q)| &\le \sum_{n=0}^{\infty} |f_{n+M} q^{n+M}| \le |q|^M N^{7N} \sum_{n=0}^{\infty} (n+M)^{3N} |q|^n \\ &\le |q|^M N^{7N} M^{3N} \sum_{n=0}^{\infty} \left(1 + \frac{n}{M}\right)^{3N} |q|^n \\ &\le |q|^M N^{7N} M^{3N} \sum_{n=0}^{\infty} (n+1)^{3N} |q|^n \\ &\le |q|^M N^{7N} M^{3N} \frac{(3N)!}{(1-|q|)^{3N+1}} \le |q|^M M^{3N} N^{10N} \frac{27^N}{(1-|q|)^{3N+1}}.\end{aligned}$$

Now (19) follows by taking c_{10} sufficiently large. ∎

In the next lemma we derive an upper bound for M in terms of N. For this, we introduce the differential operator $\mathcal{D} : \mathbb{Q}[X_1, X_2, X_3] \to \mathbb{Q}[X_1, X_2, X_3]$ given by

$$\mathcal{D} = \frac{1}{12}(X_1^2 - X_2)\frac{\partial}{\partial X_1} + \frac{1}{3}(X_1 X_2 - X_3)\frac{\partial}{\partial X_2} + \frac{1}{2}(X_1 X_3 - X_2^2)\frac{\partial}{\partial X_3}.$$

We show

LEMMA 3. *Let E be a non-zero polynomial in $\mathbb{C}[X_1, X_2, X_3]$ which is isobaric in X_1, X_2 and X_3 of weight $w(E) = w$. Then*

$$\operatorname{ord}_{z=0} E(P, Q, R) \le w^2 + w. \tag{20}$$

Proof. Suppose $\operatorname{ord}_{z=0} E(P, Q, R) = 0$. Then the assertion is trivially true since $w \ge 0$. Hence we may assume that $\operatorname{ord}_{z=0} E(P, Q, R) \ne 0$. Thus E is a non-constant polynomial and not a monomial in X_1, X_2 and X_3. Since E is isobaric, this also means that E is a polynomial in at least two of the variables X_1, X_2 and X_3. Suppose E is a polynomial in X_2 and X_3 only. Then

$$\begin{aligned}E(X_2, X_3) &= \sum_{2k_2+3k_3=w} c_{\bar{k}} X_2^{k_2} X_3^{k_3} \\ &= \left(\frac{X_3}{X_2}\right)^w \sum_{2k_2+3k_3=w} c_{\bar{k}} \left(\frac{X_2^3}{X_3^2}\right)^{k_2+k_3} = \left(\frac{X_3}{X_2}\right)^w \prod_{i=1}^{l} \left(\frac{X_2^3}{X_3^2} - \beta_i\right)\end{aligned}$$

where $\beta_1, \ldots, \beta_l$ are complex numbers and $l \le w/2$. Since

$$\operatorname{ord}_{z=0}\left(\frac{Q^3}{R^2} - \beta_i\right) = \begin{cases} 1 & \text{if } \beta_i = 1, \\ 0 & \text{if } \beta_i \neq 1, \end{cases}$$

we derive that

$$\operatorname{ord}_{z=0} E(Q,R) \le l \le w/2. \tag{21}$$

Thus (20) is satisfied whenever E is a polynomial in X_2 and X_3 only.

Now we assume that E is irreducible and not a polynomial in X_2 and X_3 only. For any polynomial E satisfying the hypothesis of Lemma 3, we have

$$\mathcal{D}E = \sum b_{\overline{k}} X_1^{k_1} X_2^{k_2} X_3^{k_3}, \qquad b_{\overline{k}} \in \mathbb{C}, \tag{22}$$

and the summation is over $\overline{k}$ with $k_1 + 2k_2 + 3k_3 = w + 1$. Thus $\mathcal{D}E$ is a polynomial isobaric in X_1, X_2 and X_3 of weight $w+1$. Further we note by virtue of (3) that

$$\theta(E(P(z), Q(z), R(z))) = (\mathcal{D}E)(P(z), Q(z), R(z)). \tag{23}$$

Suppose $\mathcal{D}E \equiv 0$. Then by (23), we conclude that $E(P(z), Q(z), R(z)) = \alpha_0 \in \mathbb{C}$. But this contradicts the result of Mahler [7]. Thus we obtain

$$\mathcal{D}E \not\equiv 0.$$

We consider two cases.

