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On the Barban–Davenport–Halberstam theorem: XI

by

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1. Introduction. Hitherto articles in this series devoted to the study of prime numbers in arithmetical progressions (I, II, IV–VIII, as enumerated in the list of publications at the end) have concentrated on those properties of the differences

$$E(x;a,k)=\theta(x;a,k)-\frac{x}{\phi(k)}=\sum_{\substack{p\le x\\ p\equiv a,\,\mathrm{mod}\,k}}\log p-\frac{x}{\phi(k)} \tag{0}$$

that are formulated in terms of all reduced residue classes a, modulo k. For example, apart from providing some confirmation of the conjecture that

$$\frac{E(x;a,k)}{x^{1/2}\log^{1/2}x/\phi^{1/2}(k)}$$

is usually substantially bounded, the results obtained have been consistent with the expectation that the above variate over all reduced residues, mod k, has a distribution function, the first, second, and third moments of which have been determined as 0, 1, and 0 (vid., in particular, VII and VIII). But so far nothing has been done in regard to the reasonable prediction that the behaviour of $E(x;a,k)$ over a set of reduced residues a, mod k, is not essentially altered when the set of a considered is contracted into those in a shorter range. We therefore now begin to remedy this deficiency in the theory by establishing analogues of the Barban–Montgomery theorem for sums such as

$$\sum_{k}\sum_{\substack{u_1<a\le u_2\\ (a,k)=1}}E^2(x;a,k), \tag{1}$$

where u_1, u_2 are either fixed or are small multiples of k.

The reduction in the number of residue classes taken into account transforms the problem considered in I into several of much greater difficulty

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that are not entirely dissimilar in character to the one about cubic moments treated in VIII. As in that memoir, we are confronted at a critical stage of the analysis with a ternary additive problem, which now, however, involves only two primes but whose resolution is not entirely easy to exploit. Yet, having likened our difficulties on the former occasion to one of Dante's experiences, we should perhaps affirm in contrast that we seldom felt during the present investigation (Purgatorio)

> "..........................as one
> Who, wandered from his track, thinks every step
> Trodden in vain till he regain the path".

Once more, the principal problem is to treat all terms arising from the original dissection of the moment so accurately that all the principal items in the required estimates emerge with remainder terms that do not vitiate them. But all points of delicacy are best appreciated at the points where they occur in the sequel.

The ultimate destination of our present researches is an asymptotic formula for the sum

$$S_\varrho(x,Q)=\sum_{k\le Q}\sum_{\substack{0<a\le\varrho k\\(a,k)=1}}E^2(x;a,k)\qquad(\varrho=\varrho(x)\le 1),\tag{2}$$

whither we travel via an investigation of sums (1) for fixed values of u_1, u_2 that occupy most of our attention. However, the intermediate results obtained during our journey are of interest in themselves and are therefore also quoted in a series of theorems.

As in previous work on this type of topic, more accurate theorems become available if the extended Riemann hypothesis be assumed. But yet again considerations of space compel us to reserve a discussion of this matter until a later occasion.

2. Notation. Owing to the length of the memoir it is not practicable to lay down a completely consistent notation. Yet, the meaning of all symbols should be clear from their context in view of the following guide.

The letters p, p_1, p_2 denote (positive) prime numbers; l, l', l_1, l_2, l_1', l_2' are positive integers save at the beginning of Section 5 when some of them may be zero; a, b, d, δ, Δ are positive integers; m is a non-negative integer.

The letters B_i are specific constants whose values are immaterial to the investigation; A, A_1 are any positive absolute constants that need not be connected, while $A_2, A_3,\ldots$ are positive absolute constants whose association with each other and with A, A_1 will be plain from the text.

The constants implied by the O-notation depend at most on those values of A, A_i that are relevant to each occasion. As usual (a,b), $[a,b]$ respectively denote the positive highest common factor and least common multiple of a and b when these are defined; $d(m)$ is the number of divisors of m, where m itself may be a multiple of an integer d.

3. Initial analysis of $S^*(x,u;Q_1,Q_2)$. The primary object $S_\varrho(x,Q)$ of our study is approached through the medium of the sums

$$S^*(x;u_1,u_2;Q_1,Q_2) = \sum_{Q_1<k\le Q_2} k \sum_{\substack{u_1<a\le u_2\\(a,k)=1}} E^2(x;a,k) \tag{3}$$

and

$$S^*(x,u;Q_1,Q_2) = \sum_{Q_1<k\le Q_2} k \sum_{\substack{0<a\le u\\(a,k)=1}} E^2(x;a,k), \tag{4}$$

to the latter of which the major part of our investigation is devoted under the assumption that

$$u<Q_1, \qquad x\log^{-A_1}x<Q_1<Q_2\le x \tag{5}$$

where A_1 is any given positive absolute constant as in Section 2. Associated with $S^*(x,u;Q_1,Q_2)$, there is also the parallel sum

$$S(x,u;Q_1,Q_2) = \sum_{Q_1<k\le Q_2} \sum_{\substack{0<a\le u\\(a,k)=1}} E^2(x;a,k) \tag{6}$$

whose behaviour will not be utilized in the derivation of our main result but is sufficiently interesting for it to be deduced from that of the former sum. Here we have already initiated a convention to the effect that the insertion of a superscript asterisk in a given notation for a sum over k means that its summand is to be affected by a weight k, an understanding that facilitates our moving to and fro between unweighted and weighted sums in order that each major entity in the analysis should be treated in the most expeditious way.

Proceeding to the preliminary analysis of $S^*(x,u;Q_1,Q_2)$, we first infer from (4) and (1) that

$$\begin{aligned}S^*&(x,u;Q_1,Q_2)\\ &= \sum_{Q_1<k\le Q_2} k \sum_{\substack{0<a\le u\\(a,k)=1}} \left(\frac{x^2}{\phi^2(k)} - \frac{2x\theta(x;a,k)}{\phi(k)} + \theta^2(x;a,k)\right)\\ &= x^2 \sum_{Q_1<k\le Q_2} \frac{k}{\phi^2(k)} \sum_{\substack{0<a\le u\\(a,k)=1}} 1 - 2x \sum_{Q_1<k\le Q_2} \frac{k}{\phi(k)} \sum_{\substack{0<a\le u\\(a,k)=1}} \theta(x;a,k)\end{aligned} \tag{7}$$

$$+\sum_{Q_1<k\le Q_2} k \sum_{\substack{0<a\le u\\(a,k)=1}} \theta^2(x;a,k)$$

$$= x^2 S_1^*(x,u;Q_1,Q_2) - 2xS_2^*(x,u;Q_1,Q_2) + S_3^*(x,u;Q_1,Q_2), \quad \text{say},$$

the sums in which will be treated according to ascending order of difficulty.

The first sum $S_1^*(x,u;Q_1,Q_2)$ is dismissed at once through the following well-known lemma, to which we shall refer again during the later estimations.

Lemma 1. *The number of positive integers not exceeding v that are coprime to k is equal to*

$$\frac{v\phi(k)}{k} + O\{d(k)\}$$

for (1) $v \ge 0$.

Thus (7) implies that

$$S_1^*(x,u;Q_1,Q_2) = u\sum_{Q_1<k\le Q_2} \frac{1}{\phi(k)} + O\biggl(\sum_{k\le Q_2} \frac{d(k)k}{\phi^2(k)}\biggr)$$
$$= u\sum_{Q_1<k\le Q_2} \frac{1}{\phi(k)} + O(\log^2 Q_2),$$

whence, remembering (5), we infer that

$$S_1^*(x,u;Q_1,Q_2) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} u\log\frac{Q_2}{Q_1} + O\biggl(\frac{u\log Q_1}{Q_1}\biggr) + O(\log^2 Q_2)$$
$$= \frac{\zeta(2)\zeta(3)}{\zeta(6)} u\log\frac{Q_2}{Q_1} + O(\log^2 x)$$

by the familiar estimate cited as part (i) of Lemma 1 in I. This we then place in (7) and complete the initial phase of the analysis by concluding that

$$S^*(x,u;Q_1,Q_2) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} ux^2\log\frac{Q_2}{Q_1} - 2xS_2^*(x,u;Q_1,Q_2) + S_3^*(x,u;Q_1,Q_2) + O(x^2\log x). \tag{8}$$

4. Estimation of $S_2^*(x,u;Q_1,Q_2)$. The item analogous to $S_2^*(x,u;Q_1,Q_2)$ in our proof of the Barban–Montgomery theorem in I is so simple that its calculation is concealed in the middle of equation (2) therein. But the change in circumstances means that the estimation of $S_2^*(x,u;Q_1,Q_2)$ cannot be performed trivially and that we must therefore apply a principle akin to one appearing in a slightly later part of I. Accordingly we find we must first concentrate our attention on the sums

(1) For $0 \le v < 1$ the result is trivial but helpful in what follows later.

$$J_2^*(x,u;Q) = S_2^*(x,u;Q,x)$$

for

$$u < Q, \qquad x\log^{-A_1} x < Q \le x, \tag{9}$$

between which and $S_2^*(x,u;Q_1,Q_2)$ there is the obvious relation

$$S_2^*(x,u;Q_1,Q_2) = J_2^*(x,u;Q_1) - J_2^*(x,u;Q_2) \tag{10}$$

implied by (7).

If we examine the conditions of summation contained in the definition of $J_2^*(x,u;Q)$ as a multiple sum in (7), we see from (9) that $a \le u \le k$, which inequality in itself means that a is the only possible prime number congruent to a, mod k, when $(a,k) > 1$. Therefore the condition $(a,k) = 1$ in the expression

$$\sum_{\substack{0<a\le u\\ (a,k)=1}} \sum_{\substack{p\le x\\ p\equiv a,\ \mathrm{mod}\ k}} \log p$$

for the inner sum in $J_2^*(x,u;Q)$ is superfluous in regard to primes for which $p - a > 0$, and we therefore get

$$\begin{aligned} J_2^*(x,u;Q) &= \sum_{Q<k\le x} \frac{k}{\phi(k)} \sum_{p\le u;\, p\nmid k} \log p \\ &\quad + \sum_{0<a\le u} \sum_{Q<k\le x} \frac{k}{\phi(k)} \sum_{\substack{a<p\le x\\ p\equiv a,\ \mathrm{mod}\ k}} \log p \\ &= J_2^{\$}(x,u;Q) + J_2^{\dagger}(x,u;Q), \quad \text{say}, \end{aligned} \tag{11}$$

wherein it is only the final term that need delay us. Indeed, the final effect of the penultimate term is easily measured, since the prime-number theorem for arithmetical progressions gives

$$\begin{aligned} & J_2^{\$}(x,u;Q_1) - J_2^{\$}(x,u;Q_2) \\ &= \sum_{Q_1<k\le Q_2} \frac{k}{\phi(k)} \left\{ u + O\left(\frac{u}{\log^A u}\right) + O(\log k) \right\} \\ &= u \sum_{Q_1<k\le Q_2} \frac{k}{\phi(k)} + O\left(\frac{u}{\log^A x} \sum_{k\le Q_2} \frac{k}{\phi(k)}\right) + O\Big(\sum_{k\le Q_2} \log k\Big) \\ &= \frac{\zeta(2)\zeta(3)}{\zeta(6)} u(Q_2 - Q_1) + O(u\log Q_2) + O\left(\frac{uQ_2}{\log^A x}\right) + O(Q_2\log Q_2) \\ &= \frac{\zeta(2)\zeta(3)}{\zeta(6)} u(Q_2 - Q_1) + O\left(\frac{x^2}{\log^A x}\right) \end{aligned} \tag{12}$$

by a variant of the already cited Lemma 1 in I and then by (9).

To treat $J_2^{\dagger}(x,u;Q)$ let us denote by $\Phi_{a,k,p}$ the conditions

$$0 < a \le u, \quad Q < k \le x, \quad a < p \le x, \quad p \equiv a, \bmod k, \tag{13}$$

appertaining to its definition, deploying the identity

$$\frac{k}{\phi(k)} = \sum_{d|k} \frac{\mu^2(d)}{\phi(d)}$$

to obtain first

$$\begin{aligned} J_2^{\dagger}(x,u;Q) &= \sum_{d\le x} \frac{\mu^2(d)}{\phi(d)} \sum_{\substack{\Phi_{a,k,p} \\ d|k}} \log p \\ &= \sum_{d\le x} \frac{\mu^2(d)}{\phi(d)} I_1(x,u,d;Q), \quad \text{say,} \end{aligned} \tag{14}$$

and then

$$I_1(x,u,d;Q) = \sum_{0<a\le u} \sum_{\substack{\Phi_{a,k,p} \\ k\equiv 0,\, \bmod d}} \log p = \sum_{0<a\le u} I_2(x,a,d;Q), \quad \text{say.} \tag{15}$$

Here $\Phi_{a,k,p}$ now has its obvious meaning as the conjunction of the last three constituents in (13) for a given value of a, which imply that $p-a=lk$ for a positive integer l such that $l<(x-a)/Q$. Combined with the conditions $p-a>lQ$ and $p\le x$, the previous two requirements are tantamount to $\Phi_{a,k,p}$ in its second rôle with the conclusion that

$$I_2(x,a,d;Q) = \sum_{l<(x-a)/Q} \sum_{\substack{lQ+a<p\le x \\ p\equiv a,\, \bmod ld}} \log p,$$

in which, by the prime-number theorem for arithmetical progressions, the inner sum is

$$\frac{x-lQ-a}{\phi(ld)} + O\left(\frac{x}{\log^{A+A_1+1} x}\right)$$

if $(a,ld)=1$ but is easily seen to be zero otherwise. Hence, by (9),

$$\begin{aligned} I_2(x,a,d;Q) &= \sum_{\substack{l<(x-a)/Q \\ (a,l)=1}} \left\{ \frac{x-lQ-a}{\phi(ld)} + O\left(\frac{x}{\log^{A+A_1+1} x}\right) \right\} \\ &= \sum_{\substack{l\le(x-a)/Q \\ (a,l)=1}} \frac{x-lQ-a}{\phi(ld)} + O\left(\frac{x}{\log^{A+1} x}\right) \end{aligned} \tag{16}$$

if $(a,d)=1$, whereas

$$I_2(x,a,d;Q)=0 \tag{17}$$

in the contrary instance.

