ACTA ARITHMETICA
XCI.4 (1999)

Chen's theorem in short intervals

by

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1. Introduction. In 1966 Chen Jingrun [1] made a considerable progress in the research of the binary Goldbach conjecture; in [2] he proved the well-known Chen's theorem: *Let N be a sufficiently large even integer. Then the equation $N = p + P_2$ is solvable, where p is a prime and P_2 is an almost prime with at most two prime factors.*

In fact, Chen's theorem can be stated in a more exact quantitative form.

In this paper we generalize Chen's theorem to short intervals.

THEOREM. *Let N be a sufficiently large even integer, $U = N^{0.972}$. Let $S(N,U)$ be the number of solutions of the equation*

$$N = p + P_2, \qquad N/2 - U \le p, P_2 \le N/2 + U.$$

Then

$$S(N,U) \ge \frac{0.001C(N)U}{\log^2 N}, \tag{1.1}$$

where

$$C(N) = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p\mid N,\, p>2}\frac{p-1}{p-2}. \tag{1.2}$$

COROLLARY. *For sufficiently large x and $y = x^{0.972}$, we have*

$$\sum_{\substack{x\le p<x+y\\ p+2=P_2}} 1 \gg \frac{Cy}{\log^2 x}, \tag{1.3}$$

where

$$C = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right). \tag{1.4}$$

1991 *Mathematics Subject Classification*: Primary 11N36.

Project supported by The National Natural Science Foundation of China (grant no. 19531010 and 19801021).

This result is comparable with Jie Wu's result [8] that (1.3) holds for $y = x^{0.973}$. Recently Salerno and Vitolo [6] obtained the exponent 0.9729.

2. Some preliminary lemmas. Let $\mathcal{A}$ be a finite set of integers, $\mathcal{P}$ an infinite set of primes and $\overline{\mathcal{P}}$ the set of primes that do not belong to $\mathcal{P}$. For $z \ge 2$, put

$$P(z) = \prod_{p<z,\, p\in\mathcal{P}} p, \qquad S(\mathcal{A},\mathcal{P},z) = \sum_{a\in\mathcal{A},\, (a,P(z))=1} 1,$$
$$\mathcal{A}_d = \{a \mid a\in\mathcal{A},\ a\equiv 0 \pmod d\}, \qquad \mathcal{P}(q) = \{p \mid p\in\mathcal{P},\ (p,q)=1\}.$$

LEMMA 1 [3]. *Let*

$$|\mathcal{A}_d| = \frac{\omega(d)}{d}X + r_d, \qquad \mu(d)\neq 0,\ (d,\overline{\mathcal{P}})=1,$$
$$\sum_{z_1\le p<z_2} \frac{\omega(p)}{p} = \log\frac{\log z_2}{\log z_1} + O\biggl(\frac{1}{\log z_1}\biggr), \qquad z_2 > z_1 \ge 2,$$

where $\omega(d)$ is a multiplicative function, $0\le\omega(p)<p$, $X>1$ is independent of d. Then

$$S(\mathcal{A},\mathcal{P},z) \ge XV(z)\biggl\{f(s) + O\biggl(\frac{1}{\log^{1/3} D}\biggr)\biggr\} - R_D,$$
$$S(\mathcal{A},\mathcal{P},z) \le XV(z)\biggl\{F(s) + O\biggl(\frac{1}{\log^{1/3} D}\biggr)\biggr\} + R_D,$$

where

$$s = \frac{\log D}{\log z}, \qquad R_D = \sum_{d<D,\, d\mid P(z)} |r_d|,$$
$$V(z) = C(\omega)\frac{e^{-\gamma}}{\log z}\biggl(1 + O\biggl(\frac{1}{\log z}\biggr)\biggr),$$
$$C(\omega) = \prod_p \biggl(1-\frac{\omega(p)}{p}\biggr)\biggl(1-\frac{1}{p}\biggr)^{-1},$$

