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Theta functions of quadratic forms over imaginary quadratic fields

by

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1. Introduction. Let Q be a positive definite $n \times n$ matrix with integral entries and even diagonal entries. It is well known that the theta function

$$\vartheta_Q(z) := \sum_{g \in \mathbb{Z}^n} \exp\{\pi i\, {}^t g Q g z\}, \qquad \operatorname{Im} z > 0,$$

is a modular form of weight $n/2$ on $\Gamma_0(N)$, where N is the level of Q, i.e. NQ^{-1} is integral and NQ^{-1} has even diagonal entries. This was proved by Schoeneberg [5] for even n and by Pfetzer [3] for odd n. Shimura [6] uses the Poisson summation formula to generalize their results for arbitrary n and he also computes the theta multiplier explicitly. Stark [8] gives a different proof by converting $\vartheta_Q(z)$ into a symplectic theta function and then using the transformation formula for the symplectic theta function. In [4], we apply Stark's method and use theta functions of indefinite quadratic forms to construct modular forms over totally real number fields. In this paper, we define theta functions attached to quadratic forms over imaginary quadratic fields. We show that these theta functions are modular forms of weight $n/2$ on some Γ_0 groups by regarding them as symplectic theta functions and then applying well known results for symplectic theta functions. In particular, the main result of [8] allows us to compute the theta multiplier for our theta functions in a very elegant way.

2. Symplectic theta functions. The symplectic group, $\mathrm{Sp}_n(\mathbb{R})$, consists of those $2n \times 2n$ real matrices

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

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(each entry is $n \times n$) such that

$$ {}^tMJM = J := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, $$

where I_n is the $n\times n$ identity matrix. The corresponding symmetric space is the Siegel upper half plane $\mathfrak{H}^{(n)}$ which consists of $n \times n$ symmetric complex matrices Z with $\operatorname{Im} Z > 0$ (positive definite). The action of M on Z is given by

$$ M \circ Z = (AZ + B)(CZ + D)^{-1}. $$

Let $\Gamma^{(n)} = \mathrm{Sp}_n(\mathbb{Z})$. The theta subgroup $\Gamma_\vartheta^{(n)}$ of $\Gamma^{(n)}$ is the set of all $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $\Gamma^{(n)}$ such that both $A\,{}^tB$ and $C\,{}^tD$ have even diagonal entries. The subgroup acts on the symplectic theta function,

$$ \vartheta\left(Z, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \sum_{m\in\mathbb{Z}^n} \exp\{\pi i[\,{}^t(m+v)Z(m+v) - 2\,{}^tmu - {}^tvu]\}, $$

where u and v are column vectors in $\mathbb{C}^n$. It is well known (see Eichler [1], for example) that for

$$ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{in } \Gamma_\vartheta^{(n)}, $$

we have

$$ \vartheta\left(M \circ Z, M\begin{pmatrix} u \\ v \end{pmatrix}\right) = \chi(M)[\det(CZ+D)]^{1/2}\vartheta\left(Z, \begin{pmatrix} u \\ v \end{pmatrix}\right), \tag{1} $$

where $\chi(M)$ is an eighth root of unity which depends upon the chosen square root of $\det(CZ + D)$, but which is otherwise independent of Z, u, and v. It is also known that $\chi(M)$ can be expressed in terms of Gaussian sums. Stark [8] determined $\chi(M)$ in the important special case where pD^{-1} is integral for some odd prime p. The main result in [8] is

Theorem 1. *Suppose $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is in $\Gamma_\vartheta^{(n)}$ where C^{-1} and D^{-1} exist. Suppose further that for some odd prime p, pD^{-1} is integral. Then* (*mod* p), *the symmetric matrix $pD^{-1}C$ has rank h where $\det(D) = \pm p^h$. Let $(pD^{-1}C)^{(h)}$ be a nonsingular* (*mod* p) *$h \times h$ principal submatrix of $pD^{-1}C$ and s be the signature* (*the number of positive eigenvalues minus the number of negative eigenvalues*) *of $C^{-1}D$. Then*

$$ \chi(M)[\det(CZ+D)]^{1/2} $$
$$ = \varepsilon_p^{-h}\left(\frac{2^h \det[(pD^{-1}C)^{(h)}]}{p}\right) e^{\pi i s/4} |\det(C)|^{1/2}\{\det[-iC^{-1}(CZ+D)]\}^{1/2}, $$

where $\varepsilon_p = 1$ for $p \equiv 1 \bmod 4$, $\varepsilon_p = i$ for $p \equiv 3 \bmod 4$, $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol, $|\det(C)|^{1/2}$ is positive and $\{\det[-iC^{-1}(CZ+D)]\}^{1/2}$ is given by analytic continuation from the principal value when $Z = -C^{-1}D + iY$.