We return to the sum $I_1(x,u,d;Q)$ in (15) and infer from (16) and (17) that

$$\begin{aligned} I_1(x,u,d;Q) &= \sum_{l<x/Q}\frac{1}{\phi(ld)}\sum_{\substack{0<a\le u,x-lQ\\(a,ld)=1}}(x-lQ-a)+O\biggl(\frac{xu}{\log^{A+1}x}\biggr)\\ &= \sum_{l<x/Q}\frac{1}{\phi(ld)}\sum_{\substack{0<a\le u,x-lQ\\(a,ld)=1}}(x-lQ-a)+O\biggl(\frac{x^2}{\log^{A+1}x}\biggr) \end{aligned}$$

in virtue of (9) and a further change in the order of additions. Next, by Lemma 1, the last inner sum above is

$$\frac{\phi(ld)(x-lQ)u}{ld}-\frac{\phi(ld)u^2}{2ld}+O\{xd(ld)\}$$

or

$$\frac{\phi(ld)(x-lQ)^2}{2ld}+O\{xd(ld)\}$$

according as $l<(x-u)/Q$ or $(x-u)/Q\le l<x/Q$. Therefore

$$\begin{aligned} (18)\quad & I_1(x,u,d;Q)\\ &= \frac{1}{2d}\sum_{l<x/Q}\frac{(x-lQ)^2}{l}-\frac{1}{2d}\sum_{l<(x-u)/Q}\frac{(x-lQ)^2+u^2-2u(x-lQ)}{l}\\ &\quad+O\biggl(x\sum_{l\le x/Q}\frac{d(ld)}{\phi(ld)}\biggr)+O\biggl(\frac{x^2}{\log^{A+1}x}\biggr)\\ &= \frac{1}{2d}\sum_{l<x/Q}\frac{(x-lQ)^2}{l}-\frac{1}{2d}\sum_{l<(x-u)/Q}\frac{(x-u-lQ)^2}{l}\\ &\quad+O\biggl(x\sum_{l\le x/Q}\frac{d(l)}{\phi(l)}\biggr)+O\biggl(\frac{x^2}{\log^{A+1}x}\biggr)\\ &= \frac{1}{2d}\sum_{l<x/Q}\frac{(x-lQ)^2}{l}-\frac{1}{2d}\sum_{l<(x-u)/Q}\frac{(x-u-lQ)^2}{l}+O\biggl(\frac{x^2}{\log^{A+1}x}\biggr), \end{aligned}$$

the difference of the last two sums in which is assessed by means of

LEMMA 2. *If* $0<h<y$ *and* (2) $y\ge 1$, *then*

$$\frac{1}{2}\sum_{l<y}\frac{(y-l)^2}{l}-\frac{1}{2}\sum_{l<y-h}\frac{(y-h-l)^2}{l}$$
$$=\tfrac{1}{2}\{y^2\log y-(y-h)^2\log(y-h)\}$$
$$+\tfrac{1}{2}B_1\{y^2-(y-h)^2\}+\tfrac{1}{2}h+O(hy^{-1/4}),$$

where the value of $\gamma-3/2$ *of* B_1 *is not yet of importance.*

As with some later explicitly or implicitly stated results we use, this theorem belongs most naturally to an order of ideas related to the Euler–Maclaurin sum formula, to which in fact we shall later briefly advert. But we prefer to establish it by a contour integral method in order to preserve a quick and unified approach to the summation of most of the series of this general type that occur, not all of which are treatable by the Euler–Maclaurin method.

By the usual method involving the calculus of residues, the left side of the proposed formula equals

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta(s+1)\frac{y^{s+2}-(y-h)^{s+2}}{s(s+1)(s+2)}\,ds\qquad (c>0)\tag{19}$$
$$=\mathfrak{R}_1+\mathfrak{R}_2+\frac{1}{2\pi i}\int_{-5/4-i\infty}^{-5/4+i\infty}\zeta(s+1)\frac{y^{s+2}-(y-h)^{s+2}}{s(s+1)(s+2)}\,ds,$$

where $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are the residues of the integrand at $s=0$ and $s=-1$. Next, since the principal part of $\zeta(s+1)/s$ is $1/s^2+\gamma/s$ in the neighbourhood of $s=0$ and since $\zeta(0)=-1/2$,

$$\mathfrak{R}_1+\mathfrak{R}_2=\tfrac{1}{2}\{y^2\log y-(y-h)^2\log(y-h)\}+B_1\{y^2-(y-h)^2\}+\tfrac{1}{2}h;$$

also the residual integral is

$$O\bigg(\int_0^\infty\frac{hy^{-1/4}}{(t+1)^{5/4-\varepsilon}}\,dt\bigg)=O(hy^{-1/4})$$

because within it $\zeta(s+1)=O\{(|t|+1)^{3/4+\varepsilon}\}$ and

$$|y^{2+s}-(y-h)^{2+s}|=O\bigg(\frac{|2+s|}{2+\sigma}\{y^{2+\sigma}-(y-h)^{2+\sigma}\}\bigg)=O\{(|t|+1)hy^{-1/4}\}.$$

(2) It is worth noting that, as for later lemmata and results of the same breed, the proof of Lemma 2 is valid if y or merely $y-h$ be less than 1 although its content then becomes relatively trivial.

The required result therefore follows; there is no advantage for the immediate application in finding an improved version through the Euler–Maclaurin sum formula.

Using the lemma twice with the values $y = x/Q_2$, $h = u/Q_2$ and $y = x/Q_1$, $h = u/Q_1$, we obtain

$$I_1(x,u,d;Q_1) - I_1(x,u,d;Q_2)$$
$$= \frac{1}{2d}\left(\{x^2-(x-u)^2\}\log\frac{Q_2}{Q_1}-(Q_2-Q_1)u+O(Q_2^{5/4}ux^{-1/4})\right)+O\left(\frac{x^2}{\log^{A+1}x}\right)$$

from (18) after taking into account a substantial cancellation of terms. Joined with (14) and the formula

$$\sum_{d=1}^{\infty}\frac{\mu^2(d)}{d\phi(d)} = \prod_p\left(1+\frac{1}{p(p-1)}\right) = \prod_p\frac{p^6-1}{p(p^2-1)(p^3-1)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)},$$

this yields

$$\begin{aligned}
J_2^{\dagger}(x,u;Q_1) &- J_2^{\dagger}(x,u;Q_2)\\
&= \frac{1}{2}\left\{\{x^2-(x-u)^2\}\log\frac{Q_2}{Q_1}-(Q_2-Q_1)u+O(Q_2^{5/4}ux^{-1/4})\right\}\\
&\quad\times\sum_{d\le x}\frac{\mu^2(d)}{d\phi(d)}+O\left(\frac{x^2}{\log^{A+1}x}\sum_{d\le x}\frac{1}{\phi(d)}\right)\\
&= \frac{\zeta(2)\zeta(3)}{2\zeta(6)}\left\{\{x^2-(x-u)^2\}\log\frac{Q_2}{Q_1}-(Q_2-Q_1)u\right\}\\
&\quad+O(x\log x)+O\left(\frac{x^2}{\log^A x}\right)+O(Q_2^{5/4}ux^{-1/4})\\
&= \frac{\zeta(2)\zeta(3)}{2\zeta(6)}\left\{\{x^2-(x-u)^2\}\log\frac{Q_2}{Q_1}-(Q_2-Q_1)u\right\}\\
&\quad+O(Q_2^{5/4}ux^{-1/4})+O\left(\frac{x^2}{\log^A x}\right).
\end{aligned}$$

Hence, by (10)–(12), we are led to the equation

$$\begin{aligned}
(20)\quad S_2^*(x,u;Q_1,Q_2) = \frac{\zeta(2)\zeta(3)}{2\zeta(6)}&\left\{\{x^2-(x-u)^2\}\log\frac{Q_2}{Q_1}+(Q_2-Q_1)u\right\}\\
&+O(Q_2^{5/4}ux^{-1/4})+O\left(\frac{x^2}{\log^A x}\right)
\end{aligned}$$

that is the final conclusion of this section.

5. The earlier analysis of $S_3(x,u;Q_1,Q_2)$. We reach the hardest parts of the analysis now we confront the sum $S_3^*(x,u;Q_1,Q_2)$ that was defined in (7). To reduce the resistance it offers we first move over to the unweighted sum $S_3(x,u;Q_1,Q_2)$ given by

$$S_3(x,u;Q_1,Q_2) = \sum_{Q_1<k\le Q_2} \sum_{\substack{0<a\le u\\(a,k)=1}} \theta^2(x;a,k) \tag{21}$$

in accordance with the conventions laid down at the beginning of Section 3, then letting

$$J_3(x,u;Q) = S_3(x,u;Q,x) \tag{22}$$

when (9) holds so that

$$S_3(x,u;Q_1,Q_2) = J_3(x,u;Q_1) - J_3(x,u;Q_2) \tag{23}$$

much as in (10). Then the square $\theta^2(x;a,k)$ in $J_3(x,u;Q)$ is equal to the sum of $\log p_1 \log p_2$ taken over all solutions of the conditions

$$p_1 - a = l_2 k, \quad p_2 - a = l_1 k, \quad a \le p_1, p_2 \le x \tag{24}$$

in primes p_1, p_2 and (non-negative) integers l_1, l_2, since as in the previous section the stipulation (9) implies that positive numbers congruent to a, mod k, are not less than a. These solutions fall into six mutually exclusive categories characterized by the features

$$\begin{array}{ll} \text{(i) } p_1 = p_2 = a; & \text{(iv) } p_1 = p_2 > a,\ l_1 = l_2 > 0; \\ \text{(ii) } p_1 = a,\ p_2 > a; & \text{(v) } 0 < l_2 < l_1; \\ \text{(iii) } p_1 > a,\ p_2 = a; & \text{(vi) } 0 < l_1 < l_2, \end{array} \tag{25}$$

from which it is deduced that

$$J_3(x,u;Q) = J_3^{\S}(x,u;Q) + 2J_3^{\ddagger}(x,u;Q) + J_3^{\$}(x,u;Q) + 2J_3^{\dagger}(x,u;Q) \tag{26}$$

where $J_3^{\S}$, $J_3^{\ddagger}$, $J_3^{\$}$, $J_3^{\dagger}$ are, respectively, the contributions of categories (i), (ii), (iv), (v) to J_3 and appear in rising order of difficulty. But, before going on, we emphasize that in future the letter l, with or without subscript, will denote a positive integer.

The sum $J_3^{\S}$ can be eliminated from the work at once, since (i), (21), and (22) imply that

$$J_3^{\S}(x,u;Q_1) - J_3^{\S}(x,u;Q_2) = \sum_{Q_1<k\le Q_2} \sum_{\substack{p\le u\\ p\nmid k}} \log^2 p = O(Q_2 u \log u) \tag{27}$$

by Chebyshev's inequality. With this, we can now take leave of the present section in order to concentrate on the remaining sums, in which we note at once that the summatory condition $(a,k)=1$ is redundant in view of the comments after (10).

6. Estimation of $J_3^\ddagger(x,u;Q)$. The estimation of $J_3^\ddagger(x,u;Q)$ is not entirely dissimilar to that of $J_2^\dagger(x,u;Q)$ above although there are some important points of difference. First, by (21), (22), (24), and (ii), $J_3^\ddagger$ is the sum of $\log p_1 \log p_2$ taken over all solutions in primes p_1, p_2 and positive integers l of

$$p_1 \le u, \quad p_2 \equiv p_1, \bmod l, \quad l < x/Q, \quad p_1 + lQ < p_2 \le x,$$

which conditions imply that $p_1 \nmid l$. Therefore

$$\begin{aligned}
(28)\quad & J_3^\ddagger(x,u;Q)\\
&= \sum_{l<x/Q} \sum_{\substack{p_1 \le \min(u, x-lQ)\\ p_1 \nmid l}} \log p_1 \sum_{\substack{p_1+lQ<p_2\le x\\ p_2 \equiv p_1,\, \bmod l}} \log p_2\\
&= \sum_{l<x/Q} \sum_{\substack{p_1 \le \min(u, x-lQ)\\ p_1 \nmid l}} \log p_1 \left\{ \frac{x-lQ-p_1}{\phi(l)} + O\left(\frac{x}{\log^A x}\right)\right\}\\
&= \sum_{l<x/Q} \frac{1}{\phi(l)} \sum_{p_1 \le \min(u,x-lQ)} (x-lQ-p_1)\log p_1\\
&\quad + O\left(x \sum_{l\le x} \frac{\log l}{\phi(l)}\right) + O\left(\frac{x^2 u}{Q\log^A x}\right)\\
&= \sum_{l<x/Q} \frac{1}{\phi(l)} \sum_{p_1 \le \min(u,x-lQ)} (x-lQ-p_1)\log p_1\\
&\quad + O\left(\frac{x^2}{\log^A x}\right)\\
&= J_3^{\ddagger\ddagger}(x,u;Q) + O\left(\frac{x^2}{\log^A x}\right), \quad \text{say,}
\end{aligned}$$

by another application of the prime-number theorem for arithmetical progressions and then by (9). Next, the inner sum in $J_3^{\ddagger\ddagger}$ being

$$\frac{1}{2}(x-lQ)^2 + O\left(\frac{x^2}{\log^{A+1} x}\right)$$

or

$$u(x-lQ) - \frac{1}{2}u^2 + O\left(\frac{x^2}{\log^{A+1} x}\right)$$

according as $(x-u)/Q \le l < x/Q$ or $l < (x-u)/Q$, we have

$$
\begin{aligned}
(29)\quad & J_3^{\ddagger\ddagger}(x,u;Q)\\
&= \frac{1}{2}\sum_{(x-u)/Q\le l<x/Q}\frac{(x-lQ)^2}{\phi(l)} + \frac{1}{2}\sum_{l\le (x-u)/Q}\frac{2u(x-lQ)-u^2}{\phi(l)}\\
&\quad + O\biggl(\frac{x^2}{\log^{A+1}x}\sum_{l<x/Q}\frac{1}{\phi(l)}\biggr)\\
&= \frac{1}{2}\sum_{l<x/Q}\frac{(x-lQ)^2}{\phi(l)}\\
&\quad - \frac{1}{2}\sum_{l\le (x-u)/Q}\frac{(x-lQ)^2-2u(x-lQ)+u^2}{\phi(l)} + O\biggl(\frac{x^2}{\log^{A}x}\biggr)\\
&= \frac{1}{2}\sum_{l<x/Q}\frac{(x-lQ)^2}{\phi(l)} - \frac{1}{2}\sum_{l<(x-u)/Q}\frac{(x-u-lQ)^2}{\phi(l)} + O\biggl(\frac{x^2}{\log^{A}x}\biggr),
\end{aligned}
$$

the last two sums in which are like those in (18) save that the denominators are now $\phi(l)$ instead of l. To calculate their difference we therefore need to replace the use of Lemma 2 by that of the analogous but simpler