γ denotes Euler's constant, and $f(s)$ and $F(s)$ are determined by the following differential-difference equation:

$$\begin{cases} F(s) = 2e^{\gamma}/s, \quad f(s) = 0, \quad 0 < s \le 2, \\ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), \quad s\ge 2. \end{cases}$$

LEMMA 2 [5]. *We have*

$$F(s)=\begin{cases}\dfrac{2e^\gamma}{s}, & 0<s\le 3,\\[2ex] \dfrac{2e^\gamma}{s}\left(1+\displaystyle\int_2^{s-1}\frac{\log(t-1)}{t}\,dt\right), & 3\le s\le 5,\end{cases}$$

$$f(s)=\begin{cases}\dfrac{2e^\gamma\log(s-1)}{s}, & 2\le s\le 4,\\[2ex] \dfrac{2e^\gamma}{s}\left(\log(s-1)+\displaystyle\int_3^{s-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(u-1)}{u}\,du\right), & 4\le s\le 6.\end{cases}$$

LEMMA 3 [7]. *Let $g(n)$ be an arithmetical function such that*

$$\sum_{n\le x}\frac{g^2(n)}{n}\ll \log^c x$$

for some $c>0$. For $(al,q)=1$, define

$$H(z,h,a,q,l)=\sum_{\substack{z\le ap<z+h\\ ap\equiv l\,(q)}}1-\frac{1}{\varphi(q)}\left(\mathrm{Li}\left(\frac{z+h}{a}\right)-\mathrm{Li}\left(\frac{z}{a}\right)\right),$$

where

$$\mathrm{Li}\,x=\int_2^x\frac{dt}{\log t}.$$

Then for any given constant $A>0$, there exists a constant $B=B(A,c)>0$ such that for $3/5<\theta\le 1$, $y=x^\theta$, $0\le\beta<(5\theta-3)/2$, $\lambda=\theta-1/2$, $D=x^\lambda\log^{-B}x$,

$$\sum_{d\le D}\max_{(l,d)=1}\max_{h\le y}\max_{x/2\le z\le x}\Big|\sum_{a\le x^\beta,\,(a,d)=1}g(a)H(z,h,a,d,l)\Big|\ll\frac{y}{\log^A x}.$$

REMARK. Let $r_1(a,h)$ and $r_2(a,h)$ be positive functions such that

$$z\le ar_1(a,h),\ ar_2(a,h)\le z+h$$

and put

$$\overline{H}(z,h,a,q,l)=\sum_{\substack{r_1(a,h)\le p<r_2(a,h)\\ ap\equiv l\,(q)}}1-\frac{1}{\varphi(q)}(\mathrm{Li}(r_2(a,h))-\mathrm{Li}(r_1(a,h))).$$

Then under the conditions in Lemma 3,

$$\sum_{d\le D}\max_{(l,d)=1}\max_{h\le y}\max_{x/2\le z\le x}\Big|\sum_{a\le x^\beta,\,(a,d)=1}g(a)\overline{H}(z,h,a,d,l)\Big|\ll\frac{y}{\log^A x}.$$

This result can be proved in the same way as Lemma 3.

LEMMA 4. *Let*

$$x > 1, \quad x^{3/5} \le y < x, \quad z = x^{1/u}, \quad Q(z) = \prod_{p<z} p.$$

Then for $u \ge u_0 > 1$, *we have*

$$\sum_{\substack{x \le n < x+y \\ (n,Q(z))=1}} 1 = w(u)\frac{y}{\log z} + O\biggl(\frac{y}{\log^2 z}\biggr),$$

where $w(u)$ *is determined by the following differential-difference equation*:

$$\begin{cases} w(u) = 1/u, & 1 \le u \le 2, \\ (uw(u))' = w(u-1), & u \ge 2. \end{cases}$$

Moreover, *we have*

$$w(u) < \frac{1}{1.7803}, \quad u \ge 4.$$

Proof. This lemma can be proved in the same way as Lemma 9.4 in [5] with Huxley's prime number theorem in short intervals in place of the prime number theorem. For the upper bound on $w(u)$, see Lemma 20 in [4].