Alternatively, if just C^{-1} exists and pC^{-1} is integral, $\det(C) = \pm p^h$, *then* $pC^{-1}D$ *(mod p) has rank h and*

$$\chi(M)[\det(CZ+D)]^{1/2}$$
$$= \varepsilon_p^{-h}\left(\frac{-2}{p}\right)^h \left(\frac{\det[(pC^{-1}D)^{(h)}]}{p}\right)|\det(C)|^{1/2}\{\det[-iC^{-1}(CZ+D)]\}^{1/2}.$$

3. Theta functions as modular forms. Let $K = \mathbb{Q}(\sqrt{d})$ be the imaginary quadratic field with discriminant $d < 0$. Let $\mathfrak{O}_K$ be the ring of integers of K and δ_K be the different of K. The algebraic conjugate of an algebraic number α is identical with its complex conjugate and denoted by $\overline{\alpha}$. Furthermore, let $\Gamma = \mathrm{SL}_2(\mathfrak{O}_K)$ and, as usual, for an integral ideal $\mathfrak{N}$, let

$$\Gamma_0(\mathfrak{N}) := \left\{ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \,\middle|\, M \in \Gamma \text{ and } \gamma \in \mathfrak{N} \right\}.$$

Our upper half space $\mathfrak{H} := \{x + yk \mid x \in \mathbb{C},\ y \in \mathbb{R}^+\}$ is the quaternionic upper half plane consisting of quaternions with no j-component and positive k-component. The matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(K)$$

acts on $\mathfrak{H}$ by $M \circ z := (\alpha z + \beta)(\gamma z + \delta)^{-1}$. Note that $M \circ z \in \mathfrak{H}$. For γ and δ in K and z in $\mathfrak{H}$, we define

$$\mathcal{N}(\gamma z + \delta) := \|\gamma z + \delta\|^2 = |\gamma x + \delta|^2 + |\gamma|^2 y^2.$$

Let Q be a symmetric $n \times n$ matrix with entries in $\mathfrak{O}_K$ defining the quadratic form $Q[x] := {}^t x Q x$, where $x \in \mathbb{C}^n$. Furthermore, let $Q\{x\} := {}^t x Q \overline{x}$ and $\overline{Q}[x] := {}^t x \overline{Q} x$. If, in addition, Q has diagonal entries which are divisible by 2, we say that Q is of *level* N ($N \in \mathfrak{O}_K$) whenever the following two conditions are satisfied:

(a) The matrix NQ^{-1} has entries in $\mathfrak{O}_K$, and 2 divides the diagonal entries of NQ^{-1}.

(b) For any $M \in \mathfrak{O}_K$, N divides M whenever MQ^{-1} has entries in $\mathfrak{O}_K$ and 2 divides the diagonal entries of MQ^{-1}.

For the vector $\lambda = {}^t(\lambda_1, \ldots, \lambda_n)$, we define $\overline{\lambda} := {}^t(\overline{\lambda}_1, \ldots, \overline{\lambda}_n)$, where $\lambda_1, \ldots, \lambda_n$ are in K. We define the theta function Θ_Q for a quadratic form by

Definition 1. Let Q be a symmetric $n \times n$ matrix with entries in $\mathfrak{O}_K$ such that 2 divides the diagonal entries of Q and such that Q is of level N. Since Q is symmetric, $Q = {}^t LL$ for an upper triangular complex matrix $L = (l_{sr})_{s,r=1,\ldots,n}$ ($l_{sr} = 0$ for $s > r$). For an ideal $\mathfrak{I} \subset \mathfrak{O}_K$ and

$z = x + yk \in \mathfrak{H}$, set

$$\Theta_Q(z) := \sum_{\lambda\in\mathfrak{I}^n} \exp\Big\{\pi i\Big[(Q[\lambda]x + \overline{Q}[\overline{\lambda}]\overline{x}) + 2i\Big(\sum_{s=1}^{n}\Big|\sum_{r=s}^{n} l_{sr}\lambda_r\Big|^2\Big)y\Big]\Big\},$$

where $\lambda = {}^t(\lambda_1, \dots, \lambda_n)$.

