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A remark on product of Dirichlet L-functions

by

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1. Introduction. While trying to understand the methods and the results of [3], especially in Section 2, we stumbled on an identity ($*$) below, which looked worth recording since we could not locate it in the literature. We would like to thank Dinesh Thakur and Dipendra Prasad for their comments.

2. The identity. For a positive integer $N \in \mathbb{N}$ set $L_N(s) = \zeta(s)$ if $N = 1$ and for $N > 1$ set

$$L_N(s) = \prod_{\chi \pmod N} L(s, \chi)$$

the product taken over all Dirichlet characters mod N. We prove

THEOREM. *For* $\mathrm{Re}(s) > 2$, *the product* $\prod_{N=1}^{\infty} L_N(s)$ *converges absolutely and we have the identity*

$$\prod_{N=1}^{\infty} L_N(s) = \frac{\zeta(s-1)}{\zeta(s)}. \tag{$*$}$$

We need the following lemma.

LEMMA. *Fix a prime* p *and let* $S_p = \{n \in \mathbb{N} \mid (n, p) = 1\}$. *For any* $n \in S_p$ *let* $f(p, n)$ *be the order of* p *modulo* n. *Then the map* $S_p \to \mathbb{N}$ *given by* $n \mapsto f(p, n)$ *is surjective with finite fibres* (*i.e.*, *the inverse image of any number is a finite set*).

Proof. Let k be any natural number. We need an n such that $p^k \equiv 1 \pmod n$ and k is the least positive integer with this property. Clearly $n = (p^k - 1)/(p - 1)$ has this property. Thus the fibre over any $k \in \mathbb{N}$ is contained in the set of divisors of $p^k - 1$ and hence is finite.

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Proof (of the Theorem). For $\mathrm{Re}(s) > 1$, we have ([2], Lemma 6, p. 72)

$$L_N(s) = \prod_{p\nmid N} (1 - p^{-f(p,N)s})^{-\phi(N)/f(p,N)} \tag{1}$$

where $f(p,N)$ is the order of p in $(\mathbb{Z}/N\mathbb{Z})^*$. For brevity, we write f for $f(p,N)$ whenever there is no cause for confusion. The convergence of $\prod_{N=1}^{\infty} L_N(s)$ is equivalent to the convergence of the series

$$\sum_{N\ge 1} \sum_{p\nmid N} \frac{\phi(N)}{f(p,N)} p^{-f(p,N)s}. \tag{2}$$

Note that for $p\nmid N$ we have $1 \le \phi(N)/f(p,N) \le \phi(N) \le N$. As $p^f \equiv 1 \pmod N$ we see that $p^f - 1 = kN$, for some positive integer k. However, several different k, N may give rise to the same integer $p^f - 1$. In any case the number of different k, N corresponds to the number of divisors of $p^f - 1$. By [1] (Theorem 315, p. 260), we have $d(p^f - 1) = o((p^f - 1)^{\varepsilon}) = o((kN)^{\varepsilon})$ for every positive ε and p sufficiently large. Consequently, the series (2) is majorised by

$$C \sum_k \frac{1}{k^{s-\varepsilon}} \sum \frac{1}{N^{s-1-\varepsilon}}$$

for some positive constant C and hence it converges for $\mathrm{Re}(s) > 2$.

Thus we can interchange the product over N and over p in (1). Hence if we set

$$L^p(s) = \prod_{N\ge 1,\, \gcd(p,N)=1} (1 - p^{-fs})^{-\phi(N)/f(p,N)}$$

we have $\prod_N L_N(s) = \prod_p L^p(s)$.

Now put $U = p^{-s}$ and take the logarithmic derivative of $L^p(s)$ with respect to U to get

$$\frac{d}{dU} \log L^p(s) = \sum_{N\ge 1,\, (p,N)=1} \phi(N) \frac{U^{f-1}}{1 - U^f}.$$

By formally writing out geometric series and interchanging order of summations we get

$$\frac{d}{dU} \log L^p(s) = \sum_{N\ge 1,\, (p,N)=1} \phi(N) \sum_{m\ge 1} U^{fm-1} = \sum_{l\ge 1} a_l U^{l-1}, \tag{3}$$

where $a_l = \sum_{f(p,N)m=l} \phi(N)$.

Now observe that by the Lemma, for any $l \in \mathbb{N}$ and a representation $l = f(p,n)m$ for some $n \in S_p, m \in \mathbb{N}$ (in the notations of the Lemma), we have a divisor of $p^l - 1$. Conversely, any divisor d of $p^l - 1$ gives a representation $l = f(p, d)m'$ for some m'. Therefore

$$a_l = \sum_{f(p,N)m=l} \phi(N) = \sum_{d|p^l-1} \phi(d) = p^l - 1.$$

Thus we have

$$\frac{d}{dU} \log L^p(s) = \sum_{l \geq 1} (p^l - 1)U^{l-1}.$$

Then by integrating we see that

$$L^p(s) = \frac{1 - p^{-s}}{1 - p^{-(s-1)}}$$

which proves the Theorem.

References

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed., Oxford Univ. Press, 1968.

[2] J.-P. Serre, *A Course in Arithmetic*, Springer International Student Edition, Narosa Publ. House, New Delhi, 1979.

[3] Y. Taniyama, *L-functions of number fields and zeta functions of abelian varieties*, J. Math. Soc. Japan 9 (1957), 330–366.

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