Lemma 3. *If* $0<h<y$ *and* $y\ge 1$, *then*

$$
\begin{aligned}
\frac{1}{2}\sum_{l<y}\frac{(y-l)^2}{\phi(l)} - \frac{1}{2}\sum_{l<y-h}\frac{(y-l-h)^2}{\phi(l)}
&= \frac{\zeta(2)\zeta(3)}{2\zeta(6)}\{y^2\log y-(y-h)^2\log(y-h)\}\\
&\quad + B_2\{y^2-(y-h)^2\} + O(hy^{3/4}).
\end{aligned}
$$

All we have to do is to revisit the asymptotic formula for

$$T(u)=\sum_{l<u}\frac{(u-l)^2}{\phi(l)}$$

contained in Lemma 1 of I (or Lemma 1 in VIII), considering how an appropriate remainder term can be made available for $T(y)-T(y-h)$. In fact, if we subtract from $T(y)-T(y-h)$ the explicit terms in our proposed formula corresponding to the first two explicit terms in the formula of I, Lemma 1, we are left with

$$\frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty}\zeta(s+1)\zeta(s+2)h(s+1)\frac{y^{s+2}-(y-h)^{s+2}}{s(s+1)(s+2)}\,ds$$

when the notation of I is deployed. This is similar to (but easier than) the residual integral in (19) except for the factors

$$\zeta(s+1) = O\{(|t|+1)^{1/8+\varepsilon}\}, \quad \zeta(s+2)h(s+1) = O(1),$$
$$y^{s+2} - (y-h)^{s+2} = O\{(|t|+1)\, hy^{3/4}\} \quad (\sigma = -1/4)$$

whose presence in the integrand leads to the estimate

$$O\biggl(\int_0^\infty \frac{hy^{3/4}}{(t+1)^{15/8+\varepsilon}}\, dt\biggr) = O(hy^{3/4})$$

for the remainder term. A considerably more accurate version of this result can in fact be made available but would not endow our present project with additional benefit.

The final part of the assessment echoes the procedure at the end of the previous section and uses the new lemma for the same sets of values $y = x/Q_2$, $h = u/Q_2$; $y = x/Q_1$, $h = u/Q_1$. Accordingly, by (28) and (29), we conclude that

$$\begin{aligned} (30)\quad & J_3^\ddagger(x,u;Q_1) - J_3^\ddagger(x,u;Q_2) \\ &= \frac{\zeta(2)\zeta(3)}{2\zeta(6)}\{x^2 - (x-u)^2\}\log\frac{Q_2}{Q_1} + O(Q_2^{1/4}ux^{3/4}) + O\biggl(\frac{x^2}{\log^A x}\biggr). \end{aligned}$$

7. Estimation of $J_3^\$(x,u;Q)$. The ascent steepens as we arrive at $J_3^\$(x,u;Q)$ even though the main peak to be scaled is not reached till the next section. Analyzing (24) as in Section 6 save that (iv) in (25) is assumed in place of (ii), we begin with the apparently innocuous equation

$$J_3^\$(x,u;Q) = \sum_{l<x/Q} \sum_{\substack{0<a\le\min(u,x-lQ)\\(a,l)=1}} \sum_{\substack{a+lQ<p\le x\\ p\equiv a,\,\mathrm{mod}\, l}} \log^2 p,$$

the innermost sum in which is expressed as

$$\frac{1}{\phi(l)}\int_{a+lQ}^{x} \log t\, dt + O\biggl(\frac{x}{\log^A x}\biggr)$$

in order to ease what would otherwise be a cumbersome summation over a. Therefore, by (9),

$$\begin{aligned} (31)\quad J_3^\$(x,u;Q) &= \sum_{l<x/Q}\frac{1}{\phi(l)} \sum_{\substack{0<a\le\min(u,x-lQ)\\(a,l)=1}} \int_{a+lQ}^{x} \log t\, dt + O\biggl(\frac{x^2u}{Q\log^A x}\biggr) \\ &= \sum_{l<x/Q}\frac{1}{\phi(l)} H(x,u,l;Q) + O\biggl(\frac{x^2}{\log^A x}\biggr), \quad \text{say.} \end{aligned}$$

Next, if $(x-u)/Q \le l < x/Q$, Lemma 1 shews that

$$H(x,u,l;Q) = \int_{lQ}^{x} \log t \sum_{\substack{0<a\le t-lQ\\(a,l)=1}} 1\,dt \tag{32}$$
$$= \frac{\phi(l)}{l}\int_{lQ}^{x}(t-lQ)\log t\,dt + O\{d(l)x\log x\}$$
$$= \frac{\phi(l)}{l}D_1(x,lQ) + O\{d(l)x\log x\}, \quad \text{say},$$

whereas, if $l < (x-u)/Q$, then

$$H(x,u,l;Q) \tag{33}$$
$$= \int_{lQ}^{u+lQ} \log t \sum_{\substack{a\le t-lQ\\(a,l)=1}} 1\,dt + \int_{u+lQ}^{x} \log t \sum_{\substack{a\le u\\(a,l)=1}} 1\,dt$$
$$= \frac{\phi(l)}{l}\int_{lQ}^{u+lQ}(t-lQ)\log t\,dt + \frac{\phi(l)}{l}\int_{u+lQ}^{x} u\log t\,dt + O\{d(l)x\log x\}$$
$$= \frac{\phi(l)}{l}\int_{lQ}^{x}(t-lQ)\log t\,dt - \frac{\phi(l)}{l}\int_{u+lQ}^{x}(t-lQ-u)\log t\,dt + O\{d(l)x\log x\}$$
$$= \frac{\phi(l)}{l}\int_{lQ}^{x}(t-lQ)\log t\,dt - \frac{\phi(l)}{l}\int_{lQ}^{x-u}(t-lQ)\log t\,dt$$
$$- \frac{\phi(l)}{l}\int_{lQ}^{x-u}(t-lQ)\{\log(t+u)-\log t\}\,dt + O\{d(l)x\log x\}$$
$$= \frac{\phi(l)}{l}D_1(x,lQ) - \frac{\phi(l)}{l}D_1(x-u,lQ) - \frac{\phi(l)}{l}D_2(x,u,lQ)$$
$$+ O\{d(l)x\log x\}, \quad \text{say}.$$

Hence (31) transforms into

$$J_3^{\$}(x,u;Q) = \sum_{l<x/Q}\frac{D_1(x,lQ)}{l} - \sum_{l<(x-u)/Q}\frac{D_1(x-u,lQ)}{l} \tag{34}$$
$$- \sum_{l<(x-u)/Q}\frac{D_2(x,u,lQ)}{l} + O\left(\frac{x^2}{\log^A x}\right)$$
$$+ O\left(x\log x\sum_{l\le x}\frac{d(l)}{\phi(l)}\right)$$

$$= J_3^{(1)}(x,Q) - J_3^{(1)}(x-u,Q) - J_3^{(2)}(x,u;Q)$$
$$+ O\left(\frac{x^2}{\log^A x}\right), \quad \text{say,}$$

to progress from which we must first evaluate the integral $D_1(x_1, lQ)$ when x_1 is either x or $x-u$.

From the definition of $D(x_1, lQ)$ in (32) and (33) for $lQ < x_1$, we have

$$\begin{aligned}
D_1(x_1, lQ) &= {}_{lQ}^{x_1}\big[\tfrac{1}{2}(t-lQ)^2\log t\big] - \frac{1}{2}\int_{lQ}^{x_1}\frac{(t-lQ)^2}{t}\,dt \\
&= \frac{1}{2}(x_1-lQ)^2\log x_1 \\
&\quad - \frac{1}{2}\left(\frac{1}{2}(x_1^2-l^2Q^2) - 2lQ(x_1-lQ) + l^2Q^2\log\frac{x_1}{Ql}\right) \\
&= \frac{1}{2}(x_1-lQ)^2\log x_1 \\
&\quad - \left(\frac{1}{4}x_1(x_1-lQ) - \frac{3}{4}lQ(x_1-lQ) + \frac{1}{2}l^2Q^2\log\frac{x_1}{Ql}\right) \\
&= Q^2\bigg\{\frac{1}{2}(x_2-l)^2\log x_1 \\
&\quad - \left(\frac{1}{4}x_2(x_2-l) - \frac{3}{4}l(x_2-l) + \frac{1}{2}l^2\log\frac{x_2}{l}\right)\bigg\}
\end{aligned}$$

on setting $x_2 = x_1/Q$. Therefore, substituting this into the formula for $J_3^{(1)}(x_1, Q)$ that is implicitly contained in (34), we get the equation

$$\begin{aligned}
\frac{1}{Q^2}J_3^{(1)}(x_1,Q) &= \frac{1}{2}\log x_1\sum_{l<x_2}\frac{(x_2-l)^2}{l} \\
&\quad - \left(\frac{1}{4}x_2\sum_{l<x_2}\frac{x_2-l}{l} - \frac{3}{4}\sum_{l<x_2}(x_2-l) + \frac{1}{2}\sum_{l_2<x}l\log\frac{x_2}{l}\right),
\end{aligned}$$

to develop which we employ the previous contour integral methods in preference to other available techniques. In consequence

$$\begin{aligned}
\frac{1}{Q^2}J_3^{(1)}(x_1,Q) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta(s+1)\frac{x_2^{s+2}\log x_1}{s(s+1)(s+2)}\,ds \\
&\quad - \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty}\zeta(s+1)x_2^{s+2} \\
&\quad \times\left(\frac{1}{s(s+1)} - \frac{3}{(s+1)(s+2)} + \frac{2}{(s+2)^2}\right)ds
\end{aligned}$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s+1) \frac{x_2^{s+2} \log x_1}{s(s+1)(s+2)} \, ds$$

$$- \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s+1) \frac{x_2^{s+2}}{s(s+1)(s+2)^2} \, ds$$

for $c > 0$ in the first place, whereupon, on writing $y = x/Q$ and $h = u/Q$ in the notation of Lemmata 2 and 3, we obtain

$$(35) \quad \frac{1}{Q^2}\{J_3^{(1)}(x,Q) - J_3^{(1)}(x-u,Q)\}$$
$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s+1) \frac{y^{s+2} \log x - (y-h)^{s+2} \log(x-u)}{s(s+1)(s+2)} \, ds$$
$$- \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s+1) \frac{y^{s+2} - (y-h)^{s+2}}{s(s+1)(s+2)^2} \, ds.$$

Were it not for our desire to have preliminary theorems of optimum sharpness, it would suffice to move the contours of integration to $\sigma = -1/4$ as in the proof of Lemma 3. As it is, we are obliged to shift at least the first contour further left to $\sigma = -5/4$, it then being convenient for notational brevity to shift the second one likewise. Next, following details related to Lemma 2, we observe that the double poles at $s = 0$ and the single poles at $s = -1$ create a term

$$\tfrac{1}{2}(y^2 \log y + B_3 y^2) \log x - \tfrac{1}{2}\{(y-h)^2 \log(y-h) + B_3 (y-h)^2\} \log(x-u)$$
$$- \left[\tfrac{1}{4}(y^2 \log y + B_4 y^2) - \tfrac{1}{4}\{(y-h)^2 \log(y-h) + B_4(y-h)^2\}\right]$$
$$+ \tfrac{1}{2}\{y \log x - (y-h)\log(x-u) - h\},$$

to which must be added a donation from the residual integrals of $O(hy^{-1/4} \log x)$ that can be absorbed in the estimate $O(h \log x)$ from the last explicit term.

From this and (35), we then infer that

$$(36) \quad J_3^{(1)}(x,Q_1) - J_3^{(1)}(x-u,Q_1) - \{J_3^{(1)}(x,Q_2) - J_3^{(1)}(x,Q_2)\}$$
$$= \frac{1}{2}\{x^2 \log x - (x-u)^2 \log(x-u)\} \log \frac{Q_2}{Q_1}$$
$$- \frac{1}{4}\{x^2 - (x-u)^2\} \log \frac{Q_2}{Q_1} + O(Q_2 u \log x),$$

thus finalizing the discussion of the influence of $J_3^{(1)}(x,Q)$ on our problem.