REMARKS. (a) For $R := {}^tL\overline{L}$, we have

$$\sum_{s=1}^{n}\Big|\sum_{r=s}^{n} l_{sr}\lambda_r\Big|^2 = {}^t\lambda R\overline{\lambda} = R\{\lambda\}.$$

Furthermore, observe that

(2) $$\overline{R}Q^{-1}R = \overline{Q} \quad \text{and} \quad {}^tR = \overline{R} > 0.$$

Hence the matrix R is a majorant of the matrix Q (in the terminology of Siegel [7]).

(b) For any algebraic integer $t \in K$, $Q[\lambda]t + \overline{Q}[\overline{\lambda}]\overline{t} = \mathrm{tr}(Q[\lambda]t)$ is an even rational integer, and thus $\Theta_Q(z)$ is invariant under linear transformations, i.e.

(3) $$\Theta_Q(z+t) = \Theta_Q(z).$$

The first task toward showing that Θ_Q is a modular form is to convert Θ_Q into a symplectic theta function ϑ. Let us introduce some helpful notation. For $\alpha \in \mathbb{C}$, define

$$\mathrm{diag}(\alpha) := \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}$$

and the $2n \times 2n$ matrix

$$\mathrm{diag}^*(\alpha) := \begin{pmatrix} \mathrm{diag}(\alpha) & & \\ & \ddots & \\ & & \mathrm{diag}(\alpha) \end{pmatrix}.$$

For $z = x + yk \in \mathfrak{H}$, let

$$Z_2 := \begin{pmatrix} x & iy \\ iy & \overline{x} \end{pmatrix}$$

and furthermore, define the $2n \times 2n$ matrix

$$Z^* := \begin{pmatrix} Z_2 & & \\ & \ddots & \\ & & Z_2 \end{pmatrix}.$$

Let $\Lambda := {}^t(\lambda_1, \overline{\lambda}_1, \dots, \lambda_n, \overline{\lambda}_n)$. Some computation gives

$$(4)\qquad {}^t\Lambda\,{}^tLZ^*L\Lambda = \Big\{(Q[\lambda]x + \overline{Q}[\overline{\lambda}]\overline{x}) + 2i\Big(\sum_{s=1}^{n}\Big|\sum_{r=s}^{n} l_{sr}\lambda_r\Big|^2\Big)y\Big\},$$

where

$$L := \begin{pmatrix} \mathrm{diag}(l_{11}) & \dots & \mathrm{diag}(l_{1n}) \\ & \ddots & \vdots \\ & & \mathrm{diag}(l_{nn}) \end{pmatrix}.$$

Let $\{\omega_1, \omega_2\}$ be an integral basis of the ideal $\mathfrak{I} \subset \mathfrak{O}_K$. The entries of the vector Λ are integers in $\mathfrak{I}$ and can be written in terms of the basis $\{\omega_1, \omega_2\}$. Hence, we can define a vector $P = {}^t(m_1, \dots, m_{2n})$ with rational integers $m_1, \dots, m_{2n}$ such that $\Lambda = WP$, where

$$(5)\qquad W := \begin{pmatrix} W_2 & & \\ & \ddots & \\ & & W_2 \end{pmatrix}$$

and

$$(6)\qquad W_2 := \begin{pmatrix} \omega_1 & \omega_2 \\ \overline{\omega}_1 & \overline{\omega}_2 \end{pmatrix}.$$

Furthermore,

$$(7)\qquad W_2^{-1} = \begin{pmatrix} \nu_1 & \overline{\nu}_1 \\ \nu_2 & \overline{\nu}_2 \end{pmatrix},$$

where $\{\nu_1, \nu_2\}$ is an integral basis for $\mathfrak{I}^{-1}\delta_K^{-1}$.

With $T := LW$ and $Z := {}^tTZ^*T$, we have

$$(8)\qquad \Theta_Q(z) = \vartheta\Big(Z, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\Big) = \sum_{m \in \mathbb{Z}^{2n}} \exp\{\pi i[{}^tmZm]\}.$$

To see that Z is actually in $\mathfrak{H}^{(2n)}$, we observe that Z is symmetric and that $Z = {}^t\overline{T}S^*Z^*T$, where

$$S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad S^* := \begin{pmatrix} S & & \\ & \ddots & \\ & & S \end{pmatrix}.$$

Also, $\frac{1}{2i}(S^*Z^* - {}^t\overline{S^*Z^*}) = yI_{2n} > 0$ and a corollary of Sylvester's theorem implies that $\mathrm{Im}\, Z > 0$.

