ACTA ARITHMETICA
XCI.4 (1999)

On intervals containing full sets of conjugates of algebraic integers

by

ARTŪRAS DUBICKAS (Vilnius)

1. Introduction. Let α be an algebraic number with $a(x-\alpha_1)\ldots\ldots(x-\alpha_d)$ as its minimal polynomial over $\mathbb{Z}$. Then α is called *totally real* if all its conjugates $\alpha_1=\alpha,\alpha_2,\ldots,\alpha_d$ are real. Also, α is called an *algebraic integer* if $a=1$. Now, define $I(d)$ as the smallest positive number with the following property: any closed real interval of length at least $I(d)$ contains a full set of conjugates of an algebraic integer of degree d. It is clear that $I(1)=1$.

THEOREM 1. *We have*

$$I(2)=\frac{1+\sqrt{5}}{2}+\sqrt{2}.$$

As we will see from the proof of this simple theorem $I(d)$ can also be computed for all small d. The purpose of this paper is to give an upper bound for $I(d)$ for large d.

In 1918, I. Schur [Sc] proved that an interval on the real axis of length smaller than 4 can contain only a finite number of full sets of conjugates of algebraic integers. T. Zaïmi [Za] gave another proof of Schur's result. His approach is based on M. Langevin's proof [La] of Favard's conjecture. Moreover, in [Za] it is proved that the length of an interval containing a full set of conjugates of an algebraic integer of degree d is greater than $4-\psi_1(d)$ with some explicitly given positive function $\psi_1(d)$ satisfying $\lim_{d\to\infty}\psi_1(d)=0$. Note that a similar result with another explicitly given function $\psi_2(d)$ also follows from [Sc].

On the other hand, R. Robinson [Ro] showed that any interval of length greater than 4 contains infinitely many full sets of conjugates of algebraic integers. Moreover, V. Ennola [En] proved that such an interval contains full sets of conjugates of algebraic integers of degree d for all d sufficiently large. Hence $\lim_{d\to\infty} I(d)=4$.

1991 *Mathematics Subject Classification*: 11R04, 11R80, 12D10.

A *lower* bound for $I(d)$ can be obtained via a Kronecker type theorem. In 1857, L. Kronecker [Kr] proved that if α is an algebraic integer all of whose conjugates lie in $[-2;2]$ then $\alpha = 2\cos(\pi r)$ with r rational. So if α is a totally real algebraic integer not of the form $2\cos(\pi r)$ with r rational, then

$$\overline{|\alpha|} = \max_{1\le j\le d} |\alpha_j| > 2.$$

In 1965, A. Schinzel and H. Zassenhaus [SZ] asked for a lower bound of the *house* $\overline{|\alpha|}$ in terms of the degree d of α. They showed that with the same hypotheses,

$$\overline{|\alpha|} > 2 + 4^{-2d-3}. \tag{1}$$

This lower bound was derived from the lower bound for $\overline{|\alpha|}$, where α is an algebraic integer which is not a root of unity. The conjectural inequality $\overline{|\alpha|} > 1 + c_1/d$ (see [SZ]) with an absolute positive constant c_1 is not yet proved. This is also the case with D. H. Lehmer's [Le] more general conjectural inequality

$$M(\alpha) = a\prod_{j=1}^{d} \max\{1, |\alpha_j|\} > 1 + c_2$$

where α is an algebraic number which is not a root of unity and c_2 is an absolute positive constant. Using the best known lower bound in Lehmer's conjecture [Lo] the author strengthened the inequality (1). We proved [Du] that if α is a totally real algebraic integer of degree d, $\alpha \ne 2\cos(\pi r)$ with r rational, and d is sufficiently large, then

$$\overline{|\alpha|} > 2 + 4.6\frac{(\log\log d)^3}{d(\log d)^4}.$$

Thus the interval

$$\left(-2\cos\left(\frac{\pi}{2d}\right);\ 2 + 4.6\frac{(\log\log d)^3}{d(\log d)^4}\right]$$

does not contain a full set of conjugates of an algebraic integer of degree d. It follows immediately that for all d sufficiently large,

$$I(d) > 4 + \frac{9}{2}\cdot\frac{(\log\log d)^3}{d(\log d)^4}. \tag{2}$$

Our main theorem gives an explicit slowly decreasing function, namely $12(\log\log d)^2/\log d$, which cannot replace $9(\log\log d)^3/2d(\log d)^4$ in (2).