We need not tarry long over the treatment emanating from $D_2(x,u,lQ)$, which by (33) equals

$$\int_{lQ}^{x-u}(t-lQ)\biggl\{\frac{u}{t}+O\biggl(\frac{u^2}{t^2}\biggr)\biggr\}\,dt = u(x-u-lQ)-ulQ\log\frac{x-u}{lQ} + O\biggl(u^2\log\frac{x-u}{lQ}\biggr)$$

under the assumption that $l<(x-u)/Q$. This and (34) then yield

$$\begin{aligned} J_3^{(2)}(x,u;Q) &= u\sum_{l<(x-u)/Q}\frac{x-u-lQ}{l} - uQ\sum_{l<(x-u)/Q}\log\frac{x-u}{lQ} \\ &\quad + O\biggl(u^2\sum_{l<x/Q}\frac{1}{l}\log\frac{x}{lQ}\biggr) \\ &= u(x-u)\log\frac{x-u}{Q} + B_5u(x-u) - u(x-u) \\ &\quad + O(Q^{1/4}ux^{3/4}) + O\biggl(u^2\log^2\frac{2x}{Q}\biggr) \\ &= u(x-u)\log\frac{x-u}{Q} + (B_5-1)u(x-u) + O(Q^{1/4}ux^{3/4}) \end{aligned}$$

since $u<Q$, wherefore

$$J_3^{(2)}(x,u;Q_1) - J_3^{(2)}(x,u;Q_2) = u(x-u)\log\frac{Q_2}{Q_1} + O(Q_2^{1/4}ux^{3/4}). \tag{37}$$

Thus we conclude that

$$\begin{aligned} &J_3^{\$}(x,u;Q_1) - J_3^{\$}(x,u;Q_2) \\ &\quad = \frac{1}{2}\{x^2\log x-(x-u)^2\log(x-u)\}\log\frac{Q_2}{Q_1} \\ &\qquad - \frac{1}{4}\{x^2-(x-u)^2\}\log\frac{Q_2}{Q_1} - u(x-u)\log\frac{Q_2}{Q_1} \\ &\qquad + O(Q_2u\log x) + O(Q_2^{1/4}ux^{3/4}) + O\biggl(\frac{x^2}{\log^A x}\biggr), \end{aligned} \tag{38}$$

whither we have arrived via (34), (36), and (37).

8. Estimation of $J_3^{\dagger}(x,u;Q)$—the preliminary stages and the application of the circle method. The difficulty of the treatment quickly attains its culmination after we appraise the consequences of (25)(v) and the other conditions of summation appertaining to the implicit meaning of $J_3^{\dagger}(x,u;Q)$ as a triple sum. First, if $(l_1,l_2)=\delta$ with the consequence that we may write $l_1 = l_1'\delta$, $l_2 = l_2'\delta$ where

$$(l_1', l_2') = 1, \tag{39}$$

then the first two items in (24) are tantamount to the pair

$$p_1 \equiv p_2 \equiv a, \bmod \delta, \tag{40}$$

and

$$l_1'\{(p_1 - a)/\delta\} = l_2'\{(p_2 - a)/\delta\},$$

the latter member of which may be restated as

$$l_1'p_1 - l_2'p_2 - l_3'a = 0 \quad (l_3' = l_1' - l_2'). \tag{41}$$

Secondly, the conditions related to k translate into the implication

$$l_2 < l_1 < x/Q$$

and the two requirements

$$p_2 > a + l_1 Q, \quad p_1 > a + l_2 Q,$$

the second one of which is implied by the first when (41) is in place. Therefore, if we recall again the final remark in Section 5 and any remaining conditions related to (21) and (22), we confirm that (41) and the inequality $p_2 \le x$ imply

$$p_1 < p_2, \quad p_1 \le x \tag{42}$$

and then complete the first phase of the calculation by deducing that

$$J_3^\dagger(x, u; Q) = \sum_{\delta < x/Q} \sum_{\substack{l_2' < l_1' < x/(Q\delta) \\ (l_1', l_2') = 1}} P(x, u, Q\delta l_1'; l_1', l_2'; \delta) \tag{43}$$

the inner summand in which is defined by letting $\Theta = \Theta_{\delta, l_1', l_2'}$ indicate the conjunction of (40) and (41) and by then setting

$$\begin{aligned} P(x, u, T; l_1', l_2'; \delta) \\ = \sum_{\substack{\Theta \\ 0 < a \le u;\, a + T < p_2 \le x}} \log p_1 \log p_2 \quad (x \log^{-A_1} x < T < x) \end{aligned} \tag{44}$$

as a sum over the variables a, p_1, p_2.

The formula needed for $P(x, u, T; l_1', l_2'; \delta)$ is obtained by a version of the circle method having some affinity with problems of a binary Goldbach type, although the presence of the variable a means we are meeting a workable binary assignment. Somewhat similar to its namesake in VIII on account of the core conditions (40) and (41), the sum is handled by using some of the results of the previous paper in combination with a Farey dissection in which very few arcs are to be regarded as *major.* It therefore suffices to sketch the demonstration, especially since the calculations over the minor arcs are mainly influenced by the presence of simple exponential sums with integer argument a.

As in VIII, we first contemplate an associated problem in which the domains of summation over each of p_1, p_2, a are not interconnected and therefore look at the sum

$$P_1(x, t_1, t_2; l'_1, l'_2; \delta) = \sum^{\Theta}_{0<a\le t_1;\, t_2<p_2\le x} \log p_1 \log p_2 \tag{45}$$

for any values of t_1, t_2 such that $0 < t_1 < t_2 \le x$, where the second part of (42) is still implicit in the summation. Since

$$\begin{aligned} P_1(x, t_1, t_2; l'_1, l'_2; \delta) &= \sum_{\substack{0<b\le\delta \\ (b,\delta)=1}} \sum_{p_1\equiv p_2\equiv a\equiv b,\, \mathrm{mod}\, \delta} \\ &= \sum_{\substack{0<b\le\delta \\ (b,\delta)=1}} P_2(x, t_1, t_2; l'_1, l'_2; b; \delta), \quad \text{say,} \end{aligned} \tag{46}$$

by the first part of (42), we are then led to introduce the exponential sums

$$\begin{gathered} f_1(\theta) = \sum_{\substack{p_1\le x \\ p_1\equiv b,\, \mathrm{mod}\, \delta}} \log p_1\, e^{2\pi i l'_1 p_1 \theta}, \quad f_2(\theta) = \sum_{\substack{t_2<p_2\le x \\ p_2\equiv b,\, \mathrm{mod}\, \delta}} \log p_2\, e^{-2\pi i l'_2 p_2 \theta}, \\ f_3(\theta) = \sum_{\substack{0<a\le t_1 \\ a\equiv b,\, \mathrm{mod}\, \delta}} e^{-2\pi i l'_3 a \theta}, \end{gathered} \tag{47}$$

in terms of which we find that

$$P_2(x, t_1, t_2; l'_1, l'_2; b; \delta) = \int_0^1 f_1(\theta) f_2(\theta) f_3(\theta)\, d\theta \tag{48}$$

in the usual way.

Assuming throughout

$$l'_2 < l'_1 < (\log^{A_1} x)/\delta \le \log^{A_1} x \quad \text{and} \quad \delta \le \log^{A_1} x \tag{49}$$

in conformity with (9), we use a Farey dissection of order $M = x \log^{-A_2} x$ that has the property that each θ in the range of integration belongs to one and only one arc, mod 1, (apart from the end points) of the form

$$|\theta - h/k| \le \vartheta_{h,k}/(Mk), \tag{50}$$

where $k \le M$, $0 < h \le k$, $(h, k) = 1$, and $1/2 \le \vartheta_{h,k} \le 1$. Next, by (47), $f_3(\theta) = O(1/\|\delta l'_3 \theta\|)$ so that

$$|f_3(\theta)| > A_3 x \log^{-A_2} x \tag{51}$$

only when $\delta l_3'\theta = m + \psi$ for some (non-negative) integer m and for $|\psi| < \frac{1}{2}x^{-1}\log^{A_2} x$, namely, only when

$$\theta = \frac{m}{\delta l_3'} + \phi \quad \text{and} \quad |\phi| < \frac{\log^{A_2} x}{2x\delta l_3'} = \frac{1}{2M\delta l_3'}$$

and hence certainly only when θ lies within an arc (50) for which $k \mid \delta l_3'$. All such arcs are then dilated to form the set $\mathfrak{M}$ of major arcs

$$|\theta - h/k| < 1/M, \qquad k \mid \delta l_3',$$

which are non-intersecting because $k \le \log^{A_1} x$; the complement of $\mathfrak{M}$ in the range of integration then forms the set $\mathfrak{m}$, on which (51) is false. We therefore determine at once the effect of $\mathfrak{m}$ on the integral in (48) because

$$\begin{aligned}\int_{\mathfrak{m}} f_1(\theta)f_2(\theta)f_3(\theta)\,d(\theta) &= O\bigg\{\frac{x}{\log^{A_2} x}\Big(\int_0^1 |f_1(\theta)|^2\,d\theta\Big)^{1/2}\Big(\int_0^1 |f_2(\theta)|^2\,d\theta\Big)^{1/2}\bigg\}\\ &= O\bigg(\frac{x}{\log^{A_2} x}\sum_{p\le x}\log^2 p\bigg) = O\bigg(\frac{x^2}{\log^{A_2-1} x}\bigg),\end{aligned}$$

the cumulative contribution of $\mathfrak{m}$ to $P_1(x, t_1, t_2; l_1', l_2'; \delta)$ via (46) being

$$O\bigg(\frac{x^2\delta}{\log^{A_2-1} x}\bigg) = O\bigg(\frac{x^2}{\log^{A_2-A_1-1} x}\bigg) = O\bigg(\frac{x^2}{\log^{A_4} x}\bigg) \tag{52}$$

for any A_4 provided that $A_2 = A_2(A_1, A_4)$ be chosen to be sufficiently large.

On the major arcs we almost treat matters as we would for a binary problem, disengaging the function $f_3(\theta)$ from the integral by performing the summation over a outside the integral sign. Accordingly the integral of $f_1(\theta)f_2(\theta)f_3(\theta)$ over $\mathfrak{M}$ is expressed as

$$\sum_{\substack{0<a\le t_1\\ a\equiv b,\ \mathrm{mod}\ \delta}} \int_{\mathfrak{M}} f_1(\theta)f_2(\theta)e^{-2\pi i l_3' a\theta}\,d\theta, \tag{53}$$

to which the contribution of the arc centred on h/k is

$$\sum_{\substack{0<a\le t_1\\ a\equiv b,\ \mathrm{mod}\ \delta}} \int_{-1/M}^{1/M} e^{-2\pi i h b l_3'/k} f_1(h/k+\phi)f_2(h/k+\phi)e^{-2\pi i a l_3'\phi}\,d\phi \tag{54}$$

because $k \mid \delta l_3'$. In this, the integral is processed by the methods in VIII and, in particular, by the formulae for $f_1(h/k+\phi)$ and $f_2(h/k+\phi)$ furnished by the work immediately following equation (50) therein. The upshot is that, setting

$$\upsilon_1(\phi) = \int_0^x e^{2\pi i l_1' z\phi}\,dz, \qquad \upsilon_2(\phi) = \int_{t_2}^x e^{-2\pi i l_2' z\phi}\,dz,$$

$$k_1 = k/(k, l_1'), \qquad k_2 = k/(k, l_2')$$

and then letting a_* denote the unique root, mod k, of the simultaneous congruences

$$\nu \equiv 0, \bmod k/(k,\delta), \qquad \nu \equiv b, \bmod (k,\delta), \tag{55}$$

when

$$(k/(k,\delta),\delta) = 1, \tag{56}$$

we discover that the integrand in (54) is

$$\prod_{j=1,2} \frac{\mu\{k_j/(k_j,\delta)\}}{\phi([k_j,\delta])} e^{2\pi i h\{a_*(l_1'-l_2')-bl_3'\}/k} \upsilon_1(\phi)\upsilon_2(\phi) e^{-2\pi i a l_3'\phi} + O\Big(\frac{x^2}{\log^{A_5} x}\Big) \tag{57}$$

if (56) hold but that it is

$$O\Big(\frac{x^2}{\log^{A_5} x}\Big) \tag{58}$$

in the contrary case. Next, in the former situation let $k^* = k/(k,\delta)$, $k^\dagger = (k,\delta)$ for brevity and deduce from (55) that

$$a_* - b \equiv 0, \bmod k^\dagger, \qquad a_* - b \equiv -b, \bmod k^*,$$

whence, by the definition of l_3' in (41),

$$\frac{h\{a_*(l_1'-l_2')-bl_3'\}}{k} = \frac{h(a_*-b)l_3'}{k} \equiv -\frac{h\overline{k}^\dagger b l_3'}{k^*}, \bmod 1,$$

where $\overline{k}^\dagger k^\dagger \equiv 1, \bmod k^*$. Then direct this to (57), which when incorporated in (54) shews the latter equals

$$\begin{aligned} &\sum_{\substack{0<a\le t_1\\ a\equiv b, \bmod \delta}} \Big\{ e^{-2\pi i h\overline{k}^\dagger b l_3'/k^*} \prod_{j=1,2} \frac{\mu\{k_j/(k_j,\delta)\}}{\phi([k_j,\delta])} \int_{-\infty}^{\infty} \upsilon_1(\phi)\upsilon_2(\phi) e^{-2\pi i a l_3'\phi}\, d\phi + O\Big(\frac{x}{\log^{A_6} x}\Big)\Big\} \\ &= \sum_{\substack{0<a\le t_1\\ a\equiv b, \bmod \delta}} e^{-2\pi i h\overline{k}^\dagger b l_3'/k^*} \prod_{j=1,2} \frac{\mu\{k_j/(k_j,\delta)\}}{\phi([k_j,\delta])} I_{l_1',l_2',a} + O\Big(\frac{x}{\log^{A_6} x} \sum_{\substack{0<a\le t_1\\ a\equiv b, \bmod \delta}} 1\Big), \quad \text{say,} \end{aligned} \tag{59}$$

by methods akin to those that substantiated (51) in VIII, the constants A_2 and then A_5 having been chosen to be sufficiently large.