3. Weighted sieve method. Let N be a sufficiently large even integer, $U = N^{0.972}$ and

$$\mathcal{A} = \{N - p \mid N/2 - U \le p \le N/2 + U\}, \tag{3.1}$$

$$\mathcal{P} = \{p \mid (p, N) = 1\}. \tag{3.2}$$

LEMMA 5 [2]. *Let* $\alpha > 0$, $\beta > 0$ *and* $\alpha + 3\beta > 1$. *Then*

$$\begin{aligned} S(N,U) \ge S(\mathcal{A}, N^{\alpha}) &- \frac{1}{2} \sum_{N^{\alpha} \le p < N^{\beta},\, (p,N)=1} S(\mathcal{A}_p, N^{\alpha}) \\ &- \frac{1}{2} \sum_{\substack{N^{\alpha} \le p_1 < N^{\beta} \le p_2 < (N/p_1)^{1/2} \\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2}, p_2) \\ &- \sum_{\substack{N^{\beta} \le p_1 < p_2 < (N/p_1)^{1/2} \\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2}, p_2) \\ &+ \frac{1}{2} \sum_{\substack{N^{\alpha} \le p_1 < p_2 < p_3 < N^{\beta} \\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1), p_2) + O(N^{1-\alpha}). \end{aligned}$$

LEMMA 6.

$$2S(N,U) \ge 2S(\mathcal{A}, N^{1/12}) - \frac{1}{2} \sum_{N^{1/12} \le p < N^{1/3.25},\, (p,N)=1} S(\mathcal{A}_p, N^{1/12})$$

$$- \frac{1}{2} \sum_{\substack{N^{1/12} \le p_1 < N^{1/3.25} \le p_2 < (N/p_1)^{1/2} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{p_1 p_2}, p_2)$$

$$- \sum_{\substack{N^{1/3.25} \le p_1 < p_2 < (N/p_1)^{1/2} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{p_1 p_2}, p_2)$$

$$- \frac{1}{2} \sum_{N^{1/12} \le p < N^{1/9.8},\, (p,N)=1} S(\mathcal{A}_p, p)$$

$$- \frac{1}{2} \sum_{N^{1/12} \le p < N^{1/3.34},\, (p,N)=1} S(\mathcal{A}_p, N^{1/12})$$

$$+ \frac{1}{2} \sum_{\substack{N^{1/12} \le p_1 < p_2 < N^{1/9.8} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{p_1 p_2}, N^{1/12})$$

$$+ \frac{1}{2} \sum_{\substack{N^{1/12} \le p_1 < N^{1/9.8} \le p_2 < U N^{-2/3} p_1^{-1} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{p_1 p_2}, N^{1/12})$$

$$- \frac{1}{2} \sum_{\substack{N^{1/9.8} \le p_1 < N^{1/3.34} \le p_2 < (N/p_1)^{1/2} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{p_1 p_2}, p_2)$$

$$- \sum_{\substack{N^{1/3.34} \le p_1 < p_2 < (N/p_1)^{1/2} \\ (p_1 p_2, N)=1}} S(\mathcal{A}_{p_1 p_2}, p_2)$$

$$- \frac{1}{2} \sum_{\substack{N^{1/12} \le p_1 < p_2 < p_3 < p_4 < N^{1/9.8} \\ (p_1 p_2 p_3 p_4, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(p_1), p_2)$$

$$- \frac{1}{2} \sum_{\substack{N^{1/12} \le p_1 < p_2 < p_3 < N^{1/9.8} \le p_4 < U N^{-2/3} p_3^{-1} \\ (p_1 p_2 p_3 p_4, N)=1}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(p_1), p_2)$$