For $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma = \mathrm{SL}_2(\mathfrak{O}_k)$, set

$$(9)\qquad M^* := \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} := \begin{pmatrix} \mathrm{diag}^*(\alpha) & \mathrm{diag}^*(\beta) \\ \mathrm{diag}^*(\gamma) & \mathrm{diag}^*(\delta) \end{pmatrix}.$$

It is easy to check that the diagram

$$\begin{tikzcd} z \arrow[r] \arrow[d] & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z \arrow[d] \\ Z^* \arrow[r] & \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} \circ Z^* \end{tikzcd}$$

is commutative. Hence

$$z \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z$$

in $\mathfrak{H}$ corresponds to

$$Z \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ Z$$

in $\mathfrak{H}^{(2n)}$, where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} {}^tTA^*\,{}^tT^{-1} & {}^tTB^*T \\ T^{-1}C^*\,{}^tT^{-1} & T^{-1}D^*T \end{pmatrix}. \tag{10}$$

When is the matrix in (10) in the theta subgroup? To answer this question, let us introduce some more notation. Assume that $S = (s_{ij})_{i,j=1,2}$ and $R = (r_{km})_{k,m=1,\ldots,n}$ are matrices with entries in K (not necessarily in $\mathfrak{O}_K$). We define the matrix

$$R \odot S := ((\mathrm{tr}(r_{km}s_{ij})_{i,j=1,2}))_{k,m=1,\ldots,n}.$$

Note that the entries of $R \odot S$ are rational numbers. Computation shows that $A = I_n \odot A'$, $B = Q \odot B'$, $C = Q^{-1} \odot C'$, $D = I_n \odot D'$, and A', B', C' and D' are given by

$$A' = \begin{pmatrix} \omega_1\nu_1\alpha & \omega_1\nu_2\alpha \\ \omega_2\nu_1\alpha & \omega_2\nu_2\alpha \end{pmatrix}, \qquad C' = \begin{pmatrix} \nu_1\nu_1\gamma & \nu_1\nu_2\gamma \\ \nu_2\nu_1\gamma & \nu_2\nu_2\gamma \end{pmatrix},$$

$$B' = \begin{pmatrix} \omega_1\omega_1\beta & \omega_1\omega_2\beta \\ \omega_2\omega_1\beta & \omega_2\omega_2\beta \end{pmatrix}, \qquad D' = \begin{pmatrix} \omega_1\nu_1\delta & \omega_2\nu_1\delta \\ \omega_1\nu_2\delta & \omega_2\nu_2\delta \end{pmatrix}.$$

From the definition of A, B, C and D, we see that ${}^tAC = {}^tCA$, ${}^tBD = {}^tDB$, and ${}^tDA - {}^tBC = I_n$ (as $\alpha\delta - \beta\gamma = 1$) and hence $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbb{R})$. In addition, $A\,{}^tB = {}^tTA^*B^*T = Q \odot (\alpha B')$ and $C\,{}^tD = T^{-1}C^*D^*\,{}^tT^{-1} = Q^{-1} \odot (\delta C')$. Thus, if γ is in the ideal $\mathfrak{I}^2\delta_K N$ (N the level of Q), then the entries of A, B, C, and D are traces of algebraic integers and hence rational integers, and $A\,{}^tB$ and $C\,{}^tD$ have even diagonal entries. Hence for

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \varGamma_0(\mathfrak{I}^2\delta_K N),$$

we have

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\vartheta^{(2n)}.$$

It is easy to verify that

$$\det(CZ+D) = \det(C^*Z^*+D^*) = \mathcal{N}(\gamma z+\delta)^n,$$

and therefore by (1) and (8),

$$\Theta_Q\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z\right) = \chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q\right)\mathcal{N}(\gamma z+\delta)^{n/2}\Theta_Q(z), \tag{11}$$

where $\chi\big(\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right), Q\big)$ is an eighth root of unity depending on $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ and Q. Thus, $\Theta_Q(z)$ is a modular form on $\Gamma_0(\mathfrak{I}^2\delta_K N)$ of weight $n/2$.