It is opportune to evaluate the integral $I_{l_1',l_2',a}$ in the standard way by Fourier's integral theorem. Arising as a double integral with variables of integration z_1, z_2, the integrand in $I_{l_1',l_2',a}$ is transformed by the substitution

$$Z_1 = l_1' z_1 - l_2' z_2, \qquad Z_2 = z_2$$

of absolute modulus l_1' so that it becomes the Fourier transform

$$\frac{1}{l_1'}\int_{-\infty}^{\infty} F(Z_1) e^{2\pi i Z_1 \phi}\, dZ_1.$$

Consequently,

$$I_{l_1',l_2',a} = \frac{1}{l_1'} F(a l_3') = \frac{x - t_2}{l_1'} \tag{60}$$

because the limits t_2 and x for z_2 imply that $0 \le z_1 \le x$ when $l_1' z_1 - l_2' z_2 - l_3' a = l_1'(z_1 - a) - l_2'(z_2 - a) = 0$ and $a \le t_1 < t_2$.

Introducing the Ramanujan sum

$$c_q(m) = \sum_{\substack{0<d\le q \\ (d,q)=1}} e^{2\pi i m d/q} = \sum_{r|q;\, r|m} \mu\left(\frac{q}{r}\right) r \tag{61}$$

to gather up what has so far been obtained, let us add (59) and the integral of (58) over the Farey fractions corresponding to the major arcs to get

$$\frac{x-t_2}{l_1'} \sum_{\substack{0<a\le t_1 \\ a\equiv b,\, \mathrm{mod}\, \delta}} \sum_{\substack{k|\delta l_3' \\ (k/(k,\delta),\delta)=1}} \phi(k^\dagger) c_{k^*}(b l_3') \prod_{j=1,2} \frac{\mu\{k_j/(k_j,\delta)\}}{\phi([k_j,\delta])} + O\left(\frac{x}{\log^{A_6} x} \sum_{\substack{0<a\le t_1 \\ a\equiv b,\, \mathrm{mod}\, \delta}} \sum_{k|\delta l_3'} \phi(\delta l_3')\right) \tag{62}$$

by (60). In the principal term here, by (56), we may assume that k^* is square-free, while $(b, k^*) \,|\, l_3'$ since $k \,|\, \delta l_3'$ and $(b,\delta) = 1$. Therefore $c_{k^*}(b l_3') = c_{k^*}(l_3')$ because of (61), and we thus see that (62) becomes

$$\frac{x-t_2}{l_1'} \sum_{\substack{0<a\le t_1 \\ a\equiv b,\, \mathrm{mod}\, \delta}} \sum_{\substack{k|\delta l_3' \\ (k/(k,\delta),\delta)=1}} \phi(k^\dagger) c_{k^*}(l_3') \prod_{j=1,2} \frac{\mu\{k_j/(k_j,\delta)\}}{\phi([k_j,\delta])} + O\left(\frac{x\delta l_3'}{\log^{A_6} x} \sum_{\substack{0<a\le t_1 \\ a\equiv b,\, \mathrm{mod}\, \delta}} 1\right)$$

$$= \frac{x-t_2}{l_1'}\mathfrak{S}_{\delta,l_1',l_2'} \sum_{\substack{0<a\le t_1\\ a\equiv b,\ \mathrm{mod}\ \delta}} 1 + O\biggl(\frac{x}{\log^{A_6-A_1} x} \sum_{\substack{0<a\le t_1\\ a\equiv b,\ \mathrm{mod}\ \delta}} 1\biggr), \quad \text{say,}$$

because of (49). Therefore, summing over b to gauge the effect of $\mathfrak{M}$ on $P_1(x,t_1,t_2;l_1',l_2';\delta)$ and then adding on (52), we conclude that

$$\begin{aligned} P_2(x,t_1,t_2;l_1',l_2';\delta) &= \frac{x-t_2}{l_1'}\mathfrak{S}_{\delta,l_1',l_2'}\biggl(\frac{\phi(\delta)t_1}{\delta} + O\{d(\delta)\}\biggr) \\ &\quad + O\biggl(\frac{xt_1}{\log^{A_6-A_1} x}\biggr) + O\biggl(\frac{x^2}{\log^{A_4} x}\biggr) \\ &= \frac{t_1(x-t_2)}{l_1'}\cdot\frac{\phi(\delta)}{\delta}\mathfrak{S}_{\delta,l_1',l_2'} + O\biggl(\frac{x^2}{\log^{A_7} x}\biggr) \end{aligned}$$

by Lemma 1 and a trivial bound for $\mathfrak{S}_{\delta,l_1',l_2'}$.

To determine the value of the singular series let us note that

$$\mathfrak{S}_{\delta,l_1',l_2'} = \frac{1}{\phi^2(\delta)} \sum_{\substack{k\mid\delta l_3'\\ (k/(k,\delta),\delta)=1}} \phi(k^\dagger)c_{k^*}(l_3') \prod_{j=1,2}\frac{\mu\{k_j/(k_j,\delta)\}}{\phi\{k_j/(k_j,\delta)\}},$$

the summand in which is a multiplicative function of k when it is taken to be zero for values of k not satisfying $(k/(k,\delta),k)=1$. Hence, by a simple argument that takes into consideration (61) and the definition of l_3' in (41), we have

$$\begin{aligned} \mathfrak{S}_{\delta,l_1',l_2'} &= \frac{1}{\phi^2(\delta)}\prod_{p^\alpha\parallel\delta}(1+\phi(p)+\ldots+\phi(p^\alpha))\prod_{p\mid l_3';\, p\nmid\delta}\biggl(1+\frac{p-1}{(p-1)^2}\biggr) \\ &= \frac{\delta}{\phi^2(\delta)}G_\delta(l_3'), \end{aligned}$$

where

$$G_\delta(l_3') = \prod_{p\mid l_3';\, p\nmid\delta}\biggl(1+\frac{1}{p-1}\biggr). \tag{63}$$

Altogether, therefore, we arrive at

$$P_1(x,t_1,t_2;l_1',l_2';\delta) = \frac{G_\delta(l_3')}{l_1'\phi(\delta)}t_1(x-t_2) + O\biggl(\frac{x^2}{\log^{A_7} x}\biggr), \tag{64}$$

to which we append the obviously zero determination of P_1 when we allow t_2 to stray into the formerly debarred territory $t_2 > x$.

We extract from (64) the behaviour of $P(x,u,T)$ in (43) by a Tauberian argument that perforce is rather longer than its analogue in VIII. First, if

$$P_3(x,u,T) = P_3(x,u,T;l_1',l_2';\delta) = \int_T^x P(x,u,T_1)\,dT_1, \tag{65}$$

then

$$P_3(x,u,T) = \sum_{\substack{0<a\le u;\, a+T<p_2\le x \\ \Theta}} \log p_1 \log p_2 \int_T^{p_2-a} dT_1$$
$$= \sum_{\substack{0<a\le u;\, a+T<p_2\le x \\ \Theta}} (p_2 - a - T)\log p_1 \log p_2,$$

whereas, if

$$P_4(x,u,T) = P_4(x,u,T;l_1',l_2';\delta) = \int_T^{x-u} P_1(x,u,u+T_1)\,dT_1 \tag{66}$$

and $u \le x - T$, then

$$P_4(x,u,T) = \sum_{\substack{0<a\le u;\, u+T<p_2\le x \\ \Theta}} \log p_1 \log p_2 \int_T^{p_2-u} dT_1$$
$$= \sum_{\substack{0<a\le u;\, u+T<p_2\le x \\ \Theta}} (p_2 - u - T)\log p_1 \log p_2.$$

Secondly, by (45),

$$\int_0^u P_1(x,t,t+T)\,dt = \sum_{\substack{0<a\le u;\, a+T<p_2\le x \\ \Theta}} \log p_1 \log p_2 \int_a^{\min(u,p_2-T)} dt, \tag{67}$$

which equals $P_3(x,u,T)$ if $u \ge x - T$ but which equals

$$\begin{aligned} &\sum_{\substack{0<a\le u;\, a+T<p_2\le u+T \\ \Theta}} (p_2 - a - T)\log p_1 \log p_2 \\ &\qquad + \sum_{\substack{0<a\le u;\, u+T<p_2\le x \\ \Theta}} (u-a)\log p_1 \log p_2 \\ &= \sum_{\substack{0<a\le u;\, a+T<p_2\le x \\ \Theta}} (p_2 - a - T)\log p_1 \log p_2 \\ &\qquad - \sum_{\substack{0<a\le u;\, u+T<p_2\le x \\ \Theta}} (p_2 - u - T)\log p_1 \log p_2 \\ &= P_3(x,u,T) - P_4(x,u,T) \end{aligned} \tag{68}$$

if $u \le x - T$. Next (64) implies that the integral on the left of (67) is either

$$\int_0^{x-T} P_1(x,t,t+T)\,dt = \frac{G_\delta(l_3')}{l_1'\phi(\delta)} \int_0^{x-T} t(x-T-t)\,dt + O\left(\frac{x^2(x-T)}{\log^{A_7} x}\right)$$
$$= \frac{G_\delta(l_3')}{6l_1'\phi(\delta)}\,(x-T)^3\,dt + O\left(\frac{x^2(x-T)}{\log^{A_7} x}\right)$$

or

$$\frac{G_\delta(l_3')}{l_1'\phi(\delta)} \int_0^{u} t(x-T-t)\,dt + O\left(\frac{x^2(x-T)}{\log^{A_7} x}\right)$$
$$= \frac{G_\delta(l_3')}{l_1'\phi(\delta)} \left\{\frac{1}{2}u^2(x-T) - \frac{1}{3}u^3\right\} + O\left(\frac{x^2(x-T)}{\log^{A_7} x}\right)$$

according as $u \ge x - T$ or $u \le x - T$, while in the latter instance it also implies that

$$P_4(x,u,T)\,dt = \frac{G_\delta(l_3')u}{l_1'\phi(\delta)} \int_T^{x-u} (x-u-T_1)\,dT_1 + O\left(\frac{x^2(x-T)}{\log^{A_7} x}\right)$$
$$= \frac{G_\delta(l_3')}{2l_1'\phi(\delta)}\,u(x-T-u)^2 + O\left(\frac{x^2(x-T)}{\log^{A_7} x}\right).$$

Therefore, with the aid of (68), we deduce that

$$P_3(x,u,T) = \frac{G_\delta(l_3')}{6l_1'\phi(\delta)}\,F(x,u,T) + O\left(\frac{x^2(x-T)}{\log^{A_7} x}\right), \tag{69}$$

where

$$F(x,u,T) = (x-T)^3 \tag{70}$$

for $u \ge x - T$ but where

$$F(x,u,T) = 3u^2(x-T) - 2u^3 + 3u(x-T-u)^2 = (x-T)^3 - (x-u-T)^3 \tag{71}$$

for $u \le x - T$. Having obtained an Abelian version of what is needed, we perform a "de la Vallée Poussin differentiation" by choosing H such that $0 < H < x - T, T$ and using the inequality

$$\frac{1}{H}\{P_3(x,u,T) - P_3(x,u,T+H)\} \le P_3(x,u,T)$$
$$\le \frac{1}{H}\{P_3(x,u,T-H) - P_3(x,u,T)\}$$

that is an inference from (65). Since the function $F(x,u,T)$ defined by (70) and (71) is a twice differentiable function having second derivatives $x - T$ or u according as $u \ge x - T$ or $u \le x - T$, this inequality with (69) leads to

$$P(x,u,T) = \frac{G_\delta(l_3')}{6l_1'\phi(\delta)} \cdot \frac{\partial}{\partial T} F(x,u,T) + O\{H(x-T)\} + O\left(\frac{x^2(x-T)}{H\log^{A_7} x}\right)$$

in view of (63). Hence, setting $H = (x-T)\log^{-A_7/2} x$ and confirming through (44) that $H < x\log^{-A_1} x < T$ for sufficiently large values of A_7, we complete [3] the estimation and thereby gain

$$P(x,u,T) = \frac{G_\delta(l_3')}{2l_1'\phi(\delta)} F_1(x,u,T) + O\left(\frac{x^2}{\log^{A_8} x}\right) \tag{72}$$

where

$$F_1(x,u,T) = \begin{cases} (x-T)^2 & \text{if } u \geq x-T, \\ (x-T)^2 - (x-u-T)^2 & \text{if } u \leq x-T. \end{cases} \tag{73}$$

Armed with (72), we are at long last able to return to (43), remarking on account of (41) that l_3' can replace l_2' in the summatory conditions and thus obtaining

$$\begin{aligned} J_3^{\dagger}(x,u;Q) &= \frac{1}{2}\sum_{\delta<x/Q}\frac{1}{\phi(\delta)}\sum_{\substack{l_3'<l_1'<x/(Q\delta)\\(l_1',l_3')=1}}\frac{1}{l_1'}F_1(x,u,Q\delta l_1')G_\delta(l_3') \\ &\quad + O\left(\frac{x^2}{\log^{A_8} x}\sum_{\delta<x/Q}\sum_{l_1'<l_3'<x/(Q\delta)} 1\right) \\ &= \frac{1}{2}\sum_{\delta<x/Q}\frac{1}{\phi(\delta)}\sum_{\substack{l_3'<l_1'<x/(Q\delta)\\(l_1',l_3')=1}}\frac{1}{l_1'}F_1(x,u,Q\delta l_1')G_\delta(l_3') \\ &\quad + O\left(\frac{x^2}{\log^{A_8-2A_1} x}\right) \\ &= J_3^{\dagger\dagger}(x,u;Q) + O\left(\frac{x^2}{\log^{A_9} x}\right), \quad \text{say,} \end{aligned} \tag{74}$$

because of (9). Since the coprimality condition is a distraction in the treatment of $J_3^{\dagger\dagger}(x,u;Q)$, we remove it by affecting the summand by the factor

$$\sum_{d|l_1';\,d|l_3'} \mu(d),$$

whence, setting $l_1' = dl_1$, $l_3' = dl_3$, and noting from (63) that

$$G_\delta(dl_3) = G_\delta(d)\,G_{d\delta}(l_3),$$

---

[3] The lower bound for T in (44) is not strictly necessary but slightly reduces the length of the demonstration.