CASE (i): $E \mid \mathcal{D}E$. Then by Lemma 4.1 of [8] and the Corollary following it, we have $E = X_2^3 - X_3^2$ and hence $\operatorname{ord}_{z=0} E(P,Q,R) = 1$ and (20) follows in this case.

CASE (ii): $E \nmid \mathcal{D}E$. Let F be the resultant of E and $\mathcal{D}E$ with respect to X_1. Then $F \not\equiv 0$ and

$$F(X_2, X_3) = UE + V\mathcal{D}E \tag{24}$$

for some polynomials U and V in $\mathbb{Z}[X_1, X_2, X_3]$. It follows from the definition of weight function and the representation of the resultant as a determinant that

$$w(F) \le (\deg_{X_1} E) w(\mathcal{D}E) + (\deg_{X_1} \mathcal{D}E) w(E) \le 2w(w+1). \tag{25}$$

Let F_0 be the sum of the monomials of F of weight $w(F)$. Then F_0 is an isobaric polynomial in X_2 and X_3 of weight $w(F)$. On comparing terms of weight $w(F)$ in (24), we get

$$F_0(X_2, X_3) = U_0 E + V_0 \mathcal{D}E \tag{26}$$

where U_0 and V_0 are isobaric polynomials. Since

$$\operatorname{ord}_{z=0} E(P,Q,R) \le \operatorname{ord}_{z=0} \mathcal{D}E(P,Q,R)$$

we derive from (26) and (21) with E replaced by F_0 and w by $w(F_0)$ that

$$\operatorname{ord}_{z=0} E(P,Q,R) \le \operatorname{ord}_{z=0} F_0(Q,R) \le \tfrac{1}{2}w(F_0) = \tfrac{1}{2}w(F),$$

which implies (20) by (25).

Thus the lemma is true whenever E is irreducible. Suppose E is reducible. We observe that E can be written as $E = E_1^{a_1}\ldots E_s^{a_s}$ where each E_i is irreducible, isobaric in X_1, X_2, X_3 and $a_1,\ldots,a_s$ are positive integers. Thus

$$\operatorname{ord}_{z=0} E(P,Q,R) = \sum_{i=1}^{s} a_i \operatorname{ord}_{z=0} E_i(P,Q,R) \le \sum_{i=1}^{s} a_i w(E_i)(w(E_i)+1)$$

since E_i's are irreducible. Now we use the fact that $w = \sum_{i=1}^{s} a_i w(E_i)$ to get

$$\operatorname{ord}_{z=0} E(P,Q,R) \le \sum_{i=1}^{s} a_i (w(E_i))^2 + w \le \Big(\sum_{i=1}^{s} a_i w(E_i)\Big)^2 + w \le w^2 + w.$$

This completes the proof of the lemma. ∎

It follows from Lemma 3 that

$$M \le 2N^2. \tag{27}$$

Following exactly the proof of Lemma 2.3 of [8] and then using (27), we obtain

LEMMA 4. *Let* $q \in \mathbb{C}$ *with* $0 < |q| < 1$. *Suppose* $N \ge c_{10}$. *Then there exists an integer* T *with* $0 \le T < c_{11} N \log N$ *for which*

$$|F^{(T)}(q)| > \exp\{-c_{12}N^2\}.$$

In the above lemma and in the sequel we use without mention the assumption that c_{10} is sufficiently large. Since the Ramanujan functions satisfy differential equations of the type (3), it is convenient to change from the ordinary differentiation on $F(z)$ to using θ on F. The next two lemmas serve this purpose. For $h \ge 1$ we see by induction on h that

$$(z^{-1}\theta)^h = z^{-h}\prod_{k=0}^{h-1}(\theta - k). \tag{28}$$

Set

$$\prod_{k=0}^{h-1}(\theta - k) = \sum_{k=1}^{h} s(h,k)\theta^k.$$

The numbers $s(h,k)$ are called the *Stirling numbers of the first kind* (see Hall [4, p. 29, Ex-2]). They satisfy the recurrence relation

$$s(h+1,k) = s(h,k-1) - hs(h,k)$$

from which we derive

(29) $$|s(h,1)| = (h-1)!, \quad s(h,h) = 1,$$
$$|s(h,k)| \le \binom{h-1}{k-1} h^{h-k} \quad \text{for } 1 < k < h.$$

