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The Lifted Root Number Conjecture for some cyclic extensions of $\mathbb{Q}$

by

Jürgen Ritter (Augsburg) and Alfred Weiss (Edmonton, Alta.)

1. Introduction. This paper gives first evidence for the Lifted Root Number Conjecture [GRW1] which refines Chinburg's Root Number Conjecture [Ct]. The general setting has also been described in [GRW2]. Here, we observe that the root number conjecture in its lifted form makes predictions about the relations between the global units and the ideal class group which go beyond what Euler systems or the Main Conjecture of Iwasawa theory are known to imply (1).

This is discussed in the simplest case, namely when $K/\mathbb{Q}$ is a cyclic extension of odd prime degree l and squarefree conductor $n = p_1 \ldots p_r$ with all primes $p_j \neq l$. Note that $K \subset \mathbb{Q}(\zeta_n)$ where, for a natural number m, ζ_m always denotes a primitive mth root of unity.

Let $G = \langle g_0 \rangle$ be the Galois group of $K/\mathbb{Q}$ and cl_K the group of ideal classes in K. Then there is a $\mathbb{Z}_l G$-module isomorphism

$$(1.1)\qquad \mathbb{Z}_l \otimes_{\mathbb{Z}} cl_K \simeq \bigoplus_{i=1}^{r-1} \mathbb{Z}_l G/\langle 1 + g_0 + \ldots + g_0^{l-1}, (g_0 - 1)^{h_i} \rangle$$

with unique natural numbers h_i (see the proof of Lemma 2.1). We fix classes $\mathfrak{C}_i$ in cl_K, $1 \leq i \leq r-1$, of order a power of l, so that the image of $\mathfrak{C}_i$ in $\mathbb{Z}_l \otimes cl_K$ generates the ith component under this isomorphism. Let $\mathfrak{p}_j$ be the prime of K above p_j and write

$$(1.2)\qquad [\mathfrak{p}_j] = \prod_{i=1}^{r-1} \mathfrak{C}_i^{b_{ij}(g_0-1)^{h_i-1}}, \qquad 1 \leq j \leq r,$$

in $\mathbb{Z}_l \otimes cl_K$. Define B_k to be $(-1)^{k+1}$ times the determinant of the matrix (b_{ij}) with the kth column deleted $(1 \leq k \leq r)$ (2) . These elements B_k serve

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(1) At the same time, it fits into a more general setting (see [Bu]).
(2) If $r = 1$, we must set $B_1 = 1$.

as a bridge between cl_K and the cyclotomic unit group $\mathcal{E}_K$ in the unit group E_K of K. Indeed, set

$$\xi_K = \prod_{1\neq d|n} N_{\mathbb{Q}(\zeta_n)/K}(1-\zeta_n^{n/d}).$$

Then

$$\xi_K^{g_0-1} = \alpha_\infty^{(g_0-1)^{h+1}} \tag{1.3}$$

with a unique $\alpha_\infty \in K$ (up to rational factors) such that the norm $N_{K/\mathbb{Q}}(\alpha_\infty)$ of α_∞ is not an lth power. It is readily seen that B_j and the $\mathfrak{p}_j$-value $v_{\mathfrak{p}_j}(\alpha_\infty)$ of α_∞ are proportional modulo l, independent of j. The Lifted Root Number Conjecture now predicts this ratio to equal a certain number c defined in Lemma 2.3 (and is, in this case, equivalent to that equality). In Section 2 we also review the classical material that has been referred to here. To get a clear picture of what is going on, note that the above h is such that $|\mathbb{Z}_l \otimes cl_K| = l^h$ and that $\xi_K^{g_0-1}$ generates $\mathbb{Z}_l \otimes \mathcal{E}_K$ in $\mathbb{Z}_l \otimes E_K = \langle \alpha_\infty^{g_0-1}\rangle$.

The actual connection to the Lifted Root Number Conjecture is explained in Section 3. However, it would go beyond the scope of this paper to go into detail here, so the reader is referred to [GRW1] ([3]). In this section we characterize certain maps $\Delta S' \xrightarrow{\varphi} E_{S'}$ which when injective have cohomologically trivial cokernel. Here, $K/\mathbb{Q}$ may be replaced by an arbitrary cyclic extension K/k of number fields, and S' is a finite, sufficiently large $\mathrm{Gal}(K/k)$-set of primes of K, $\Delta S'$ the augmentation submodule in the free G-module $\mathbb{Z}S'$ on the $\mathbb{Z}$-basis $\mathfrak{p} \in S'$, and finally $E_{S'}$ the group of S'-units in K. We close Section 3 by restating the Lifted Root Number Conjecture in terms of the cokernel of an injective φ.