we deduce that

$$\begin{aligned}
J_3^{\dagger\dagger}(x,u;Q) &= \frac{1}{2}\sum_{\delta<x/Q}\frac{1}{\phi(\delta)}\sum_{d<x/(Q\delta)}\frac{\mu(d)}{d}\\
&\quad\times\sum_{l_3<l_1<x/(Qd\delta)}\frac{1}{l_1}F_1(x,u,Qd\delta l_1)G_\delta(dl_3)\\
&= \frac{1}{2}\sum_{\delta<x/Q}\frac{1}{\phi(\delta)}\sum_{d<x/(Q\delta)}\frac{\mu(d)G_\delta(d)}{d}\\
&\quad\times\sum_{0<l_3<l_1<x/(Qd\delta)}\frac{1}{l_1}F_1(x,u,Qd\delta l_1)G_{d\delta}(l_3)\\
&= \frac{1}{2}\sum_{\Delta<x/Q}I(\Delta)\sum_{l_3<l_1<x/(Q\Delta)}\frac{1}{l_1}F_1(x,u,Q\Delta l_1)G_\Delta(l_3)
\end{aligned}$$

where

$$I(\Delta)=\sum_{d\delta=\Delta}\frac{\mu(d)G_\delta(d)}{d\phi(\delta)}.$$

But, by (63), the multiplicative function $I(\Delta)$ is given by the determination

$$I(p^\alpha)=\begin{cases}1/(p-1)-1/(p-1)=0 & \text{if } \alpha=1,\\ 1/\phi(p^\alpha)-1/(p\phi(p^{\alpha-1}))=0 & \text{if } \alpha>1\end{cases}$$

so that $I(\Delta)=0$ unless $\Delta=1$. Hence we arrive at the equation

$$J_3^{\dagger\dagger}(x,u;Q)=\frac{1}{2}\sum_{l_3<l_1<x/Q}\frac{1}{l_1}F_1(x,u,Ql_1)G_1(l_3) \tag{75}$$

and complete the first half of the treatment of $J_3^\dagger(x,u;Q)$.

9. Estimations of $J_3^\dagger(x,u;Q)$ and $S_3(x,u;Q_1,Q_2)$—the second stages. Examining (75), (63), and (73), we discern the sum

$$U(v)=\frac{1}{2}\sum_{l_3<l_1<v}\frac{(v-l_1)^2 l_3}{l_1\phi(l_3)}$$

and find it present in the relation

$$J_3^{\dagger\dagger}(x,u;Q)=Q^2\{U(y)-U(y-h)\}, \tag{76}$$

where $y=x/Q$ and $h=u/Q$ as before. An investigation of $U(y)$ must therefore follow, the principal difficulty being to profit from the smoothing element that is latent in the formation of the sum.

First, dropping the subscript from l_3 to lighten the notation, we have

$$U(v) = \frac{1}{2}\sum_{l<v}\frac{l}{\phi(l)}\sum_{l<l_1<v}\frac{(v-l_1)^2}{l_1} = \frac{1}{2}\sum_{l<v}\frac{l}{\phi(l)}V(v,l), \quad \text{say}, \tag{77}$$

and

$$\frac{1}{2}V(v,l) = \frac{1}{2}V_1(v,v) - \frac{1}{2}V_1(v,l), \tag{78}$$

where

$$\frac{1}{2}V_1(v,w) = \frac{1}{2}\sum_{l\le w}\frac{(v-l)^2}{l}$$

for $w \le v$. To proceed from here, we must amplify and exploit the theory behind Lemma 2, first using the Euler–Maclaurin sum formula to shew that, for positive integers w,

$$\begin{aligned} \frac{1}{2}V_1(v,w) &= \frac{1}{2}v^2\sum_{l\le w}\frac{1}{l} - v\sum_{l\le w}1 + \frac{1}{2}\sum_{l\le w}l \\ &= \frac{1}{2}v^2\left\{\log w + \gamma + \frac{1}{2w} + C(w)\right\} - vw + \frac{1}{4}w^2 + \frac{1}{4}w \end{aligned} \tag{79}$$

where

$$C(w) = O(1/w^2). \tag{80}$$

This implies that, if we set

$$R_2(v) = \frac{1}{2}V_1(v,v) - \frac{1}{2}v^2\log v - \frac{1}{2}v^2\left(\gamma - \frac{3}{2}\right) - \frac{1}{2}v, \tag{81}$$

then, for positive integral values of v,

$$R_2(v) = \frac{1}{2}v^2\,C(v) \tag{82}$$

in contrast to the estimate

$$R_2(y) - R_2(y-h) = O(hy^{-1/4}) \tag{83}$$

supplied by Lemma 2. Hence, by (78), (79) and (81), we gain the equation

$$\begin{aligned} \frac{1}{2}V(v,l) &= \frac{1}{2}v^2\log\frac{v}{l} - \frac{3}{4}v^2 + vl - \frac{1}{4}l^2 - \frac{v^2}{4l} + \frac{1}{2}v - \frac{1}{4}l \\ &\quad + R_2(v) - \frac{1}{2}v^2\,C(l) \\ &= \frac{1}{2}v^2\log\frac{v}{l} - \frac{3}{4}v(v-l) + \frac{1}{4}l(v-l) + \frac{1}{4}(v-l) - \frac{v}{4l}(v-l) \\ &\quad + R_2(v) - \frac{1}{2}v^2\,C(l) \end{aligned}$$

and then, by way of (77) and (80), the intermediate estimate

$$
\begin{aligned}
(84)\quad U(v) &= \frac{1}{2}v^2\sum_{l<v}\frac{l}{\phi(l)}\log\frac{v}{l} - \frac{3}{4}v\sum_{l<v}\frac{(v-l)l}{\phi(l)} + \frac{1}{4}\sum_{l<v}\frac{(v-l)l^2}{\phi(l)} \\
&\quad + \frac{1}{4}\sum_{l<v}\frac{(v-l)l}{\phi(l)} - \frac{1}{4}v\sum_{l<v}\frac{(v-l)}{\phi(l)} + \sum_{l<v}\frac{R_2(v)l}{\phi(l)} \\
&\quad - \frac{1}{2}v^2\sum_{l<v}\frac{C(l)l}{\phi(l)} \\
&= \left\{\frac{1}{2}v^2\sum_{l<v}\frac{l}{\phi(l)}\log\frac{v}{l} - \frac{3}{4}v\sum_{l<v}\frac{(v-l)l}{\phi(l)} + \frac{1}{4}\sum_{l<v}\frac{(v-l)l^2}{\phi(l)}\right. \\
&\quad \left. + \frac{1}{4}\sum_{l<v}\frac{(v-l)l}{\phi(l)} - \frac{1}{4}v\sum_{l<v}\frac{(v-l)}{\phi(l)}\right\} + B_6v^2 \\
&\quad + \left\{\sum_{l<v}\frac{R_2(v)l}{\phi(l)} + \frac{1}{2}v^2\sum_{l\geq v}\frac{C(l)l}{\phi(l)}\right\} \\
&= U_1(v) + B_6v^2 + U_2(v), \quad \text{say,}
\end{aligned}
$$

from which flows the required formula for $U(y+h) - U(y)$.

The first component $U_1(v)$ is studied by previous contour integral methods in partnership with the function

$$
(85)\qquad f(s) = \sum_{l=1}^{\infty}\frac{1}{l^{s-1}\phi(l)} = \zeta(s)\zeta(s+1)h(s) \qquad (\sigma > 1)
$$

that appeared in I. With this procedure, we deduce that

$$
\begin{aligned}
(86)\quad U_1(v) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\left\{\frac{1}{2s^2} - \frac{3}{4s(s+1)} + \frac{1}{4(s+1)(s+2)}\right\}v^{2+s}\,ds \\
&\quad - \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\left\{\frac{1}{(s-1)s} - \frac{1}{s(s+1)}\right\}v^{1+s}\,ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{v^{2+s}}{s^2(s+1)(s+2)}\,ds \\
&\quad - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{v^{1+s}}{2(s-1)s(s+1)}\,ds \\
&= \frac{1}{2\pi i}\{\mathcal{I}_1(v) - \mathcal{I}_2(v)\}, \quad \text{say,}
\end{aligned}
$$

for $c > 1$ in the first place. Next, by (85) and by previous calculations that shew that $h(0) = 1$ (vid. proof of Lemma 1 in I) and that the principal part of $\zeta(s+1)/s^2$ at $s=0$ is $1/s^3+\gamma/s^2+B_7/s$, the residues of the integrand in $\mathcal{I}_1(v)$ at $s=1$ and $s=0$ are, respectively,

$$\frac{1}{6}\zeta(2)h(1)v^3 = \frac{\zeta(2)\zeta(3)v^3}{6\zeta(6)}$$

and

$$\begin{aligned}(87)\qquad &\left\{\frac{1}{2}\cdot\frac{d^2}{ds^2}\left(\frac{\zeta(s)h(s)v^{2+s}}{(s+1)(s+2)}\right)+\gamma\frac{d}{ds}\left(\frac{\zeta(s)h(s)v^{2+s}}{(s+1)(s+2)}\right)\right\}_{s=0}+B_8v^2\\ &= \frac{1}{4}\zeta(0)h(0)v^2\log^2 v\\ &\quad -\frac{1}{2}\zeta(0)h(0)\left(\frac{\zeta'(0)}{\zeta(0)}+\frac{h'(0)}{h(0)}+\gamma-\frac{3}{2}\right)v^2\log v+B_9v^2\\ &= -\frac{1}{8}v^2\log^2 v-\frac{1}{4}\left(\frac{\zeta'(0)}{\zeta(0)}+\frac{h'(0)}{h(0)}+\gamma-\frac{3}{2}\right)v^2\log v+B_9v^2\\ &= -\frac{1}{8}v^2\log^2 v-\frac{1}{4}C_1v^2\log v+B_9v^2, \quad \text{say;}\end{aligned}$$

in like manner the residue of the integrand in $\mathcal{I}_2(v)$ at $s=1$ is

$$(88)\qquad \frac{1}{4}\zeta(2)h(1)v^2\log v+B_{10}v^2=\frac{\zeta(2)\zeta(3)}{4\zeta(6)}v^2\log v+B_{10}v^2.$$

Also, if $\mathcal{I}_1^*(v)$ and $\mathcal{I}_2^*(v)$ denote the integrals obtained by moving the lines of integration in $\mathcal{I}_1(v)$ and $\mathcal{I}_2(v)$ to $\sigma=-1/4$ and $\sigma=3/4$, respectively, we obtain

$$(89)\qquad \mathcal{I}_1^*(y)-\mathcal{I}_1^*(y-h)=O(hy^{3/4}),\qquad \mathcal{I}_2^*(y)-\mathcal{I}_2^*(y-h)=O(hy^{3/4})$$

in emulation of the treatment of the residual integral in the proof of Lemma 2.

The main problem associated with $U_2(v)$ is that we need an estimate for $U_2(y)-U_2(y-h)$ that involves a factor h, a consideration that impelled the entrance of $C(w)$ into earlier formulae. From the definition of $U_2(v)$ in (84), we have

$$\begin{aligned}(90)\qquad U_2(y)-U_2(y-h)&=\{R_2(y)-R_2(y-h)\}\sum_{l<y-h}\frac{l}{\phi(l)}\\ &\quad+\frac{1}{2}\{y^2-(y-h)^2\}\sum_{l\ge y}\frac{C(l)l}{\phi(l)}\\ &\quad+\sum_{y-h\le l<y}\frac{l}{\phi(l)}\{R_2(y)-(y-h)^2C(l)\}\end{aligned}$$

$$= O(hy^{3/4}) + O(h) + \sum_{y-h\le l<y} \frac{l}{\phi(l)}\{R_2(y) - (y-h)^2 C(l)\}$$
$$= O(hy^{3/4}) + \sum_{y-h\le l<y} \frac{l}{\phi(l)}\{R_2(y) - (y-h)^2 C(l)\}$$

by (83) and (80). Next, dismissing the case where the final sum above is empty, we see there is one term only within it because $h < 1$ and then that the corresponding value of l satisfies $y - l \le h$ and $(y-h)^2 = l^2 + O(lh)$. Hence, in this instance, the last component of (90) is

$$\frac{l}{\phi(l)}\{R_2(y) - \tfrac{1}{2}l^2 C(l)\} + O\left(\frac{h}{\phi(l)}\right) = \frac{l}{\phi(l)}\{R_2(y) - R_2(l)\} + O\left(\frac{h}{\phi(l)}\right)$$
$$= O(hy^{-1/4+\varepsilon}) = O(hy^{3/4})$$

by (82) and (83) again, and we conclude that

$$U_2(y) - U_2(y-h) = O(hy^{3/4}). \tag{91}$$

The estimate we seek for $J_3^\dagger(x,u;Q_1) - J_3^\dagger(x,u;Q_2)$ is at long last within reach. Starting with the equation

$$J_3^{\dagger\dagger}(x,u;Q_1) - J_3^{\dagger\dagger}(x,u;Q_2) = Q_1^2\{U(x/Q_1) - U(x/Q_1 - u/Q_1)\}$$
$$- Q_2^2\{U(x/Q_2) - U(x/Q_2 - u/Q_2)\}$$