4. The eighth root of unity. It remains to determine $\chi\big(\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right), Q\big)$ explicitly. Assume that δ is a first degree prime in $\mathfrak{O}_k$ of norm p (meaning that p is a positive odd prime in $\mathbb{Z}$). In this case, pD^{-1} is integral. Note that $\det(D) = \det(D^*) = p^n$, and thus by Theorem 1, $pD^{-1}C$ has rank n (mod p). Hence for $Q^{-1} = (r_{il})_{i,l=1,\ldots,n}$, we find that

$$(pD^{-1}C)^{(n)} = \begin{pmatrix} \operatorname{tr}(r_{11}\nu_1\nu_1 p\delta^{-1}\gamma) & \ldots & \operatorname{tr}(r_{1n}\nu_1\nu_1 p\delta^{-1}\gamma) \\ \vdots & & \vdots \\ \operatorname{tr}(r_{1n}\nu_1\nu_1 p\delta^{-1}\gamma) & \ldots & \operatorname{tr}(r_{nn}\nu_1\nu_1 p\delta^{-1}\gamma) \end{pmatrix}$$

and

$$\det(pD^{-1}C)^{(n)} \equiv (p\delta^{-1}\gamma)^n(\nu_1\nu_1)^n\det(Q)^{-1} \pmod{\delta}.$$

Some computation shows that

$$|\det(C)|^{1/2}\{\det[-iC^{-1}(CZ+D)]\}^{1/2}e^{\pi is/4} = \mathcal{N}(\gamma z+\delta)^{n/2}.$$

Hence

$$\chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q\right) = \varepsilon_p^{-n}\left(\frac{(p\delta^{-1}2\gamma)^n\det(Q)}{\delta}\right) \tag{12}$$

and in the special case where n is even,

$$\chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q\right) = \left(\frac{(-1)^{n/2}\det(Q)}{\delta}\right). \tag{13}$$

We have proved

THEOREM 2. *Suppose that* $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{I}^2\delta_K N)$, *where* δ *is a first degree prime in* $\mathfrak{O}_K$ *of norm* p. *For* $z \in \mathfrak{H}$, *we have*

$$\Theta_Q\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z\right) = \varepsilon_p^{-n}\left(\frac{(p\delta^{-1}2\gamma)^n\det(Q)}{\delta}\right)\mathcal{N}(\gamma z+\delta)^{n/2}\Theta_Q(z), \tag{14}$$

where $\varepsilon_p = 1$ *for* $p \equiv 1 \bmod 4$ *and* $\varepsilon_p = i$ *for* $p \equiv 3 \bmod 4$.

Actually, we have determined the eighth root of unity more explicitly than it seems. In (3), we showed that for all algebraic integers t, $\Theta_Q(z+t) = \Theta_Q(z)$. It follows from (11) that for $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma_0(\mathfrak{I}^2\delta_K N)$ and for all algebraic integers t,

$$\chi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}, Q\right) = \chi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, Q\right). \tag{15}$$

Furthermore, Hecke [2] gives a proof of Dirichlet's primes in progression theorem for number fields. Hence for algebraic integers γ and δ with $(\gamma, \delta) = 1$, the arithmetic progression $\{\gamma t + \delta\}_{t\in\mathfrak{O}_k}$ contains infinitely many first degree primes (in a general number field, the progression contains infinitely many totally positive first degree primes), and the theta multiplier is determined explicitly after locating a first degree prime with odd norm in the arithmetic progression $\{\gamma t + \delta\}_{t\in\mathfrak{O}_k}$.

There is a special case which should also be mentioned. Let δ be a prime in $\mathfrak{O}_K$ with $\mathcal{N}(\delta) = p^2$, where p is an odd prime in $\mathbb{Z}$. As before, we observe that pD^{-1} has rational integers as entries. Also, $\det(D) = \det(D^*) = p^{2n}$ and by Theorem 1, $pD^{-1}C$ has rank $2n \pmod p$. Thus,

$$\det(pD^{-1}C) = (\mathcal{N}(\det(Q)))^{-1}(d(\mathcal{N}(\mathcal{I})^2))^{-n}(\mathcal{N}(\gamma^n)).$$

Hence

$$\begin{aligned}\chi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, Q\right) &= \varepsilon_p^{-2n}\left(\frac{2^{2n}d^n\mathcal{N}(\gamma^n)\mathcal{N}(\det(Q))}{p}\right)\\ &= (-1)^n\left(\frac{(-1)^n\mathcal{N}(\gamma^n\det(Q))}{p}\right).\end{aligned}$$

In the special case where n is even, we see that

$$\chi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, Q\right) = \left(\frac{\mathcal{N}((-1)^{n/2}\det(Q))}{p}\right). \tag{16}$$

The result from (13) matches the result from (16) since an element a is a square in $\mathbb{F}_{p^2}$ (the field of p^2 elements) iff $\mathcal{N}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(a)$ is a square in $\mathbb{F}_p$ (the field of p elements). This can be seen by observing that the mapping $N : \mathbb{F}_{p^2}^* \to \mathbb{F}_p^*$ given by $a \to N(a) := \mathcal{N}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(a)$ is an epimorphism.

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