The next section recalls the notion of a Ramachandra map $\varphi_\infty : \Delta S_\infty \to E_K$, where S_∞ is the set of all infinite primes of K. The Ramachandra φ_∞ has been used in [RW] to prove the so-called Strong Stark Conjecture for absolutely abelian K (in which 2 is unramified). Here we now extend it to a φ as in Section 3 and show that, for our purposes, it suffices to work with a G-set S of primes which is large in the restricted sense that the order of the S-class group of K is prime to $|G|$.

Section 5 then gives the construction of an isomorphism $\varphi : \mathbb{Z}_l \otimes \Delta S \to \mathbb{Z}_l \otimes E_S$ in our example. This φ extends φ_∞, with S the set generated by $\{\infty, \mathfrak{p}_1, \ldots, \mathfrak{p}_r, \mathfrak{q}_1, \ldots, \mathfrak{q}_{r-1}, \mathfrak{q}_0\}$, where ∞ is a fixed infinite prime, $\mathfrak{q}_i \in \mathfrak{C}_i$ $(1 \le i \le r-1)$ and $\mathfrak{q}_0$ a suitably chosen prime which is inert over $\mathbb{Q}$. The Lifted Root Number Conjecture amounts to certain l-adic congruences between the Tate–Stark numbers $A_\varphi(\chi)$, where χ runs through the characters of G.

([3]) In fact, only the second part of the proof of Proposition 3.2 requires more than is in [GRW2].

We need to insert a short section, §6, on Euler systems before we can complete the calculation in a restricted situation.

THEOREM. *The Lifted Root Number Conjecture holds true for $K/\mathbb{Q}$ when $r \le 2$.*

Section 6 recalls some basic facts regarding the Euler system

$$Q \mapsto \xi_Q = N_{\mathbb{Q}(\zeta_n,\zeta_Q)/K(\zeta_Q)}(1-\zeta_n\zeta_Q)$$

with Q running through the squarefree products of rational primes q splitting in K. We employ it, in Section 7, to get a prime $\mathfrak{q}_1 \in \mathfrak{C}_1$ for which the corresponding Kolyvagin number κ_{q_1} provides congruences modulo $\mathfrak{p}_j$ ($j = 1,2$) that lead to the proof of the theorem.

In the case $r = 2$ we can arrange that κ_{q_1} has norm 1 and so a (g_0-1)th root of it is an $\alpha_{\mathfrak{q}_1}$ in the sense of Section 5. By means of local symbols we relate the $\mathfrak{p}_j$-value of $\alpha_{\mathfrak{q}_1}$ and the congruence class mod $\mathfrak{p}_j$ of $\kappa_{\mathfrak{q}_1}$. For $r \ge 3$ it seems necessary to take repeated (g_0-1)th roots and so such congruences on κ-values would not be decisive.

2. Conjecture (C). We maintain the notation of the introduction and set

- $\widehat{G} = \widehat{g}_0 = \sum_{\nu=0}^{l-1} g_0^\nu$,
- P_K = group of principal ideals of K,
- I_K = group of all ideals of K,

and correspondingly with K replaced by $\mathbb{Q}$.

LEMMA 2.1 (4). (a) $\mathbb{Z}_l \otimes E_K \simeq \mathbb{Z}_l G/\widehat{G}$.

(b) *The* $\mathfrak{p}_j$, $1 \le j \le r$, *constitute an* $\mathbb{F}_l$*-basis of* $I_K^G/I_{\mathbb{Q}}$.

(c) $P_K^G/P_{\mathbb{Q}}$ *has order* l.