LEMMA 5. *Let $q \in \mathbb{C}$ with $0 < |q| < 1$. Suppose $N \ge c_{10}$. Then there exists an integer T' with $0 \le T' \le T$ such that*

$$|\theta^{T'} F(q)| > \exp\{-c_{13}N^2\}.$$

Proof. Let $T = 0$ or 1. Then we take $T' = T$. Thus $\theta^{T'} F(q) = F(q)$ or $qF'(q)$. Now we use Lemma 4 to get the inequality in the lemma. Thus we assume that $T \ge 2$. Suppose

(30) $$|\theta^t F(q)| \le \frac{|q|^T}{(T+1)^T} \exp\{-c_{12}N^2\} \quad \text{for } 1 \le t \le T.$$

By (28),

$$F^{(T)}(z) = (z^{-1}\theta)^T (F(z)) = z^{-T} \sum_{k=1}^{T} s(T,k)(\theta^k F(z)).$$

Hence by (29) and (30), we have

$$|F^{(T)}(q)|$$
$$\le |q|^{-T} \left\{ (T-1)! + \sum_{k=2}^{T-1} \binom{T-1}{k-1} T^{T-k} + 1 \right\} \frac{|q|^T}{(T+1)^T} \exp\{-c_{12}N^2\}$$
$$< \exp\{-c_{12}N^2\}$$

which contradicts Lemma 4. Thus there exists an integer T' with $1 \le T' \le T$ such that

$$|\theta^{T'} F(q)| > \frac{|q|^T}{(T+1)^T} \exp\{-c_{12}N^2\}.$$

Now the result follows since $T < c_{11} N \log N$ and $N > c_{10}$. ■

For any integer $t \ge 1$, we write

(31) $$\theta^t = \sum_{k=1}^{t} S(t,k) z^k \frac{d^k}{dz^k}$$

where $S(t,k) \in \mathbb{Z}$. We observe that for any integer $k \ge 1$,

$$\theta\left(z^k \frac{d^k}{dz^k}\right) = kz^k \frac{d^k}{dz^k} + z^{k+1} \frac{d^{k+1}}{dz^{k+1}}.$$

Hence we note from (31) that $S(t,1) = S(t,t) = 1$ and

(32) $$S(t,k) = kS(t-1,k) + S(t-1,k-1)$$

where we take $S(h,k)=0$ whenever $k>h$. In fact, $S(t,k)$ are known as *Stirling numbers of the second kind* (see Hall [4]). From the recurrence relation (32) one can easily derive by induction on t and k that

$$|S(t,k)| \le \frac{1}{(k-1)!}(2k)^{t-1} \quad \text{for } 1 \le k \le t. \tag{33}$$

LEMMA 6. *Let $q \in \mathbb{C}$ with $0<|q|<1$. Suppose $N \ge c_{10}$ and T' is chosen as in Lemma 5. Then*

$$|\theta^{T'}F(q)| < \exp\{-c_{14}N^2\}.$$

Proof. Suppose $T'=0$. Then the lemma is valid by Lemma 2 and (18). Hence we assume that $T' \ge 1$. By (31) and (33), we get

$$|\theta^{T'}F(q)| \le \sum_{k=1}^{T'} |S(T',k)q^k F^{(k)}(q)| \le \sum_{k=1}^{T'} \frac{(2T')^{T'-1}}{(k-1)!}|q|^k |F^{(k)}(q)|.$$

We estimate $|F^{(k)}(q)|$ by the formula

$$F^{(k)}(q) = \frac{k!}{2\pi i}\int_C \frac{F(z)}{(z-q)^{k+1}}\,dz$$

where C is the circle $|z-q| = r-|q|$ with $|q|<r<1$ and r chosen depending only on q. Then on C we have $|z| \le |z-q|+|q| = r$. Hence by Lemma 2 with q replaced by z, we get

$$|F^{(k)}(q)| \le \frac{k!r^M M^{3N}N^{11N+4}}{(r-|q|)^k}.$$

Thus

$$|\theta^{T'}F(q)| \le (2T')^{T'} r^M M^{3N} N^{11N+4} \sum_{k=1}^{T'} \left(\frac{|q|}{r-|q|}\right)^k.$$