Theorem 2. *There is an infinite sequence S of positive integers such that for $d \in S$ any interval of length greater than or equal to*

$$4 + 12\frac{(\log\log d)^2}{\log d}$$

contains a full set of conjugates of an algebraic integer of degree d.

Clearly, for $d \in S$ we have the inequality

$$I(d) \le 4 + 12\frac{(\log\log d)^2}{\log d}.$$

Our proof of Theorem 2 is based on the following statement.

LEMMA. *Let u, v, w be three fixed positive integers. Then there is an infinite sequence $S(u,v,w)$ of positive integers such that every $d \in S(u,v,w)$ is divisible by*

$$w(vq(d)^u)^{q(d)}q(d)!,$$

where

$$q(d) = \left[\frac{\log d}{(u+1)\log\log d}\right].$$

Here and below $[\ldots]$ denotes the integral part. We will also show that the sequence S in Theorem 2 can be taken to be all sufficiently large elements of $S(2,16,2)$.

Now we will prove the Lemma, Theorem 2 and Theorem 1.

2. Proof of the Lemma. Put for brevity

$$f(x) = \frac{\log x}{\log\log x}.$$

For $x \ge 16$ the function $f(x)$ is increasing. Let $k \ge 2$ be an integer. Then the equation (in x)

$$\frac{f(x)}{u+1} = k$$

has a unique solution which we denote by x_k. Clearly, $x_2 > 5503$ and the sequence x_k is increasing. We now prove that

$$x_{k+1} > x_k \log x_k.$$

Indeed, if $x_{k+1} \le x_k \log x_k$ then

$$\begin{aligned} u+1 &= (u+1)(k+1) - (u+1)k = f(x_{k+1}) - f(x_k) \\ &\le f(x_k \log x_k) - f(x_k) = \frac{\log(x_k \log x_k)}{\log\log(x_k \log x_k)} - \frac{\log x_k}{\log\log x_k}. \end{aligned}$$

Put $y_k = \log\log x_k$ for brevity. By the last inequality we have

$$\begin{aligned} 2 \le u+1 &\le \frac{y_k + e^{y_k}}{\log(y_k + e^{y_k})} - \frac{e^{y_k}}{y_k} = \frac{y_k^2 + y_k e^{y_k} - e^{y_k}\log(y_k + e^{y_k})}{y_k \log(y_k + e^{y_k})} \\ &= \frac{y_k^2 - e^{y_k}\log(1 + y_k e^{-y_k})}{y_k \log(y_k + e^{y_k})} < \frac{y_k^2 - e^{y_k}\log(1 + y_k e^{-y_k})}{y_k^2}. \end{aligned}$$

The last expression is less than 1, a contradiction.

Set

$$N_k = \{[x_k]+1, [x_k]+2, \dots, [x_k \log x_k]\}.$$

Clearly, for $n \in N_k$,

$$q(n) = \left[\frac{f(n)}{u+1}\right] = k.$$

Note that for all n sufficiently large the expression

$$r(n) = w(vq(n)^u)^{q(n)} q(n)\,!$$

is less than

$$\begin{aligned} w(vq(n)^u)^{q(n)} q(n)^{q(n)} \le\ & \exp\biggl(\log w + \frac{\log v \log n}{(u+1)\log\log n} \\ & + \frac{\log n}{\log\log n}\log\left(\frac{\log n}{(u+1)\log\log n}\right)\biggr) \\ <\ & \exp(\log n) = n, \end{aligned}$$

so that for all $n \in N_k$,

$$r(n) = w(vk^u)^k\, k! \le [x_k].$$

Hence, at least one of the integers $[x_k]+1, [x_k]+2, \dots, 2[x_k]$ is divisible by $w(vk^u)^k\, k!$. Since $2[x_k] \le [x_k \log x_k]$, at least one element of N_k belongs to $S(u,v,w)$. The Lemma is proved.