(d) (1.1) *holds.*

For the proof observe that $\mathbb{Z}_l G/\widehat{G} \simeq \mathbb{Z}_l[\zeta_l]$ is a discrete valuation ring with prime element the image of $g_0 - 1$. As $l \ne 2$, K is totally real and $\widehat{G}$ annihilates $\mathbb{Z}_l \otimes E_K$. Thus (a) is a consequence of Dirichlet's unit theorem. Therefore $H^1(G, E_K) = \mathbb{F}_l$ and $H^2(G, E_K) = 0$. Thus, from $E_K \rightarrowtail K^\times \twoheadrightarrow P_K$, we see that $H^1(G, P_K) = 0$ and $P_K^G/P_{\mathbb{Q}}$ has order l, proving (c); (b) is obvious. For (d) use $P_K \rightarrowtail I_K \twoheadrightarrow cl_K$ in order to arrive at $P_K^G/P_{\mathbb{Q}} \rightarrowtail I_K^G/I_{\mathbb{Q}} \twoheadrightarrow cl_K^G$, whence $cl_K^G \simeq \mathbb{F}_l^{r-1}$ by (b) and (c). Since $\widehat{G}$ annihilates cl_K there exist unique numbers s and $h_1 \ge \ldots \ge h_s \ge 1$ such that $\mathbb{Z}_l \otimes cl_K \simeq \bigoplus_{i=1}^s \mathbb{Z}_l G/\langle \widehat{G}, (g_0-1)^{h_i}\rangle$. Taking fixed points shows $s = r - 1$.

(4) The lemma collects well-known facts (see e.g. [Cc] or [La, XIII,4]), which also follow from the theory of genus fields [Fr].

LEMMA 2.2 (5). *There exists* $h \ge 0$ *so that* $\xi_K^{g_0-1} = \alpha_\infty^{(g_0-1)^{h+1}}$ *with* $\alpha_\infty \in K^\times$ *satisfying*:

(i) $\alpha_\infty^{\widehat{G}} \notin \mathbb{Q}^{\times l}$.
(ii) $\mathrm{supp}(\alpha_\infty) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ *and* (α_∞) *generates* $P_K^G/P_{\mathbb{Q}}$.
(iii) α_∞ *is unique up to rational factors.*

Above, $\mathrm{supp}(\alpha_\infty)$ is the set of prime divisors of the principal ideal (α_∞) generated by α_∞.

The proof of the lemma is based on the fact that $\xi_K^{g_0-1} \neq 1$, which is due to Ramachandra [Wa, Theorem 8.3]. Since $\xi_K^{g_0-1} \in E_K$, there is, by Lemma 2.1(a), a maximal $h \ge 0$ with $\xi_K^{g_0-1} = v^{(g_0-1)^h}$ and $v \in E_K$. As $\xi_K^{g_0-1}$ has norm 1 and l is odd, we may assume that v has norm 1. Hence there exists $\alpha_\infty \in K^\times$ with $\alpha_\infty^{g_0-1} = v$. In particular $(\alpha_\infty) \in P_K^G$. Suppose that (α_∞) is in the image of $P_{\mathbb{Q}}$ in P_K^G, i.e., $\alpha_\infty = a \cdot v_1$ with $a \in \mathbb{Q}^\times$ and a unit v_1. Then $v = v_1^{g_0-1}$ contradicts the maximality of h. Consequently, (α_∞) generates $P_K^G/P_{\mathbb{Q}}$ and is a product of the $\mathfrak{p}_j$ times a rational number. Modifying α_∞ by the inverse of this rational number proves (i) and (ii).

If $h = 0$, then (iii) is obvious. If $h > 0$, then $\alpha_1^{(g_0-1)^{h+1}} = \alpha_\infty^{(g_0-1)^{h+1}}$ leads to $\alpha_1^{(g_0-1)^h} = \alpha_\infty^{(g_0-1)^h} \cdot a$ for some $a \in \mathbb{Q}^\times$, and taking norms yields $a^l = 1$, hence $a = 1$. This argument can be repeated.

As in the introduction (see (1.2)), we write $[\mathfrak{p}_j] = \prod_{i=1}^{r-1} \mathfrak{C}_i^{b_{ij}(g_0-1)^{h_i-1}}$ in $\mathbb{Z}_l \otimes cl_K$ with integers b_{ij}. The proof of Lemma 2.1 shows that the matrix $(b_{ij})_{\substack{1\le i\le r-1\\ 1\le j\le r}}$ has rank $r-1$ over $\mathbb{F}_l$, whence the row vector $B_k = (-1)^{k+1}\det(b_{ij})$ is non-zero modulo l (6). It satisfies $(b_{ij}) \cdot (B_1, \dots, B_r)^T = (0, \dots, 0)$. Since, by (1.2), (b_{ij}) is the matrix of the $\mathbb{F}_l$-linear map $I_K^G/I_{\mathbb{Q}} \twoheadrightarrow cl_K^G$, the ideal $\mathfrak{p}_1^{B_1} \dots \mathfrak{p}_r^{B_r}$ is principal. By Lemma 2.2(ii) we see that there is a $\widetilde{c} \in \mathbb{Z}$, $\widetilde{c} \not\equiv 0 \bmod l$ such that