$$+ O(N^{11/12})$$

$$= 2\Sigma - \frac{1}{2}\Sigma_1 - \frac{1}{2}\Sigma_2 - \Sigma_3 - \frac{1}{2}\Sigma_4 - \frac{1}{2}\Sigma_5 + \frac{1}{2}\Sigma_6 + \frac{1}{2}\Sigma_7 - \frac{1}{2}\Sigma_8 - \Sigma_9 - \frac{1}{2}\Sigma_{10} - \frac{1}{2}\Sigma_{11} + O(N^{11/12}).$$

Proof. By Buchstab's identity, we have

$$S(\mathcal{A}, N^{1/9.8}) = S(\mathcal{A}, N^{1/12}) - \frac{1}{2} \sum_{N^{1/12} \le p < N^{1/9.8},\, (p,N)=1} S(\mathcal{A}_p, p) \tag{3.3}$$

$$
\begin{aligned}
&- \frac{1}{2} \sum_{N^{1/12} \le p < N^{1/9.8},\, (p,N)=1} S(\mathcal{A}_p, N^{1/12}) \\
&+ \frac{1}{2} \sum_{\substack{N^{1/12} \le p_1 < p_2 < N^{1/9.8} \\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2}, N^{1/12}) \\
&- \frac{1}{2} \sum_{\substack{N^{1/12} \le p_1 < p_2 < p_3 < N^{1/9.8} \\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3}, p_1),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{N^{1/9.8} \le p < N^{1/3.34},\, (p,N)=1} S(\mathcal{A}_p, N^{1/9.8}) \\
&\le \sum_{N^{1/9.8} \le p < UN^{-3/4},\, (p,N)=1} S(\mathcal{A}_p, N^{1/12}) \\
&- \sum_{\substack{N^{1/12} \le p_1 < N^{1/9.8} \le p_2 < UN^{-2/3}p_1^{-1} \\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2}, N^{1/12}) \\
&+ \sum_{\substack{N^{1/12} \le p_1 < p_2 < N^{1/9.8} \le p_3 < UN^{-2/3}p_2^{-1} \\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3}, p_1) \\
&+ \sum_{UN^{-3/4} \le p < N^{1/3.34},\, (p,N)=1} S(\mathcal{A}_p, N^{1/12}),
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
&\sum_{\substack{N^{1/12} \le p_1 < p_2 < p_3 < N^{1/3.25} \\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1), p_2) \\
&- \sum_{\substack{N^{1/12} \le p_1 < p_2 < p_3 < N^{1/9.8} \\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3}, p_1) \\
&- \sum_{\substack{N^{1/12} \le p_1 < p_2 < N^{1/9.8} \le p_3 < UN^{-2/3}p_2^{-1} \\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3}, p_1) \\
&\ge - \sum_{\substack{N^{1/12} \le p_1 < p_2 < p_3 < p_4 < N^{1/9.8} \\ (p_1p_2p_3p_4,N)=1}} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1), p_2) \\
&- \sum_{\substack{N^{1/12} \le p_1 < p_2 < p_3 < N^{1/9.8} \le p_4 < UN^{-2/3}p_3^{-1} \\ (p_1p_2p_3p_4,N)=1}} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1), p_2).
\end{aligned} \tag{3.5}
$$

By Lemma 5 with $(\alpha,\beta) = (1/12, 1/3.25)$ and $(\alpha,\beta) = (1/9.8, 1/3.34)$ and (3.3)–(3.5), we complete the proof of Lemma 6.

4. Proof of the Theorem. In this section, the sets $\mathcal{A}$ and $\mathcal{P}$ are defined by (3.1) and (3.2) respectively, and $\theta = 0.972$.