We use $T' \le T < c_{11} N \log N$, (27), (18) and $r<1$ in the above estimate to complete the proof. ∎

By a simple induction, we see that the identity in (23) with $E=A$ can be extended as

$$\theta^h(A(P(z),Q(z),R(z))) = (\mathcal{D}^h A)(P(z),Q(z),R(z)) \quad \text{for } h \ge 1. \tag{34}$$

For T' as in Lemma 5, we set

$$A_N(X_1,X_2,X_3) = 12^{T'}(\mathcal{D}^{T'}A)(X_1,X_2,X_3). \tag{35}$$

Then $A_N(X_1,X_2,X_3) \in \mathbb{Z}[X_1,X_2,X_3]$ and by (34),

$$A_N(P(z),Q(z),R(z)) = 12^{T'}\theta^{T'}(F(z)).$$

Hence on using Lemmas 5 and 6, for $N \ge c_{10}$, $q \in \mathbb{C}$ with $0<|q|<1$ we get

$$\exp\{-c_{13}N^2\} < |A_N(P(q),Q(q),R(q))| < \exp\{-c_{15}N^2\}. \tag{36}$$

Further we show

LEMMA 7. *For $N > c_{10}$, we have* $\deg A_N \le c_{16} N \log N$, $\log H(A_N) \le c_{17} N \log^2 N$.

Proof. We observe that

$$\mathcal{D}^t A = \sum s_{\bar{k}} X_1^{k_1} X_2^{k_2} X_3^{k_3}, \qquad s_{\bar{k}} \in \mathbb{Q}, \tag{37}$$

is isobaric in X_1, X_2 and X_3 of weight $N + t$. Hence $\deg A_N \le N + T'$. Now the estimate for the degree follows since $T' < c_{11} N \log N$. To bound $H(A_N)$, we note that

$$A \ll H(A)(X_1 + X_2 + X_3)^N$$

where $H(A)$ satisfies (10). Hence

$$\mathcal{D}^{T'} A \ll H(A)(N + T')^{T'} (X_1 + X_2 + X_3)^{N+T'}.$$

Thus $H(A_N) \le (12^{T'} H(A)(N + T')^{T'} 3^{N+T'})$. Now the estimate follows. ∎

Proof of the Proposition. By Remark 2, it is enough to prove the Proposition when $\omega = \omega_1$. We set $\eta(\omega_1) = \eta$ and $q = e^{2\pi i \omega_2/\omega}$. From (35) and (37) we observe that

$$A_N(P(q), Q(q), R(q)) = 12^{T'} \sum s_{\bar{k}} P(q)^{k_1} Q(q)^{k_2} R(q)^{k_3}$$

where the summation is over $\bar{k}$ with $k_1 + 2k_2 + 3k_3 = N + T'$. Further by (36), not all $s_{\bar{k}}$ are zero. From the relations in (4), we get

$$\begin{aligned} &A_N(P(q), Q(q), R(q)) \\ &\quad = 12^{T'} \sum s_{\bar{k}} \left(3\frac{\omega}{\pi} \cdot \frac{\eta}{\pi}\right)^{k_1} \left(\frac{3}{4}\left(\frac{\omega}{\pi}\right)^4 g_2\right)^{k_2} \left(\frac{27}{8}\left(\frac{\omega}{\pi}\right)^6 g_3\right)^{k_3} \\ &\quad = 12^{T'} \left(\frac{\omega}{\pi}\right)^{2(N+T')} \sum s_{\bar{k}} \left(3\frac{\eta}{\omega}\right)^{k_1} \left(\frac{3}{4} g_2\right)^{k_2} \left(\frac{27}{8} g_3\right)^{k_3} \\ &\quad = 12^{T'} \left(\frac{\omega}{\pi}\right)^{2(N+T')} A_N\left(3\frac{\eta}{\omega}, \frac{3}{4} g_2, \frac{27}{8} g_3\right). \end{aligned}$$

Then we set

$$B_N(X_1, X_2, X_3) = 2^{N+T'} A_N\left(3X_1, \frac{3}{4}X_2, \frac{27}{8}X_3\right).$$