$$-v_{\mathfrak{p}_j}(\alpha_\infty) \equiv \widetilde{c} \cdot B_j \bmod l$$

for $1 \le j \le r$, where $v_{\mathfrak{p}}(x)$, for a prime ideal $\mathfrak{p}$ of K and an $x \in K$, denotes the $\mathfrak{p}$-value of x.

LEMMA 2.3. *Let* $\mathfrak{q}_i \in \mathfrak{C}_i$, $1 \le i \le r-1$, *be primes of prime absolute norm* q_i *which are different from* $p_1, \dots, p_r$. *Furthermore, let* q_0 *be a rational prime with Artin symbol* $(q_0, K/\mathbb{Q}) = g_0$. *Form the matrix* (c_{ij}) *by means of the local norm residue symbols*

(5) Compare (1.3) in Section 1.
(6) Recall that $B_1 = 1$ if $r = 1$.

$$(q_i, K_{\mathfrak{p}_j}/\mathbb{Q}_{p_j}) = g_0^{c_{ij}}, \quad 0 \le i \le r-1,\ 1 \le j \le r\ (^7).$$

Define $c = \det c_{ij}$. *Then* $c \not\equiv 0 \bmod l$.

Note that due to Chebotarev's density theorem such prime ideals $\mathfrak{q}_i$ exist. Note also that the c_{ij}, and equally well the b_{ij} and the B_j, depend on the choice of the $\mathfrak{C}_i$.

Before turning to the proof of Lemma 2.3 we specify the conjecture that has been indicated in the introduction:

(C) $$\widetilde{c} \equiv c \bmod l.$$

Proof (of Lemma 2.3). Define $\widetilde{K}$ to be the Hilbert l-class field of K and let $\widehat{K}/\mathbb{Q}$ be the maximal abelian subextension of $\widetilde{K}/\mathbb{Q}$ ([8]). The Artin symbol $(\ ,\widetilde{K}/K) : \mathbb{Z}_l \otimes cl_K \to \mathrm{Gal}(\widetilde{K}/K)$ is an isomorphism and so provides the exact sequence $\mathbb{Z}_l \otimes cl_K \rightarrowtail \mathrm{Gal}(\widetilde{K}/\mathbb{Q}) \twoheadrightarrow G$. Since G is cyclic, it follows that also $\mathbb{Z}_l \otimes cl_K/cl_K^{g_0-1} \rightarrowtail \mathrm{Gal}(\widehat{K}/\mathbb{Q}) \twoheadrightarrow G$ is exact, whence $(\ ,\widetilde{K}/K)$: $\mathbb{Z}_l \otimes cl_K/cl_K^{g_0-1} \xrightarrow{\sim} \mathrm{Gal}(\widehat{K}/K)$.

Set $\sigma_i = (q_i, \widehat{K}/\mathbb{Q}), 0 \le i \le r-1$. The choice of the $\mathfrak{q}_i$ guarantees that $\sigma_i = (\mathfrak{q}_i, \widehat{K}/K), 1 \le i \le r-1$, is an $\mathbb{F}_l$-basis of $\mathrm{Gal}(\widehat{K}/K)$. Observe here that l annihilates $\mathbb{Z}_l \otimes cl_K/cl_K^{g_0-1}$ because the ideals $\langle g_0-1, \widehat{G}\rangle$ and $\langle g_0 - 1, l\rangle$ coincide. In particular, $\mathrm{Gal}(\widehat{K}/\mathbb{Q})$ has order l^r. Since $\widehat{K}$ contains the composite of the subextensions of degree l of all $\mathbb{Q}(\zeta_{p_j}),\ 1 \le j \le r$, it therefore coincides with it and $\mathrm{Gal}(\widehat{K}/\mathbb{Q})$ is l-elementary. As a consequence, $\sigma_0, \sigma_1, \ldots, \sigma_{r-1}$ is an $\mathbb{F}_l$-basis of $\mathrm{Gal}(\widehat{K}/\mathbb{Q})$ because σ_0 restricts to g_0 on K, and the map