1) *Evaluation of* $\Sigma, \Sigma_1, \Sigma_4$–$\Sigma_7$. By Lemmas 1–3 and some routine arguments as in [8], we get

$$\Sigma \geq (1+o(1))\frac{16C(N)U}{(2\theta-1)\log^2 N} \times \left(\log(12\theta-7) + \int_2^{12\theta-8} \frac{\log(s-1)}{s}\log\frac{12\theta-7}{s+1}\,ds\right) \geq 26.94255\frac{C(N)U}{\log^2 N}, \tag{4.1}$$

$$\Sigma_1 \leq (1+o(1))\frac{16C(N)U}{(2\theta-1)\log^2 N}\left(\log\frac{24\theta-14}{6.5\theta-5.25} + \int_2^{12\theta-8} \frac{\log(s-1)}{s}\log\frac{(12\theta-7)(12\theta-7-s)}{s+1}\,ds\right) \leq 40.10478\frac{C(N)U}{\log^2 N}, \tag{4.2}$$

$$\Sigma_4 \leq (1+o(1))\frac{16C(N)U}{(2\theta-1)\log^2 N}\left(\log\frac{24\theta-14}{19.6\theta-11.8} \times \left(1 + \int_2^{9.8\theta-6.9} \frac{\log(s-1)}{s}\,ds\right) + \int_{9.8\theta-6.9}^{12\theta-8} \frac{\log(s-1)}{s}\log\frac{12\theta-7}{s+1}\,ds\right) \leq 5.03558\frac{C(N)U}{\log^2 N}, \tag{4.3}$$

$$\Sigma_5 \leq (1+o(1))\frac{16C(N)U}{(2\theta-1)\log^2 N}\left(\log\frac{24\theta-14}{6.68\theta-5.34} + \int_2^{12\theta-8} \frac{\log(s-1)}{s}\log\frac{(12\theta-7)(12\theta-7-s)}{s+1}\,ds\right) \leq 38.80741\frac{C(N)U}{\log^2 N}, \tag{4.4}$$

$$\Sigma_6 \geq (1+o(1))\frac{16C(N)U}{\log^2 N} \times \int_{1/12}^{1/9.8}\int_{t_1}^{1/9.8} \frac{\log(12\theta-7-12(t_1+t_2))}{t_1t_2(2\theta-1-2(t_1+t_2))}\,dt_1\,dt_2,$$

$$\Sigma_7 \geq (1+o(1))\frac{16C(N)U}{\log^2 N} \times \int_{1/12}^{1/9.8}\int_{1/9.8}^{\theta-2/3-t_1} \frac{\log(12\theta-7-12(t_1+t_2))}{t_1t_2(2\theta-1-2(t_1+t_2))}\,dt_1\,dt_2,$$

$$\Sigma_6+\Sigma_7 \geq 3.15074\frac{C(N)U}{\log^2 N}. \tag{4.5}$$

2) *Evaluation of* $\Sigma_{10},\ \Sigma_{11}$. We have

$$\begin{aligned}\Sigma_{10} &= \sum_{\substack{N^{1/12}\leq p_1<p_2<p_3<p_4<N^{1/9.8}\\ (p_1p_2p_3p_4,N)=1}} \sum_{\substack{a\in\mathcal{A},\, p_1p_2p_3p_4|a\\ (a,p_1^{-1}NP(p_2))=1}} 1 + O(N^{11/12}) \\ &= \sum_{\substack{N^{1/12}\leq p_1<p_2<p_3<p_4<N^{1/9.8}\\ (p_1p_2p_3p_4,N)=1}} \sum_{\substack{p=N-p_1p_2p_3p_4n\\ E_1\leq n\leq E_2,\, (n,p_1^{-1}NP(p_2))=1}} 1 \\ &\quad + O(N^{11/12}) \\ &= \Sigma_{10}'+O(N^{11/12}),\end{aligned} \tag{4.6}$$