that stems from (76), we bring (84)–(91) into action in turn to get

$$\begin{aligned}
&J_3^{\dagger\dagger}(x,u;Q_1) - J_3^{\dagger\dagger}(x,u;Q_2)\\
&= \frac{\zeta(2)\zeta(3)}{6\zeta(6)}\left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)\{x^3 - (x-u)^3\}\\
&\quad - \frac{1}{8}\left\{x^2\left(\log^2\frac{x}{Q_1} - \log^2\frac{x}{Q_2}\right) - (x-u)^2\left(\log^2\frac{x-u}{Q_1} - \log^2\frac{x-u}{Q_2}\right)\right\}\\
&\quad - \frac{1}{4}\left(C_1 + \frac{\zeta(2)\zeta(3)}{\zeta(6)}\right)\{x^2 - (x-u)^2\}\log\frac{Q_2}{Q_1} + O(Q_2^{1/4}ux^{3/4})
\end{aligned}$$

after some initial simplification. Hence, by (74) and some further simplification, we secure the equation

$$\begin{aligned}
&J_3^\dagger(x,u;Q_1) - J_3^\dagger(x,u;Q_2)\\
&\qquad = \frac{\zeta(2)\zeta(3)}{6\zeta(6)}\left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)\{x^3 - (x-u)^3\}\\
&\qquad\quad - \frac{1}{4}\{x^2\log x - (x-u)^2\log(x-u)\}\log\frac{Q_2}{Q_1}
\end{aligned}$$

$$+\frac{1}{8}\{x^2-(x-u)^2\}(\log^2 Q_2-\log^2 Q_1)$$
$$-\frac{1}{4}\biggl(C_1\frac{\zeta(2)\zeta(3)}{\zeta(6)}\biggr)\{x^2-(x-u)^2\}\log\frac{Q_2}{Q_1}+O(Q_2^{3/4}ux^{3/4})+O\biggl(\frac{x^2}{\log^A x}\biggr)$$

and then attain this section's goal by deducing that

$$\begin{aligned}(92)\qquad S_3(x,u;Q_1,Q_2)={}&\frac{\zeta(2)\zeta(3)}{3\zeta(6)}\biggl(\frac{1}{Q_1}-\frac{1}{Q_2}\biggr)\{x^3-(x-u)^3\}\\&+\frac{1}{4}\{x^2-(x-u)^2\}(\log^2 Q_2-\log^2 Q_1)\\&+\frac{1}{2}\biggl(\frac{\zeta(2)\zeta(3)}{\zeta(6)}-C_1-\frac{1}{2}\biggr)\\&\times\{x^2-(x-u)^2\}\log\frac{Q_2}{Q_1}-u(x-u)\log\frac{Q_2}{Q_1}\\&+O(Q_2^{1/4}ux^{3/4})+O(Q_2u\log x)+O\biggl(\frac{x^2}{\log^A x}\biggr),\end{aligned}$$

to which we come via (23), (26), (27), (30), (38), and some cancellation between terms.

10. Analysis of $S^*(x,u;Q_1,Q_2)$ completed and the initial theorems. The evaluation of $S^*(x,u;Q_1,Q_2)$ easily results from (92) and what went before. First, proceeding from $S_3(x,u;Q_1,Q_2)$ in (21) to $S_3^*(x,u;Q_1,Q_2)$ in (7) by partial summation, let us transform (92) into ([4])

$$\begin{aligned}&S_3^*(x,u;Q_1,Q_2)\\&\quad=\frac{\zeta(2)\zeta(3)}{3\zeta(6)}\{x^3-(x-u)^3\}\log\frac{Q_2}{Q_1}\\&\qquad+\frac{1}{2}\{Q_2\log Q_2-Q_1\log Q_1-(Q_2-Q_1)\}\{x^2-(x-u)^2\}\\&\qquad+\frac{1}{2}\biggl(\frac{\zeta(2)\zeta(3)}{\zeta(6)}-C_1-\frac{1}{2}\biggr)(Q_2-Q_1)\{x^2-(x-u)^2\}\\&\qquad-(Q_2-Q_1)u(x-u)\\&\qquad+O(Q_2^{5/4}ux^{5/4}+O(Q_2^2u\log x)+O\biggl(\frac{x^3}{\log^A x}\biggr)\end{aligned}$$

when (5) holds. Next place this and (20) in (8) to deduce that

([4]) A comment in reverse is made in a succeeding footnote. To get a given value of A in what follows, the previous value of A in (91) need only be taken to be $A-A_3$, where A is the new value.

$$
\begin{aligned}
&S^*(x,u;Q_1,Q_2)\\
&\quad= \frac{\zeta(2)\zeta(3)}{3\zeta(6)}[3ux^2 - 3x\{x^2-(x-u)^2\} + x^3 - (x-u)^3]\log\frac{Q_2}{Q_1}\\
&\qquad + \tfrac{1}{2}(Q_2\log Q_2 - Q_1\log Q_1)\{x^2-(x-u)^2\}\\
&\qquad + \frac{\zeta(2)\zeta(3)}{2\zeta(6)}(Q_2-Q_1)\{x^2-(x-u)^2-2ux\}\\
&\qquad - \tfrac{1}{2}(Q_2-Q_1)\big[\big(C_1+\tfrac{3}{2}\big)\{x^2-(x-u)^2\} + 2u(x-u)\big]\\
&\qquad + O(Q_2^{5/4}ux^{3/4}) + O(Q_2^2u\log x) + O\bigg(\frac{x^3}{\log^A x}\bigg)\\
&\quad= \frac{\zeta(2)\zeta(3)u^3}{3\zeta(6)}\log\frac{Q_2}{Q_1} + (Q_2\log Q_2 - Q_1\log Q_1)\big\{xu - \tfrac{1}{2}u^2\big\}\\
&\qquad + \frac{\zeta(2)\zeta(3)}{2\zeta(6)}(Q_2-Q_1)u^2 - (Q_2-Q_1)\big\{\big(C_1+\tfrac{5}{2}\big)ux - \big(\tfrac{1}{2}C_1 - \tfrac{7}{4}\big)u^2\big\}\\
&\qquad + O(Q_2^{5/4}ux^{3/4} + O(Q_2^2u\log x) + O\bigg(\frac{x^3}{\log^A x}\bigg)\\
&\quad= (Q_2\log Q_2 - Q_1\log Q_1)ux - (C_2+1)(Q_2-Q_1)ux\\
&\qquad + O(Q_2^{5/4}ux^{3/4} + O(Q_2^2u\log x) + O\bigg(\frac{x^3}{\log^A x}\bigg),
\end{aligned}
$$

wherein

$$
C_2 = C_1 + \frac{3}{2} = \frac{\zeta'(0)}{\zeta(0)} + \gamma + \sum_p \frac{\log p}{p(p-1)} \tag{93}
$$

by the implicit definition of C_1 in (87) and by the value of $h'(0)/h(0)$ in the proof of Lemma 1, VIII. Hence we obtain

THEOREM 1. *Defining* $E(x;a,k)$ *as in the Introduction*, *let us write*

$$
S^*(x,u;Q_1,Q_2) = \sum_{Q_1<k\le Q_2} k \sum_{\substack{0<a\le u\\(a,k)=1}} E^2(x;a,k)
$$

and suppose A, A_1 *are any positive absolute constants. Then*, *for* $x\log^{-A_1}x < Q_1 < Q_2 \le x$ *and* $u \le Q_1$, *we have*

$$
\begin{aligned}
S^*(x,u;Q_1,Q_2) &= (Q_2\log Q_2 - Q_1\log Q_1)ux\\
&\quad - (C_2+1)(Q_2-Q_1)ux + O(Q_2^{5/4}ux^{3/4})\\
&\quad + O(Q_2^2u\log x) + O\bigg(\frac{x^3}{\log^A x}\bigg),
\end{aligned} \tag{94}
$$

where C_2 *is defined by* (93) *above.*

It is readily confirmed that the main terms in this formula together amount to u times those in any formula for

$$\sum_{Q_1<k\le Q_2} \sum_{\substack{0<a\le k\\(a,k)=1}} E^2(x;a,k)$$

that is supplied by one of the more accurate enunciations of the Barban–Montgomery theorem (vid., for example, I, in which the value of D_2 in Theorem 1 therein is seen to be C_2+1 by the calculations appertaining to (11) in VIII—vid. also [1]). Already, therefore, we observe a phenomenon that is consistent with our conjectures in the Introduction regarding the sum of $E^2(x;a,k)$ over incomplete sets of residues, mod k.

Among other comments about this theorem, we should mention that the lower bound of summation must be imposed in the present context because of our emphasis on the behaviour of $E(x;a,k)$ over a set of a for which $a<k$. Also we should note that, if Q_1 be not too close to Q_2, then the larger principal term of the formula predominates for $Q_2=o(x)$ even when the first two remainder terms are replaced by the more compact but inferior $O(Q_2^{5/4}ux^{3/4}\log x)$, a substitution, however, that is to be avoided if the remainder terms shall not eclipse the lesser principal term in any part of the natural range $Q_2<x/\log x$.

A matter of some substance concerns the sum $S^*(x;u_1,u_2;Q_1,Q_2)$ defined in (4), for which Theorem 1 provides a useful asymptotic formula when $u_1<u_2<Q$ and u_2-u_1 is not too small compared with u_2. Yet the inconvenient latter condition is in fact superfluous because modifications in the proof of Theorem 1 enable one to shew without undue difficulty that the substitution of u_2-u_1 for u in the right-hand side of (94) furnishes a valid asymptotic formula for $S^*(x;u_1,u_2;Q_1,Q_2)$. Indeed, although to have essayed at the beginning to go straight for a proof of the more general result would have obscured an already complicated exposition, the attentive reader will readily apprehend the alterations needed, including in particular the use of differences of the type

$$\begin{aligned}\Phi_Q(x-u_1)-\Phi_Q(x-u_2)\\ &= \sum_{l\le(x-u_1)/Q}(x-u_1-lQ)^2a_l - \sum_{l\le(x-u_2)/Q}(x-u_2-lQ)^2a_l\end{aligned}$$

instead of $\Phi_Q(x)-\Phi_Q(x-u)$ as before; the hardest aspect of the revised treatment probably concerns the analogue of $J_3^{(2)}(x,u;Q)$ in (34). In summation, we therefore state

Theorem 2. *Let*

$$S^*(x;u_1,u_2;Q_1,Q_2)=\sum_{Q_1<k\le Q_2} k \sum_{\substack{u_1<a\le u_2\\(a,k)=1}} E^2(x;a,k)$$

and suppose A, A_1 *are any positive absolute constants. Then, for* $x\log^{-A_1}x < Q_1 < Q_2 \le x$ *and* $0 \le u_1 < u_2 \le Q_1$, *we have*

$$\begin{aligned} S^*(x;u_1,u_2;Q_1,Q_2) = {} & (Q_2\log Q_2 - Q_1\log Q_1)(u_2-u_1)x \\ & - (C_2+1)(Q_2-Q_1)(u_2-u_1)x \\ & + O\{Q_2^{5/4}(u_2-u_1)x^{3/4}\} \\ & + O\{Q_2^2(u_2-u_1)\log x\} + O\biggl(\frac{x^3}{\log^A x}\biggr), \end{aligned}$$

where C_2 *is defined in* (93) *above.*

The proof of our final theorem about $S_\varrho(x,Q)$ will stem from Theorem 2 both in its general form and in its special form as Theorem 1. However, Theorem 1 alone suffices for the derivation of a slightly weaker form of this last proposition, which therefore is approachable by those not wishing to verify the demonstration of Theorem 2 through the programme indicated above.

But ere we end this section we must fulfil an earlier promise to state an asymptotic formula for $S(x,u;Q_1,Q_2)$, which, being the unweighted form of $S^*(x,u;Q_1,Q_2)$, is in appearance closer than the latter to the sum in the Barban–Montgomery theorem. Obtained from Theorem 1 by partial summation ([5]), this is contained in

Theorem 3. *Let* $S(x,u;Q_1,Q_2)$ *be defined as in* (6). *Then, subject to the conditions laid down in Theorem* 1, *we have*

$$\begin{aligned} S(x,u;Q_1,Q_2) = {} & \frac{1}{2}(\log^2 Q_2 - \log^2 Q_1)ux - C_2ux\log\frac{Q_2}{Q_1} + O(Q_2^{1/4}ux^{3/4}) \\ & + O(Q_2u\log x) + O\biggl(\frac{x^2}{\log^A x}\biggr). \end{aligned}$$

11. Asymptotic formula for $S_\varrho(x,Q)$. Theorem 2 is applied in this section through the agency of the sums

$$S^\dagger(x;u_1,u_2;Q_1,Q_2) = \sum_{Q_1<k\le Q_2}\frac{1}{k}\sum_{\substack{u_1<k\le u_2\\(a,k)=1}} E^2(x;a,k) \tag{95}$$

and

$$S^\dagger(x,u;Q_1,Q_2) = S^\dagger(x;0,u;Q_1,Q_2),$$

which by means of partial summation are seen to conform to the estimate in

([5]) To obtain Theorem 3 for a given value of A we need to use the value $A+A_1$ for its counterpart in Theorem 1. A comparable comment applies to the derivation of Lemma 4 below.