$$\prod_{j=1}^r U_{p_j}/U_{p_j}^l \to \mathrm{Gal}(\widehat{K}/\mathbb{Q}), \quad (u_j) \mapsto \prod_{j=1}^r (u_j, \widehat{K}_{\widehat{\mathfrak{p}}_j}/\mathbb{Q}_{p_j})$$

is an isomorphism. Here, U_{p_j} is the unit group in $\mathbb{Q}_{p_j}$ and $\widehat{\mathfrak{p}}_j$ a prime of $\widehat{K}$ above $\mathfrak{p}_j$. Note that the 1-units in $\mathbb{Q}_{p_j}$ are all lth powers, so $[U_{p_j} : U_{p_j}^l] = l$.

The isomorphism takes q_i, viewed in $\prod_{j=1}^r U_{p_j}/U_{p_j}^l$ on the diagonal, to

$$\prod_{j=1}^r (q_i, \widehat{K}_{\widehat{\mathfrak{p}}_j}/\mathbb{Q}_{p_j}) = (q_i, \widehat{K}_{\widehat{\mathfrak{q}}_i}/\mathbb{Q}_{q_i})^{-1} = \sigma_i^{-1}, \quad 0 \le i \le r-1,$$

by reciprocity. This shows that $q_0, \ldots, q_{r-1}$ is an $\mathbb{F}_l$-basis of $\prod_{j=1}^r U_{p_j}/U_{p_j}^l$.

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(7) For a number field L and a prime $\mathfrak{p}$ of L, $L_{\mathfrak{p}}$ denotes the completion of L at $\mathfrak{p}$.

(8) $\widehat{K}$ is the genus field of $K/\mathbb{Q}$.

Since $\prod_{j=1}^{r}(\ ,K_{\mathfrak{p}_j}/\mathbb{Q}_{p_j}):\prod_{j=1}^{r}U_{p_j}/U_{p_j}^{l}\to G^r$ is an isomorphism, the standard basis of G^r is the image of certain $\prod_{s=0}^{r-1}q_s^{x_{is}}$, i.e.,

$$\Big(\prod_{s=0}^{r-1}q_s^{x_{is}},K_{\mathfrak{p}_j}/\mathbb{Q}_{p_j}\Big)=\begin{cases}g_0 & \text{if } i=j,\\ 1 & \text{if } i\neq j.\end{cases}$$

This implies

$$\prod_{s=0}^{r-1}(q_s,K_{\mathfrak{p}_j}/\mathbb{Q}_{p_j})^{x_{is}}=g_0^{\sum_{s=0}^{r-1}x_{is}c_{sj}}=g_0^{\delta_{ij}}$$

and finishes the proof ([9]).

3. The Lifted Root Number Conjecture for K. In this section K/k is a cyclic extension of number fields with group $G=\langle g_0\rangle$ where g_0 is the Frobenius automorphism of some fixed prime $\mathfrak{q}_0$ of K which is inert over k. We let S' denote a finite G-set of primes of K containing $\mathfrak{q}_0$, all infinite primes, all ramified primes for the extension K/k, and enough primes to generate the class group cl_K. Our aim is to characterize certain maps $\Delta S'\xrightarrow{\varphi}E_{S'}$, which whenever injective have cohomologically trivial cokernel, and to restate the Lifted Root Number Conjecture in terms of them.

Lemma 3.1. *Let $\mathfrak{p}$ be a prime of K, $g_{\mathfrak{p}}$ a generator of its decomposition group $G_{\mathfrak{p}}$ (with respect to k) and $a_{\mathfrak{p}}\in k_{\mathfrak{p}}^{\times}$ so that $(a_{\mathfrak{p}},K_{\mathfrak{p}}/k_{\mathfrak{p}})=g_{\mathfrak{p}}$ (with $k_{\mathfrak{p}}$ denoting the completion of k in $K_{\mathfrak{p}}$). Then the extension class of the bottom row sequence in the push-out diagram*