3. Proof of Theorem 2. Let d be a sufficiently large positive integer from $S(2,16,2)$. Suppose $[A;B]$ is a real interval such that

$$B - A \ge 4 + 12\frac{(\log\log d)^2}{\log d}.$$

Let also

$$q = \left[\frac{\log d}{3\log\log d}\right].$$

We take two integers p_1 and p_2 in the intervals

$$\left[Aq; \left(A + \frac{4\log\log d}{\log d}\right)q\right) \quad \text{and} \quad \left(\left(B - \frac{4\log\log d}{\log d}\right)q; Bq\right]$$

respectively. Then $[p_1/q; p_2/q] \subset [A;B]$ and

$$\frac{p_2}{q} - \frac{p_1}{q} \ge B - A - \frac{8\log\log d}{\log d} > 4 + 12\frac{(\log\log d)^2 - \log\log d}{\log d}.$$

We will show that the interval $[p_1/q; p_2/q]$ contains a full set of conjugates of an algebraic integer of degree d. Define

$$\varrho = \frac{p_1+p_2}{2q}, \qquad \lambda = \frac{p_2-p_1}{4q}.$$

Following [Ro] and [En] an irreducible monic polynomial of degree d with all d roots in the interval $[p_1/q; p_2/q] = [\varrho - 2\lambda; \varrho + 2\lambda]$ can be constructed by means of the Chebyshev polynomials

$$T_m(x) = x^m + \sum_{j=1}^{[m/2]} (-1)^j \frac{m}{j4^j} \binom{m-j-1}{j-1} x^{m-2j}.$$

In $[-1; 1]$ these are also given by the formula

$$T_m(x) = 2^{1-m} \cos(m \arccos x).$$

Set

$$P_m(x) = (2\lambda)^m T_m\left(\frac{x-\varrho}{2\lambda}\right).$$

We write

$$P_d(x) = x^d + \sum_{j=1}^{d} c_{d,j}\, x^{d-j}.$$

The denominators of the rational numbers ϱ and 2λ are both at most $2q$. Hence the coefficients $c_{d,1}, c_{d,2}, \ldots, c_{d,q}$ are all even integers if d is divisible by $2q!4^q(2q)^{2q}$. This is exactly the case, since $d \in S(2, 16, 2)$ (see the Lemma). All the polynomials $P_m(x)$ are monic, except for $P_0(x) = 2$. So in $[-1; 1)$ there are numbers $b_{q+1}, b_{q+2}, \ldots, b_d$ such that the polynomial

$$Q_d(x) = P_d(x) + \sum_{j=1+q}^{d} b_j P_{d-j}(x) = x^d + \sum_{j=0}^{d-1} a_j x^j$$

has all coefficients a_k integral and even, a_0 not being divisible by 4. Therefore, $Q_d(x)$ is irreducible by Eisenstein's criterion.