We observe that $B_N(X_1, X_2, X_3) \in \mathbb{Z}[X_1, X_2, X_3]$ and $\deg B_N \le N + T' \le c_{18} N \log N$. To calculate $H(B_N)$, we note that

$$H(B_N) \le H(A_N) c_{19}^{N+T'}.$$

Now the required bound follows from Lemma 7. We observe that

$$|B_N(\eta/\omega, g_2, g_3)| = |2^{N+T'} 12^{-T'} (\pi/\omega)^{2(N+T')} A_N(P(q), Q(q), R(q))|.$$

Now we use (36) to get the required lower and upper bounds for $|B_N(\eta/\omega, g_2, g_3)|$. This completes the proof of the Proposition. ■

4. Proof of Corollary 3. By Remark 2, it is enough to prove the corollary when $\omega = \omega_1$. As earlier, we set $\eta(\omega_1) = \eta$ and $q = e^{2\pi i\omega_2/\omega}$. Let $B(X) \in \mathbb{Z}[X]$ be any non-zero polynomial with $H(B) \le H$ and $\deg B \le d$.

First we assume that B is irreducible. Let ξ be the root of B which is nearest to $\eta(\omega)/\omega$. Then by a result of Diaz and Mignotte [1, Corollary 2], we have

$$|\eta(\omega)/\omega - \xi| \le (H^2 d(d+1)^{3/2})^{d-1} |B(\eta(\omega)/\omega)|,$$

and Corollary 1 yields the desired result.

Now let B be reducible over $\mathbb{Q}[X]$. Write

$$B(X) = B_1(X)\ldots B_s(X)$$

where $B_1(X), \ldots, B_s(X)$ are irreducible polynomials with $B_i(X)$ having height $\le H_i$ and degree $\le d_i$ for $1 \le i \le s$. Then we have

$$|B_i(\eta(\omega)/\omega)| > \exp\{-c_{20}(\log^2 H_i(\log\log H_i)^2 + d_i^2 \log^4 d_i)\} \quad \text{for } 1 \le i \le s.$$

Thus

$$|B(\eta(\omega)/\omega)| > \exp\Big\{-c_{21}\Big(\sum_{i=1}^{s} \log^2 H_i(\log\log H_i)^2 + \sum_{i=1}^{s} d_i^2 \log^4 d_i\Big)\Big\}. \tag{38}$$

Now we observe that

$$\sum_{i=1}^{s} d_i^2 \log^4 d_i \le \Big(\sum_{i=1}^{s} d_i^2\Big) \log^4 d \le d^2 \log^4 d \tag{39}$$

since $\sum_{i=1}^{s} d_i = d$. Further from (8), we get

$$\begin{aligned} \sum_{i=1}^{s} \log^2 H_i(\log\log H_i)^2 &\le c_{22}\Big(\sum_{i=1}^{s} \log H_i\Big)^2 (\log(\log H + d))^2 \\ &\le c_{23}(\log H + d)^2((\log\log H)^2 + \log^2 d) \\ &\le c_{24}(\log^2 H + d^2)((\log\log H)^2 + \log^2 d) \\ &\le c_{25}(\log^2 H(\log\log H)^2 + d^2 \log^2 d). \end{aligned} \tag{40}$$

Now we use (39) and (40) in (38) to obtain the result of Corollary 3. ■

Remark 3. Let $\alpha_1, \alpha_2, \alpha_3$ be algebraic numbers satisfying the hypothesis of the Theorem. By following the proof of the Theorem with B_N replaced by A_N, it is clear from Lemma 7 and (36) that for any $q \in \mathbb{C}$ with $0 < |q| < 1$,

$$\begin{aligned} &\max(|P(q) - \alpha_1|, |Q(q) - \alpha_2|, |R(q) - \alpha_3|) \\ &\qquad > \exp\{-C_{14}((h^* d^*(\log h^* d^*))^2 + (d^*)^2 \log^4 d^*)\} \end{aligned}$$

where C_{14} is an effectively computable number depending only on q. The above bound can be improved if we use the results of Nesterenko [8]. We follow the proof of the Theorem with B_N replaced by B'_N mentioned in Remark 1 and the inequality (5) replaced by

$$t \le \delta \frac{(N+1)^2}{\log(N+1)}.$$

Then we get

$$\max(|P(q)-\alpha_1|, |Q(q)-\alpha_2|, |R(q)-\alpha_3|) > \exp\{-C_{15}((h^* d^*(\log h^* d^*))^{3/2} + (d^*)^{3/2}\log^3 d^*)\}. \ \blacksquare$$

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