$$\begin{array}{ccccc}\mathbb{Z} & \overset{\widehat{G}_{\mathfrak{p}}}{\rightarrowtail} & \mathbb{Z}G_{\mathfrak{p}} & \overset{g_{\mathfrak{p}}-1}{\twoheadrightarrow} & \Delta G_{\mathfrak{p}}\\ \downarrow & & \downarrow & & \|\\ K_{\mathfrak{p}}^{\times} & \rightarrowtail & V_{\mathfrak{p}} & \twoheadrightarrow & \Delta G_{\mathfrak{p}}\end{array},\qquad \textit{with respect to the map } 1\mapsto a_{\mathfrak{p}},$$

corresponds to the local fundamental class of $K_{\mathfrak{p}}/k_{\mathfrak{p}}$ under the canonical isomorphisms

$$\mathrm{Ext}^1_{G_{\mathfrak{p}}}(\Delta G_{\mathfrak{p}},K_{\mathfrak{p}}^{\times})\simeq H^1(G_{\mathfrak{p}},\mathrm{Hom}(\Delta G_{\mathfrak{p}},K_{\mathfrak{p}}^{\times}))\simeq H^2(G_{\mathfrak{p}},K_{\mathfrak{p}}^{\times}).$$

For a proof see [Sn, pp. 52–53].

The exact sequence $\mathbb{Z}\overset{\widehat{G}}{\rightarrowtail}\mathbb{Z}G\overset{g_0-1}{\twoheadrightarrow}\Delta G$ tensored with $\Delta S'$ yields the new exact sequence

$$(\Delta)\qquad \Delta S'\rightarrowtail\Delta S'\otimes\mathbb{Z}G\twoheadrightarrow\Delta S'\otimes\Delta G.$$

Let S'_* be a set of G-representatives for S' and set $g_{\mathfrak{p}}=g_0^{[G:G_{\mathfrak{p}}]}$ for $\mathfrak{p}\in S'_*$, so $\langle g_{\mathfrak{p}}\rangle=G_{\mathfrak{p}}$.

([9]) δ_{ij} is the Kronecker symbol.

PROPOSITION 3.2. *Assume that for each* $\mathfrak{p} \in S'_*$, $\mathfrak{p} \neq \mathfrak{q}_0$, *we are given an element* $\alpha_\mathfrak{p} \in K^{G_\mathfrak{p}} \cap E_{S'}$ *satisfying*

$$(\alpha_\mathfrak{p}, K_{\mathfrak{p}'}/(K^{G_\mathfrak{p}})_{\mathfrak{p}'}) = \begin{cases} g_\mathfrak{p} & \text{for } \mathfrak{p}' = \mathfrak{p}, \\ 1 & \text{for } \mathfrak{p}' \neq \mathfrak{p}, \mathfrak{q}_0, \end{cases}$$

where $\mathfrak{p}'$ *runs through the primes of* K. *Then the* G-*map* $\varphi : \Delta S' \to E_{S'}$ *defined by* $\mathfrak{p} - \mathfrak{q}_0 \mapsto \alpha_\mathfrak{p}$ *for* $\mathfrak{p} \in S'_*$, $\mathfrak{p} \neq \mathfrak{q}_0$, *takes the extension class in* $\mathrm{Ext}^1_G(\Delta S' \otimes \Delta G, \Delta S')$ *of* (Δ) *to the Tate class* $\tau_{S'} \in \mathrm{Ext}^1_G(\Delta S' \otimes \Delta G, E_{S'})$.

REMARK. More precisely, tensoring the augmentation sequence $\Delta G \rightarrowtail \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$ with $\Delta S'$ induces an isomorphism

$$\mathrm{Ext}^1_G(\Delta S' \otimes \Delta G, E_{S'}) \to \mathrm{Ext}^2_G(\Delta S', E_{S'})$$

sending $\tau_{S'}$ to what is usually regarded as the Tate class [GRW1].

Proof (of Proposition 3.2). We begin by picking for each $\mathfrak{p} \in S'_*$ an element $a_\mathfrak{p}$ in $(K^{G_\mathfrak{p}})_\mathfrak{p}$ so that $(a_\mathfrak{p}, K_\mathfrak{p}/(K^{G_\mathfrak{p}})_\mathfrak{p}) = g_\mathfrak{p}$. To $a_\mathfrak{p}$ we then assign the idèle $a_{(\mathfrak{p})}$ in the S'-idèle group $J_{K^{G_\mathfrak{p}},S'}$ of $K^{G_\mathfrak{p}}$, which has component 1 everywhere except at the prime $\mathfrak{p} \cap K^{G_\mathfrak{p}}$ where the component shall be $a_\mathfrak{p}$. The element $\alpha_\mathfrak{p}$ viewed as principal idèle will be denoted by $\alpha_{(\mathfrak{p})}$.