where

$$\begin{aligned} &E_1=\frac{N/2-U}{p_1p_2p_3p_4},\qquad E_2=\frac{N/2+U}{p_1p_2p_3p_4},\\ &\Sigma_{10}' = \sum_{\substack{N^{1/12}\leq p_1<p_2<p_3<N^{1/9.8}\\ (p_1p_2p_3,N)=1}} \sum_{\substack{E_3\leq n\leq E_4\\ (n,p_1^{-1}NP(p_2))=1}} \sum_{\substack{p=N-(p_1p_2p_3n)p_4\\ E_5<p_4<E_6}} 1,\\ &E_3=\frac{N/2-U}{p_1p_2p_3N^{1/9.8}},\qquad E_4=\frac{N/2+U}{p_1p_2p_3^2},\\ &E_5=\max\left(p_3,\frac{N/2-U}{np_1p_2p_3}\right),\qquad E_6=\min\left(N^{1/9.8},\frac{N/2+U}{p_1p_2p_3n}\right).\end{aligned} \tag{4.7}$$

Now we consider the set

$$\mathcal{E}=\{e\mid e=np_1p_2p_3, N^{1/12}\leq p_1<p_2<p_3<N^{1/9.8}, (p_1p_2p_3,N)=1,\\ E_3\leq n\leq E_4, (n,p_1^{-1}NP(p_2))=1\}.$$

By the definition of the set $\mathcal{E}$, it is easy to see that for every $e\in\mathcal{E},\ p_1,p_2,p_3$ are determined by e uniquely. Let $p_3=r(e)$. Then $N^{1/12}<r(e)<N^{1/9.8}$.

Let

$$\mathcal{L} = \{l \mid l = N - ep_4,\ e \in \mathcal{E},\ E_7 < p_4 < E_8\},$$

where

$$E_7 = \max\left(r(e), \frac{N/2 - U}{e}\right), \qquad E_8 = \min\left(N^{1/9.8}, \frac{N/2 + U}{e}\right).$$

We have

$$N^{1/4} < e < N^{11/12},\ e \in \mathcal{E}, \qquad N/2 - U \le l \le N/2 + U,\ l \in \mathcal{L}.$$

Σ_{10}' does not exceed the number of primes in $\mathcal{L}$, hence

(4.8) $$\Sigma_{10}' \le S(\mathcal{L}, z), \qquad z \le N^{1/4}.$$

By Lemma 1 we get

(4.9) $$S(\mathcal{L}, D^{1/2}) \le (1 + o(1)) \frac{8C(N)|\mathcal{L}|}{(2\theta - 1)\log N} + R_1 + R_2,$$

where

$$D = N^{\theta - 1/2} \log^{-B} N \qquad (B = B(4) > 0),$$

$$R_1 = \sum_{d \le D,\, (d,N)=1} \Bigg| \sum_{\substack{e \in \mathcal{E} \\ (e,d)=1}} \Bigg(\sum_{\substack{E_7 < p_4 < E_8 \\ ep_4 \equiv N\ (d)}} 1 - \frac{1}{\varphi(d)} (\mathrm{Li}(E_8) - \mathrm{Li}(E_7)) \Bigg) \Bigg|,$$

$$R_2 = \sum_{d \le D,\, (d,N)=1} \frac{1}{\varphi(d)} \sum_{\substack{e \in \mathcal{E} \\ (e,d)>1}} \sum_{E_7 < p_4 < E_8} 1.$$

Let

$$g(a) = \sum_{e=a,\, e \in \mathcal{E}} 1.$$

Then

$$R_1 = \sum_{d \le D,\, (d,N)=1} \Bigg| \sum_{\substack{N^{1/4} < a < N^{11/12} \\ (a,d)=1}} g(a) \Bigg(\sum_{\substack{E_9 < p_4 < E_{10} \\ ap_4 \equiv N\ (d)}} 1 - \frac{1}{\varphi(d)} (\mathrm{Li}(E_{10}) - \mathrm{Li}(E_9)) \Bigg) \Bigg|,$$

$$R_2 = \sum_{d \le D,\, (d,N)=1} \frac{1}{\varphi(d)} \sum_{\substack{N^{1/4} < a < N^{11/12} \\ (a,d) \ge N^{1/12}}} g(a) \sum_{E_9 < p_4 < E_{10}} 1,$$

$$E_9 = \max\left(r(a), \frac{N/2 - U}{a}\right), \qquad E_{10} = \min\left(N^{1/9.8}, \frac{N/2 + U}{a}\right).$$