Lemma 4. *Subject to the conditions imposed in Theorem* 2, *we have*

$$\begin{aligned}S^{\dagger}(x;u_1,u_2;Q_1,Q_2) = {} & \left(\frac{\log Q_1}{Q_1} - \frac{\log Q_2}{Q_2}\right)(u_2-u_1)x \\ & - (C_2-1)\left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)(u_2-u_1)x \\ & + O\{Q_1^{-3/4}(u_2-u_1)x^{3/4}\} \\ & + O\left((u_2-u_1)\log x \log\frac{Q_2}{Q_1}\right) + O\left(\frac{x}{\log^A x}\right).\end{aligned}$$

The transition from the sums just defined to the sum

$$S_{\varrho}(x;Q_1,Q_2) = \sum_{Q_1<k\le Q_2} \sum_{\substack{0<a\le \varrho k \\ (a,k)=1}} E^2(x;a,k) \tag{96}$$

for $\varrho = \varrho(x) \le 1$ begins with the integrals

$$W_{\varrho}(x;Q_1,Q_2) = \int_{Q_1}^{Q_2} S^{\dagger}(x,\varrho t;t,Q_2)\,dt \tag{97}$$

and

$$X_{\varrho}(x;Q_1,Q_2) = \int_{0}^{Q_1} S^{\dagger}(x,\varrho t;Q_1,Q_2)\,dt, \tag{98}$$

the sum of which is

$$\begin{aligned}&\int_{Q_1}^{Q_2} \sum_{t<k\le Q_2} \frac{1}{k} \sum_{\substack{0<a\le \varrho t \\ (a,k)=1}} E^2(x;a,k)\,dt + \int_{0}^{Q_2} \sum_{Q_1<k\le Q_2} \sum_{\substack{0<a\le \varrho t \\ (a,k)=1}} E^2(x;a,k)\,dt \\ &\qquad = \sum_{Q_1<k\le Q_2} \frac{1}{k}\Bigg(\sum_{\substack{0<a\le \varrho k \\ (a,k)=1}} E^2(x;a,k) \int_{\max(a/\varrho,Q_1)}^{k} dt \\ &\qquad\qquad + \sum_{\substack{0<a\le \varrho Q_1 \\ (a,k)=1}} E^2(x;a,k) \int_{a/\varrho}^{Q_1} dt\Bigg) \\ &\qquad = \sum_{Q_1<k\le Q_2} \frac{1}{k}\Bigg\{ \sum_{\substack{0<a\le \varrho k \\ (a,k)=1}} \left(k - \max\left(\frac{a}{\varrho},Q_1\right)\right) E^2(x;a,k)\end{aligned}$$

$$+ \sum_{\substack{0<a\le\varrho Q_1\\(a,k)=1}} \left(Q_1 - \frac{a}{\varrho}\right) E^2(x;a,k)\Bigg\}$$
$$= \frac{1}{\varrho} \sum_{Q_1<k\le Q_2} \sum_{\substack{0<a\le\varrho k\\(a,k)=1}} \left(\varrho - \frac{a}{k}\right) E^2(x;a,k).$$

But, from (96) and similar reasoning,

$$S_\varrho^{(1)}(x;Q_1,Q_2) = \int_0^\varrho S_\sigma(x;Q_1,Q_2)\,d\sigma = \sum_{Q_1<k\le Q_2} \sum_{\substack{0<a\le\varrho k\\(a,k)=1}} E^2(x;a,k) \int_{a/k}^{\varrho} d\sigma$$
$$= \sum_{Q_1<k\le Q_2} \sum_{\substack{0<a\le\varrho k\\(a,k)=1}} \left(\varrho - \frac{a}{k}\right) E^2(x;a,k)$$

so that

$$\begin{aligned} S_\varrho^{(1)}(x;Q_1,Q_2) &= \varrho\{W_\varrho(x;Q_1,Q_2) + X_\varrho(x;Q_1,Q_2)\} \\ &= \varrho Z_\varrho(x;Q_1,Q_2), \quad \text{say.} \end{aligned} \tag{99}$$

To infer the required features of $S_\varrho^{(1)}(x;Q_1,Q_2)$ from this equation we suppose in the first instance that

$$\log^{-A-1} x \le \varrho \le 1 - \log^{-A-1} x$$

and avail ourselves of the inequality

$$\begin{aligned} \frac{1}{H}\{S_\varrho^{(1)}(x;Q_1,Q_2) - S_{\varrho-H}^{(1)}(x;Q_1,Q_2)\} & \\ \le S_\varrho(x;Q_1,Q_2) & \\ \le \frac{1}{H}\{S_{\varrho+H}^{(1)}(x;Q_1,Q_2) - S_\varrho^{(1)}(x;Q_1,Q_2)\} & \end{aligned} \tag{100}$$

for

$$H = \tfrac{1}{2}\log^{-A-1} x. \tag{101}$$

Next, concentrating for brevity on the right-hand side of (100) with the help of (99), we proceed from the equation

$$\begin{aligned} & S_{\varrho+H}^{(1)}(x;Q_1,Q_2) - S_\varrho^{(1)}(x;Q_1,Q_2) \\ &\quad = (\varrho+H)Z_{\varrho+H}(x;Q_1,Q_2) - \varrho Z_\varrho(x;Q_1,Q_2) \\ &\quad = (\varrho+H)\{Z_{\varrho+H}(x;Q_1,Q_2) - Z_\varrho(x;Q_1,Q_2)\} + HZ_\varrho(x;Q_1,Q_2) \end{aligned} \tag{102}$$

to the estimation of $Z_\varrho(x;Q_1,Q_2)$ by a special case of Lemma 4. Accordingly, since condition (5) on Q_1, Q_2 is still being imposed, we infer from (97) and

(98) that both

$$\begin{aligned}
W_\varrho(x;Q_1,Q_2) = \int_{Q_1}^{Q_2} & x\varrho\biggl\{\biggl(\frac{\log t}{t}-\frac{\log Q_2}{Q_2}\biggr)t-(C_2-1)\biggl(\frac{1}{t}-\frac{1}{Q_2}\biggr)t\biggr\}\,dt\\
&+O\biggl(x^{3/4}\varrho\int_{Q_1}^{Q_2} t^{1/4}\,dt\biggr)+O\biggl(\varrho\log x\int_{Q_1}^{Q_2} t\log\frac{Q_2}{t}\,dt\biggr)\\
&+O\biggl(\frac{x^2}{\log^{2A+1}x}\biggr)\\
= x\varrho&\biggl[t\log t-t-\frac{t^2\log Q_2}{2Q_2}-(C_2-1)t+\frac{(C_2-1)t^2}{2Q_2}\biggr]_{Q_1}^{Q_2}\\
&+O(Q_2^{5/4}\varrho x^{3/4})+O(Q_2^2\varrho\log x)+O\biggl(\frac{x^2}{\log^{2A+1}x}\biggr)\\
= x\varrho&\biggl(\frac{1}{2}Q_2\log Q_2-Q_1\log Q_1-\frac{Q_1^2\log Q_2}{2Q_2}\\
&-\frac{1}{2}(C_2+1)Q_2+C_2Q_1-\frac{(C_2-1)Q_1^2}{2Q_2}\biggr)\\
&+O(Q_2^{5/4}\varrho x^{3/4})+O(Q_2^2\varrho\log x)+O\biggl(\frac{x^2}{\log^{2A+1}x}\biggr)
\end{aligned}$$

and

$$\begin{aligned}
X_\varrho(x;Q_1,Q_2) = x\varrho\int_{Q_1}^{Q_2} & t\biggl\{\frac{\log Q_1}{Q_1}-\frac{\log Q_2}{Q_2}-(C_2-1)\biggl(\frac{1}{Q_1}-\frac{1}{Q_2}\biggr)\biggr\}\,dt\\
&+O(Q_1^{5/4}\varrho x^{3/4})+O\biggl(Q_1^2\varrho\log x\log\frac{Q_2}{Q_1}\biggr)\\
&+O\biggl(\frac{x^2}{\log^{2A+1}x}\biggr)\\
= \frac{1}{2}x\varrho&\biggl(Q_1\log Q_1-\frac{Q_1^2\log Q_2}{Q_2}-(C_2-1)Q_1+\frac{(C_1-1)Q_1^2}{Q_2}\biggr)\\
&+O(Q_2^{5/4}\varrho x^{3/4})+O(Q_2^2\varrho\log x)+O\biggl(\frac{x^2}{\log^{2A+1}x}\biggr),
\end{aligned}$$

which in combination yield

$$\begin{aligned}
(103)\quad Z_\varrho(x;Q_1,Q_2) = {}&\frac{1}{2}x\varrho\{Q_2\log Q_2-Q_1\log Q_1-(C_2+1)(Q_2-Q_1)\}\\
&+O(Q_2^{5/4}\varrho x^{3/4})+O(Q_2^2\varrho\log x)+O\biggl(\frac{x^2}{\log^{2A+1}x}\biggr)
\end{aligned}$$

by (99). Similarly, since the definitions (97), (98) and (95) imply that

$$W_{\varrho+H}(x;Q_1,Q_2) - W_\varrho(x;Q_1,Q_2) = \int_{Q_1}^{Q_2} S^\dagger(x;\varrho t,(\varrho+H)t;t,Q_2)\,dt$$

and

$$X_{\varrho+H}(x;Q_1,Q_2) - X_\varrho(x;Q_1,Q_2) = \int_{0}^{Q_2} S^\dagger(x;\varrho t,(\varrho+H)t;Q_1,Q_2)\,dt$$

and since the formulae given by Lemma 4 for the integrands are the same as for the case $\varrho = 0$, the method of deriving (103) immediately produces

$$\begin{aligned} &Z_{\varrho+H}(x;Q_1,Q_2) - Z_\varrho(x;Q_1,Q_2) \\ &\quad = \frac{1}{2}xH\{Q_2\log Q_2 - Q_1\log Q_1 - (C_2+1)(Q_2-Q_1)\} \\ &\qquad + O(Q_2^{5/4}Hx^{3/4}) + O(Q_2^2 H\log x) + O\bigg(\frac{x^2}{\log^{2A+1}x}\bigg). \end{aligned} \tag{104}$$

Equipped with (103) and (104), we return to (100) and (102) and first deduce that

$$\begin{aligned} S_\varrho(x;Q_1,Q_2) \le{}& x\varrho\{Q_2\log Q_2 - Q_1\log Q_1 - (C_2+1)(Q_2-Q_1)\} \\ &+ (Q_2^{5/4}\varrho x^{3/4}) + O(Q_2^2\varrho\log x) \\ &+ O\bigg(\frac{x^2}{H\log^{2A+1}x}\bigg) + O(Hx^2\log x), \end{aligned}$$

whence

$$\begin{aligned} S_\varrho(x;Q_1,Q_2) \le{}& x\varrho\{Q_2\log Q_2 - Q_1\log Q_1 - (C_2+1)(Q_2-Q_1)\} \\ &+ O(Q_2^{5/4}\varrho x^{3/4}) + O(Q_2^2\varrho\log x) + O\bigg(\frac{x^2}{\log^A x}\bigg), \end{aligned}$$

by (101). The left side of (100) gives rise in like manner to a comparable inequality in the other direction and we therefore obtain

$$\begin{aligned} S_\varrho(x;Q_1,Q_2) ={}& x\varrho\{Q_2\log Q_2 - Q_1\log Q_1 - (C_2+1)(Q_2-Q_1)\} \\ &+ O(Q_2^{5/4}\varrho x^{3/4}) + O(Q_2^2\varrho\log x) + O\bigg(\frac{x^2}{\log^A x}\bigg) \end{aligned} \tag{105}$$

under the stated conditions.

Two easy steps are needed to complete the treatment of the sum $S_\varrho(x,Q)$ introduced in (2), where restriction (9) is no longer apposite. First, if $Q \le x\log^{-A-1}x$, then by Gallagher's form of the Barban–Davenport–Halberstam theorem (vid. the Introduction in I) we have trivially

$$S_\varrho(x,Q) \le S_1(x,Q) = O(x^2\log^{-A}x)$$

and thus

$$S_\varrho(x,Q) = x\varrho\{Q\log Q - (C_2+1)Q\} + O\left(\frac{x^2}{\log^A x}\right),$$

whereas, in the opposite case, we write $Q_1 = x\log^{-A-1} x, Q_2 = Q$ in (5) to get

$$\begin{aligned} S_\varrho(x,Q) &= S_\varrho(x,Q_1) + S_\varrho(x;Q_1,Q) = S_\varrho(x;Q_1,Q) + O\left(\frac{x^2}{\log^A x}\right) \\ &= x\varrho\{Q\log Q - (C_2+1)Q\} + O(Q^{5/4}\varrho x^{3/4}) \\ &\quad + O(Q^2\varrho\log x) + O\left(\frac{x^2}{\log^A x}\right) \end{aligned}$$

by repeating the previous reasoning and then by (105), the restriction $Q \le x$ being then all that is needed. Lastly, being valid for $\varrho = \varrho_0 = 1 - \log^{-A-1} x$ and also for $\varrho = 1$ by the Barban–Montgomery theorem, this formula is seen to be true for $\varrho_0 < \varrho \le 1$ because $S_{\varrho_0}(x,Q) \le S_\varrho(x,Q) \le S_1(x,Q)$ and $1-\varrho \le 1-\varrho_0 = \log^{-A-1} x$; similarly, but more easily, we extend the range downwards to $\varrho = 0$. Thus we have reached our objective in establishing

Theorem 4. *For $\varrho = \varrho(x)$ satisfying $0 \le \varrho \le 1$, let*

$$S_\varrho(x,Q) = \sum_{k\le Q} \sum_{\substack{0<a\le\varrho k \\ (a,k)=1}} E^2(x;a,k),$$

where $E(x;a,k)$ is defined in the Introduction. Then, for $Q \le x$,

$$\begin{aligned} S_\varrho(x,Q) &= x\varrho\{Q\log Q - (C_2+1)Q\} + O(Q^{5/4}\varrho x^{3/4}) \\ &\quad + O(Q^2\varrho\log x) + O\left(\frac{x^2}{\log^A x}\right), \end{aligned}$$

in which A is any positive absolute constant and C_2 is defined by (92) *above.*

This result does not exhaust the theorems in the genre to which it belongs. But enough has been said to indicate how the others might be established.

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