We claim:

$$a_{(\mathfrak{p})} \equiv a_{(\mathfrak{q}_0)}\alpha_{(\mathfrak{p})} \bmod N_{K/K^{G_\mathfrak{p}}} J_K.$$

This is checked for each prime $\mathfrak{p}' \cap K^{G_\mathfrak{p}}$ at a time. Note that $\mathfrak{p}$ and $\mathfrak{q}_0$ are non-split in $K/K^{G_\mathfrak{p}}$.

At $\mathfrak{p}' \neq \mathfrak{p}, \mathfrak{q}_0$ the two idèles $a_{(\mathfrak{p})}$ and $a_{(\mathfrak{q}_0)}$ are 1, and $\alpha_{(\mathfrak{p})}$ is a local norm. At $\mathfrak{p}' = \mathfrak{p}$ the two idèles $a_{(\mathfrak{p})}$ and $\alpha_{(\mathfrak{p})}$ differ by a local norm and $a_{(\mathfrak{q}_0)}$ is 1. At $\mathfrak{p}' = \mathfrak{q}_0$ the reciprocity law implies $(\alpha_\mathfrak{p}, K_{\mathfrak{q}_0}/(K^{G_\mathfrak{p}})_{\mathfrak{q}_0}) = g_\mathfrak{p}^{-1}$, $a_{(\mathfrak{p})}$ is 1, and $(a_{\mathfrak{q}_0}, K_{\mathfrak{q}_0}/(K^{G_{\mathfrak{q}_0}})_{\mathfrak{q}_0}) = g_0$ becomes $g_0^{[G:G_\mathfrak{p}]} = g_\mathfrak{p}$ in $\mathrm{Gal}(K_{\mathfrak{q}_0}/(K^{G_\mathfrak{p}})_{\mathfrak{q}_0})$ as follows from the commutativity of

$$\begin{array}{ccc} k_{\mathfrak{q}_0}^\times & \xrightarrow{(\ ,K_{\mathfrak{q}_0}/k_{\mathfrak{q}_0})} & \mathrm{Gal}(K_{\mathfrak{q}_0}/k_{\mathfrak{q}_0}) \\ \downarrow & & \downarrow {\scriptstyle [G:G_\mathfrak{p}]=t} \\ (K^{G_\mathfrak{p}})_{\mathfrak{q}_0}^\times & \xrightarrow{(\ ,K_{\mathfrak{q}_0}/(K^{G_\mathfrak{p}})_{\mathfrak{q}_0})} & \mathrm{Gal}(K_{\mathfrak{q}_0}/(K^{G_\mathfrak{p}})_{\mathfrak{q}_0}) \end{array}$$

with t denoting the transfer map [Se, VII,8].

Since outside of S' the extension K/k is unramified and since local units are norms in local unramified extensions, we will even find $\beta_{(\mathfrak{p})} \in J_{K,S'}$ such that

$$a_{(\mathfrak{p})} N_{K/K^{G_\mathfrak{p}}}(\beta_{(\mathfrak{p})}) = a_{(\mathfrak{q}_0)}\alpha_{(\mathfrak{p})}.$$

Recall that here $\mathfrak{p} \in S'_*, \mathfrak{p} \neq \mathfrak{q}_0$. We temporarily set $\alpha_{(\mathfrak{q}_0)} = \beta_{(\mathfrak{q}_0)} = 1$.

The rest of the proof of the proposition consists of combining these data with the construction of a Tate sequence (see e.g. [We, Chapter 5]).