It is easy to show that $g(a) \le 1$. Now

$$R_2 \ll \sum_{d\le D}\frac{1}{\varphi(d)}\sum_{\substack{N^{1/4}<a<N^{11/12}\\(a,d)\ge N^{1/12}}}\frac{N}{a} \tag{4.10}$$
$$\ll \sum_{d\le D}\frac{1}{\varphi(d)}\sum_{m|d,\, m\ge N^{1/12}}\sum_{\substack{a<N^{11/12}\\(a,d)=m}}\frac{N}{a}$$
$$\ll N\log N\sum_{d\le D}\frac{1}{\varphi(d)}\sum_{m|d,\, m\ge N^{1/12}}\frac{1}{m}$$
$$\ll N\log N\sum_{N^{1/12}\le m\le D}\frac{1}{m\varphi(m)}\sum_{d\le D/m}\frac{1}{\varphi(d)}$$
$$\ll N^{11/12}\log^2 N.$$

By the remark following Lemma 3 we get

$$R_1 \ll N/\log^4 N. \tag{4.11}$$

Now by Lemma 4 and the prime number theorem, we have

$$|\mathcal{L}| = \sum_{e\in\mathcal{E}}\sum_{E_7<p_4<E_8} 1 \tag{4.12}$$
$$= \sum_{\substack{N^{1/12}\le p_1<p_2<p_3<p_4<N^{1/9.8}\\(p_1p_2p_3p_4,N)=1}}\sum_{\substack{N/2-U\le np_1p_2p_3p_4\le N/2+U\\(n,p_1^{-1}NP(p_2))=1}} 1$$
$$+ O(N^{11/12})$$
$$< \frac{1}{1.7803}\left(1+O\left(\frac{1}{\log N}\right)\right)$$
$$\times \sum_{N^{1/12}\le p_1<p_2<p_3<p_4<N^{1/9.8}}\frac{2U}{p_1p_2p_3p_4\log p_2} + O(N^{11/12})$$
$$= \frac{2U}{1.7803\log N}\int_{1/12}^{1/9.8}\frac{dt_1}{t_1}\int_{t_1}^{1/9.8}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\frac{1}{9.8t_2}dt_2$$
$$+ O\left(\frac{U}{\log^2 N}\right).$$

By (4.6)–(4.12) we get

$$\Sigma_{10} \le (1+o(1))\frac{16C(N)U}{1.7803(2\theta-1)\log^2 N} \tag{4.13}$$

$$\times \int_{1/12}^{1/9.8} \frac{dt_1}{t_1} \int_{t_1}^{1/9.8} \frac{1}{t_2}\left(\frac{1}{t_1} - \frac{1}{t_2}\right) \log \frac{1}{9.8t_2}\, dt_2.$$

By a similar method we get

$$\Sigma_{11} \le (1+o(1))\frac{16C(N)U}{1.7803(2\theta-1)\log^2 N} \tag{4.14}$$

$$\times \int_{1/12}^{1/9.8} \frac{dt_1}{t_1} \int_{t_1}^{1/9.8} \frac{1}{t_2}\left(\frac{1}{t_1} - \frac{1}{t_2}\right) \log\left(9.8\left(\theta - \frac{2}{3} - t_2\right)\right) dt_2.$$

