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A characterization of some additive arithmetical functions, III

by

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I. Introduction. In 1946, P. Erdős [2] proved that if a real-valued additive arithmetical function f satisfies the condition: $f(n+1)-f(n)\to 0$, $n\to\infty$, then there exists a constant C such that $f(n)=C\log n$ for all n in $\mathbb{N}^*$. Later, I. Kátai [3, 4] was led to conjecture that it was possible to determine additive arithmetical functions f and g satisfying the condition: there exist a real number l, a, c in $\mathbb{N}^*$, and integers b, d such that $f(an+b)-g(cn+d)\to l$, $n\to\infty$. This problem has been treated essentially by analytic methods ([1], [7]). In this article, we shall provide, in an elementary way, a characterization of real-valued additive arithmetical functions f and g satisfying the condition:

(H) there exist a and b in $\mathbb{N}^*$ with $(a,b)=1$ and a finite set Ω such that

$$\lim_{n\to\infty}\min_{\omega\in\Omega}|f(an+b)-g(n)-\omega|=0. \tag{1}$$

II. Results. We have the following result:

Theorem. *Let f and g be real-valued additive arithmetical functions satisfying the condition* (H). *Then there exists a constant C such that the set of values of the sequences $g(n)-C\log n$, n in $\mathbb{N}^*$, and $f(n)-C\log n$, $(n,a)=1$, is finite.*

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III. Proof of the Theorem. We shall always assume that $f(p^k)=0$ for all primes p dividing a and all k, since a change of these values does not affect the assumption of the Theorem.

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The proof relies essentially on the following result which can be obtained elementarily:

Theorem A. *Let f and g be real-valued additive arithmetical functions satisfying the condition*: *there exist a and b in $\mathbb{N}^*$ with $(a,b) = 1$ such that $f(an+b) - g(n) = O(1)$. Then there exists a constant C such that $g(n) - C\log n$ and $f(n) - C\log n$ remain bounded.*

Proof. For an announcement of this result, see [6], and for a proof, [5]. ∎

Assertion 1. *Let f and g satisfy the hypothesis* (H)*. Then there exist a constant C and bounded additive arithmetical functions g' and f' such that $f(n) = C\,\mathrm{Log}\, n + f'(n)$, $g(n) = C\,\mathrm{Log}\, n + g'(n)$.*

Proof. By our hypothesis, there exists a finite set Ω such that

$$\lim_{n\to\infty}\min_{\omega\in\Omega}|f(an+b) - g(n) - \omega| = 0.$$

This gives us that $f(an+b) - g(n)$ is bounded, and Assertion 1 is an immediate consequence of Theorem A. ∎

We only have to prove

Proposition. *Let f and g be bounded real-valued additive arithmetical functions satisfying the condition* (H)*. Then the set of values of $f(n)$ and $g(n)$ is finite.*

To prove this, we introduce the functions $f_p(n) = f(p^k)$ if $p^k \parallel n$ for each prime p, and similarly g_p. Let V_p denote the set of values of the function $f_p(an+b) - g_p(n)$. The next assertion is trivial:

Assertion 2. *An additive function $h(n)$ is bounded if and only if*

$$\sum_p \max_k |h(p^k)| < \infty. \tag{2}$$

Main Lemma. *Let $q \ge 2$ and v be any element of the set $\sum_{p\le q} V_p$. Then v is a limit point of $f(an+b) - g(n)$.*

Proof. We have $v = \sum_{p\le q} v_p$ with $v_p = f_p(an_p + b) - g_p(n_p)$. For every n we have

$$f(an+b) - g(n) = \sum_p (f_p(an+b) - g_p(n)).$$

We want to find infinitely many values of n for which this is in $(v-\varepsilon, v+\varepsilon)$. To achieve this, it is sufficient that

$$f_p(an+b) - g_p(n) = f_p(an_p + b) - g_p(n_p) \tag{3}$$

for all $p \le q$, and

$$\sum_{p>q} |f_p(an+b) - g_p(n)| < \varepsilon. \tag{4}$$

For (3), it is sufficient that n contains p with the same exponent as n_p, and $an+b$ with the same exponent as an_p+b. Both are satisfied if

$$n \equiv n_p \pmod{p^{k_p+1}}, \tag{5}$$

where k_p is the exponent of p in $n_p(an_p+b)$.

To treat (4), we first select a Q so that

$$\sum_{p>Q} \max_k |f(p^k)| + \max_k |g(p^k)| < \varepsilon$$

(Assertion 2). Then the contribution of primes $p>Q$ in (4) is $<\varepsilon$. Thus (4) will hold if we achieve that

$$f_p(an+b) - g_p(n) = 0 \tag{6}$$

for all $q<p<Q$. To do this, it is sufficient that $p\nmid n(an+b)$. Since this excludes two residue classes for $p\nmid a$ and one for $p\mid a$, we can find b_p such that the assumption

$$n \equiv b_p \pmod{p} \tag{7}$$

guarantees (6).

Hence every n that satisfies the congruence (5) for all $p\le q$ and (7) for all $q<p\le Q$ also satisfies $|f(an+b)-g(n)-v|<\varepsilon$. There are infinitely many such numbers, and consequently v is a limit point. ∎

End of proof of the Proposition. We know that the sumset $\sum_{p\le q} V_p$ is contained in Ω for all q; so the cardinalities of these sets are bounded. This means that each V_p is finite, and with a finite number of exceptions $|V_p|=1$.

Write

$$F_p = \{0, f(p), f(p^2), \ldots\}, \qquad G_p = \{0, g(p), g(p^2), \ldots\}.$$

The property that f and g have only a finite number of values is equivalent to saying that F_p and G_p are always finite and are equal to $\{0\}$ except for finitely many primes. For most primes this will follow from the lemma below.

LEMMA. *For $p\nmid 2ab$ we have $V_p = F_p \cup (-G_p)$.*

Proof. If $p\nmid b$, then we cannot have both $p\mid n$ and $p\mid an+b$, thus $V_p\subset F_p\cup(-G_p)$ follows. To show the other inclusion, for a given $k\ge 1$ take an n such that $p^k \parallel n$; then $p\nmid an+b$ and so

$$f_p(an+b) - g_p(n) = -g_p(n) = -g(p^k).$$

Similarly, as $p\nmid a$ we can find an n with $p^k \parallel an+b$ and we infer $f(p^k)\in V_p$. Finally, since $p>2$ we can find an n with $p\nmid n(an+b)$ to show $0\in V_p$. ∎

For $p\mid 2ab$ a complete description of V_p would split into several subcases, but for our aims the following weaker assumption suffices.

LEMMA. *If V_p is finite, so are F_p and G_p.*

Proof. Write $p^r \parallel b$. By considering $n = p^k$ we see that $f(p^r) - g(p^k) \in V_p$ for $k > r$, thus $g(p^k) \in -V_p + f(p^r)$ and G_p is finite.

If $p \mid a$, then $F_p = \{0\}$ by assumption. If $p \nmid a$, then we can find n such that $p^k \parallel an + b$ and we obtain $f(p^k) - g(p^r) \in V_p$, $f(p^k) \in V_p + g(p^r)$. ∎

These lemmas show that F_p, G_p are always finite, and they are $\{0\}$ whenever $p \nmid 2ab$ and $|V_p| = 1$, hence for all but finitely many p. This concludes the proof of the Theorem. ∎

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