In the interval $[\varrho - 2\lambda; \varrho + 2\lambda]$ the maximum of the absolute value of $P_m(x)$ equals $2\lambda^m$. Consequently, in this interval we bound

$$|Q_d(x) - P_d(x)| \le 2 \sum_{j=1+q}^{d} \lambda^{d-j} = 2\, \frac{\lambda^{d-q} - 1}{\lambda - 1} < \frac{2\lambda^d}{\lambda^q(\lambda - 1)}.$$

Since

$$q > \frac{\log d}{3 \log\log d} - 1 \quad \text{and} \quad \lambda > 1 + 3\, \frac{(\log\log d)^2 - \log\log d}{\log d},$$

for large d we have

$$\begin{aligned} q \log \lambda &> q(\lambda - 1)\left(1 - \frac{\lambda - 1}{2}\right) \\ &> \left(\log\log d - 1 - \frac{3(\log\log d)^2}{\log d}\right)\left(1 - \frac{3(\log\log d)^2}{2\log d}\right) \\ &> \log\log d - 2. \end{aligned}$$

Therefore,

$$\lambda^q(\lambda-1) > \frac{\log d}{e^2}(\lambda-1) > \frac{(\log\log d)^2}{3} > 1.$$

Hence, in the interval $\varrho - 2\lambda \le x \le \varrho + 2\lambda$ we have

$$|Q_d(x) - P_d(x)| < 2\lambda^d. \tag{3}$$

Suppose $\xi_1 < \ldots < \xi_{d+1}$ are the points in $[\varrho - 2\lambda; \varrho + 2\lambda]$ such that $|P_d(\xi_j)| = 2\lambda^d$. By our choice, d is even. So $P_d(\xi_j)$ is positive for odd j and negative for even j. From (3) we see that at each of the points $\xi_1, \ldots, \xi_{d+1}$ the signs of the values of $Q_d(x)$ and $P_d(x)$ coincide. Hence in each of the d intervals $(\xi_i; \xi_{i+1})$, where $i = 1, \ldots, d$, there is a zero of $Q_d(x)$. This proves Theorem 2.

4. Proof of Theorem 1. We first prove that

$$I(2) \ge \frac{1+\sqrt{5}}{2} + \sqrt{2}.$$

It suffices to show that no interval $[A; B]$ with $A > 1 - \sqrt{2}$, $B < (3+\sqrt{5})/2$ contains both conjugates of an algebraic integer of degree two.

Indeed, suppose that

$$\frac{p - \sqrt{p^2 - 4q}}{2} > 1 - \sqrt{2}, \tag{4}$$

$$\frac{p + \sqrt{p^2 - 4q}}{2} < \frac{3+\sqrt{5}}{2} \tag{5}$$

with integers p, q such that $p^2 - 4q$ is a positive integer which is not a perfect square. Then

$$\sqrt{p^2 - 4q} < \frac{3+\sqrt{5}}{2} - 1 + \sqrt{2} = \frac{1+\sqrt{5}}{2} + \sqrt{2} < 3.1,$$

so that $p^2 - 4q \le 9$. Since $p^2 - 4q$ modulo 4 is zero or one and $p^2 - 4q$ is not a perfect square, it equals 5 or 8. In the case $p^2 - 4q = 5$ inequality (5) implies that $p \le 2$. Also, p is odd, hence $p \le 1$. Then

$$\frac{p - \sqrt{p^2 - 4q}}{2} \le \frac{1-\sqrt{5}}{2} < 1 - \sqrt{2},$$

which contradicts (4).

If $p^2 - 4q = 8$, then inequality (4) implies that $p > 2$. Hence $p \ge 4$, since p is even. Then

$$\frac{p + \sqrt{p^2 - 4q}}{2} \ge \frac{4+\sqrt{8}}{2} > \frac{3+\sqrt{5}}{2},$$

which contradicts (5).

To prove the inequality

$$I(2) \le \frac{1+\sqrt{5}}{2} + \sqrt{2}$$

we show that any closed interval $[A;B]$ of length $(1+\sqrt{5})/2+\sqrt{2}$ contains both conjugates of an algebraic integer of degree two. Without loss of generality, we assume that $A \in (-1;0]$, since the intervals $[A;B]$ and $[A+z;B+z]$ with z an integer either both contain or both do not contain any set of conjugates of an algebraic integer. There are three possibilities: $A \in (-1;(1-\sqrt{5})/2]$, $A \in ((1-\sqrt{5})/2;1-\sqrt{2}]$ and $A \in (1-\sqrt{2};0]$. In the first case we have

$$B > \frac{1+\sqrt{5}}{2} + \sqrt{2} - 1 = \frac{\sqrt{5}-1}{2} + \sqrt{2}$$

and $[A;B]$ contains both roots of $x^2 - x - 1$.