By (4.14) and (4.13) we obtain

$$\Sigma_{10} + \Sigma_{11} \le (1+o(1))\frac{16C(N)U}{1.7803(2\theta-1)\log^2 N} \tag{4.15}$$

$$\times \int_{1/12}^{1/9.8} \frac{dt_1}{t_1} \int_{t_1}^{1/9.8} \frac{1}{t_2}\left(\frac{1}{t_1} - \frac{1}{t_2}\right) \log\left(\frac{\theta - 2/3}{t_2} - 1\right) dt_2$$

$$\le 0.10950 \frac{C(N)U}{\log^2 N}.$$

3) *Evaluation of* $\Sigma_2, \Sigma_3, \Sigma_8, \Sigma_9$. By Lemma 3 and the arguments used in [8], we get

$$\Sigma_2 \le (1+o(1))\frac{16C(N)U}{(2\theta-1)\log N} \tag{4.16}$$

$$\times \sum_{N^{1/12} \le p_1 < N^{1/3.25} \le p_2 < (N/p_1)^{1/2}} \frac{1}{p_1 p_2 \log \frac{N}{p_1 p_2}}$$

$$\le (1+o(1))\frac{16C(N)U}{(2\theta-1)\log^2 N} \int_{2.25}^{11} \frac{\log\left(2.25 - \frac{3.25}{t+1}\right)}{t}\, dt$$

$$\le 13.65253 \frac{C(N)U}{\log^2 N},$$

$$\Sigma_3 \le (1+o(1))\frac{16C(N)U}{(\theta-1)\log N} \sum_{N^{1/3.25} \le p_1 < p_2 < (N/p_1)^{1/2}} \frac{1}{p_1 p_2 \log \frac{N}{p_1 p_2}} \tag{4.17}$$

$$\le (1+o(1))\frac{16C(N)U}{(2\theta-1)\log^2 N} \int_{2}^{2.25} \frac{\log(t-1)}{t}\, dt$$

$$\le 0.22664 \frac{C(N)U}{\log^2 N},$$

$$(4.18)\qquad \Sigma_8 \le (1+o(1))\frac{16C(N)U}{(2\theta-1)\log N} \times \sum_{N^{1/9.8}\le p_1<N^{1/3.34}\le p_2<(N/p_1)^{1/2}} \frac{1}{p_1p_2\log\frac{N}{p_1p_2}}$$
$$\le (1+o(1))\frac{16C(N)U}{(2\theta-1)\log^2 N}\int_{2.34}^{8.8}\frac{\log\left(2.34-\frac{3.34}{t+1}\right)}{t}\,dt$$
$$\le 11.95581\frac{C(N)U}{\log^2 N},$$

$$(4.19)\qquad \Sigma_9 \le (1+o(1))\frac{16C(N)U}{(2\theta-1)\log N} \times \sum_{N^{1/3.34}\le p_1<p_2<(N/p_1)^{1/2}} \frac{1}{p_1p_2\log\frac{N}{p_1p_2}}$$
$$\le (1+o(1))\frac{16C(N)U}{(2\theta-1)\log^2 N}\int_{2}^{2.34}\frac{\log(t-1)}{t}\,dt$$
$$\le 0.39820\frac{C(N)U}{\log^2 N}.$$

4) *Proof of the Theorem.* By (4.1)–(4.5) and (4.15)–(4.19) we get

$$2S(N,U) \ge \Big(2\cdot 26.94255 - \frac{40.10478}{2} - \frac{13.65253}{2} - 0.22664 - \frac{5.03558}{2} - \frac{38.80741}{2} + \frac{3.15074}{2} - \frac{11.95581}{2} - 0.39820 - \frac{0.10950}{2}\Big)\frac{C(N)U}{\log^2 N}$$
$$> \frac{0.002C(N)U}{\log^2 N},$$

and so

$$S(N,U) > \frac{0.001C(N)U}{\log^2 N}.$$

The Theorem is proved.

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Received on 8.4.1998
and in revised form on 6.4.1999 (3362)