In the second case,

$$B > \frac{1+\sqrt{5}}{2} + \sqrt{2} + \frac{1-\sqrt{5}}{2} = 1 + \sqrt{2}$$

and $[A;B]$ contains both roots of $x^2 - 2x - 1$.

Finally, in the third case,

$$B > \frac{1+\sqrt{5}}{2} + \sqrt{2} + 1 - \sqrt{2} = \frac{3+\sqrt{5}}{2}$$

and $[A;B]$ contains both roots of $x^2 - 3x + 1$. Theorem 1 is proved.

We now show how to compute $I(d)$ for "small" d. As we already noticed, there is no loss of generality to assume that the left endpoints of the intervals lie in $(-1;0]$. So the right endpoinds are bounded above, say, by 5. However the interval $(-1;5)$ contains only a finite number of sets of conjugates of algebraic integers of degree d. Suppose there are M such sets. Clearly, $M \ge 1$, since $[0;4]$ contains such a set. Let also $\beta_1, \ldots, \beta_M$ be the smallest conjugates in these sets and let $\gamma_1, \ldots, \gamma_M$ be the largest ones. Since for $d \ge 2$ we have $I(d) > 3$, the intersection of the intervals $[\beta_i;\gamma_i]$ and $[\beta_j;\gamma_j]$ is nonempty. So

$$I(d) = \max\{\gamma_j - \beta_i\}, \tag{6}$$

where the maximum is taken over all i, j, $1 \le i, j \le M$, such that for each s, $1 \le s \le M$, either $\beta_s < \beta_i$ or $\gamma_s > \gamma_j$. If d is small, then M is not very large. Having all M polynomials of degree d with all roots in $(-1;5)$ one can apply formula (6) in order to find $I(d)$ explicitly. From (6) it is also clear that we can replace the word "closed" by "half-closed" (interval) in the definition of $I(d)$.

This research was partially supported by a grant from Lithuanian Foundation of Studies and Science.

References

[Du] A. Dubickas, *On the maximal conjugate of a totally real algebraic integer*, Lithuanian Math. J. 37 (1997), 13–19.

[En] V. Ennola, *Conjugate algebraic integers in an interval*, Proc. Amer. Math. Soc. 53 (1975), 259–261.

[Kr] L. Kronecker, *Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten*, J. Reine Angew. Math. 53 (1857), 173–175.

[La] M. Langevin, *Solution des problèmes de Favard*, Ann. Inst. Fourier (Grenoble) 38 (1988), no. 2, 1–10.

[Le] D. H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. of Math. 34 (1933), 461–479.

[Lo] R. Louboutin, *Sur la mesure de Mahler d'un nombre algébrique*, C. R. Acad. Sci. Paris 296 (1983), 707–708.

[Ro] R. M. Robinson, *Intervals containing infinitely many sets of conjugate algebraic integers*, in: Studies in Mathematical Analysis and Related Topics, Stanford Univ. Press, 1962, 305–315.

[SZ] A. Schinzel and H. Zassenhaus, *A refinement of two theorems of Kronecker*, Michigan Math. J. 12 (1965), 81–85.

[Sc] I. Schur, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z. 1 (1918), 377–402.

[Za] T. Zaïmi, *Minoration du diamètre d'un entier algébrique totalement réel*, C. R. Acad. Sci. Paris 319 (1994), 417–419.

Department of Mathematics and Informatics
Vilnius University
Naugarduko 24
2600 Vilnius, Lithuania
E-mail: arturas.dubickas@maf.vu.lt

Received on 16.3.1999
and in revised form on 17.6.1999 (3572)