For each $\mathfrak{p} \in S'_*$ we take the diagram of Lemma 3.1 with the middle vertical map denoted by $\mu_\mathfrak{p}$. Inducing these up to G and building the direct sum over S'_* we get

$$\begin{array}{ccccc} \mathbb{Z}S' & \rightarrowtail & \bigoplus_{S'_*} \mathrm{ind}_{G_\mathfrak{p}}^G \mathbb{Z}G_\mathfrak{p} & \twoheadrightarrow & \bigoplus_{S'_*} \mathrm{ind}_{G_\mathfrak{p}}^G \Delta G_\mathfrak{p} \\ \downarrow & & \downarrow & & \| \\ J_{K,S'} & \rightarrowtail & V & \twoheadrightarrow & \bigoplus_{S'_*} \mathrm{ind}_{G_\mathfrak{p}}^G \Delta G_\mathfrak{p} \end{array}$$

where we have glued on the unit idèles outside S' in J and V. We modify the left vertical map by sending $\mathfrak{p} \in S'_*$ to $a_{(\mathfrak{p})} N_{K/K^{G_\mathfrak{p}}} \beta_{(\mathfrak{p})}$, and the middle one by sending the free G-module generator $\mathrm{ind}(1_\mathfrak{p})$ of $\mathrm{ind}_{G_\mathfrak{p}}^G \mathbb{Z}G_\mathfrak{p}$ to $\mu_\mathfrak{p}(1_\mathfrak{p})\beta_{(\mathfrak{p})}$, where now $\beta_{(\mathfrak{p})} \in J_{K,S'}$ is read in V. Then the new diagram still commutes. It is the top face in

(D)
$$\begin{array}{cccccccccc}
 & & \mathbb{Z}S & \rightarrow & & \bigoplus \mathrm{ind}\, \mathbb{Z}G_\mathfrak{p} & \longrightarrow & & & \bigoplus \mathrm{ind}\, \Delta G_\mathfrak{p} \\
 & \swarrow & | & & \swarrow & | & & & \swarrow\!= & \\
J_{K,S'} & & \rightarrow & V & & \rightarrow & & \bigoplus \mathrm{ind}\, \Delta G_\mathfrak{p} & & \downarrow \\
 & & \downarrow & | & & \downarrow & & | & & \\
\downarrow & & \mathbb{Z} & \rightarrow & & \mathbb{Z}G & & \rightarrow & & \Delta G \\
 & \swarrow & & \downarrow & \swarrow & & & \downarrow & \swarrow\!= & \\
C_K & & \rightarrow & \mathfrak{V} & & \longrightarrow & & \Delta G & &
\end{array}$$

The bottom face of (D) is the diagram of Lemma 3.1 for $\mathfrak{p} = \mathfrak{q}_0$ composed with the push-out diagram along the natural map from $K_{\mathfrak{q}_0}^\times$ into the idèle class group C_K of K:

$$\begin{array}{ccccc} K_{\mathfrak{q}_0}^\times & \rightarrowtail & V_{\mathfrak{q}_0} & \twoheadrightarrow & \Delta G_{\mathfrak{q}_0} \\ \downarrow & & \downarrow & & \| \\ C_K & \rightarrowtail & \mathfrak{V} & \twoheadrightarrow & \Delta G \end{array}$$

Remember that $G_{\mathfrak{q}_0} = G$.

By the compatibility of local and global fundamental classes the bottom row has extension class corresponding to the global fundamental class.

The commutative diagram

$$\begin{array}{ccccc} \mathbb{Z} & \overset{\widehat{G}_\mathfrak{p}}{\rightarrowtail} & \mathbb{Z}G_\mathfrak{p} & \overset{g_\mathfrak{p}-1}{\longrightarrow} & \Delta G_\mathfrak{p} \\ \| & & \downarrow & & \downarrow \\ \mathbb{Z} & \overset{\widehat{G}}{\rightarrowtail} & \mathbb{Z}G & \overset{g_0-1}{\longrightarrow} & \Delta G \end{array}$$

with middle arrow $x \mapsto x(1 + g_0 + \ldots + g_0^{[G:G_\mathfrak{p}]-1})$ induces the back face in (D). The right face of (D) clearly commutes. The left face commutes because the idèle class of $a_{(\mathfrak{p})} N_{K/K^{G_\mathfrak{p}}} \beta_{(\mathfrak{p})}$, for $\mathfrak{p} \in S'_*$, $\mathfrak{p} \neq \mathfrak{q}_0$, is the same as that of $a_{(\mathfrak{q}_0)}$.

On observing that the left half of the top face in (D) is a push-out square for V we obtain a unique map $V \to \mathfrak{V}$ making the whole diagram commute.

As S' is sufficiently large, $J_{K,S'} \to C_K$ is surjective; since $\mathfrak{q}_0 \in S'_*$, $\bigoplus_{S'_*} \operatorname{ind}_{G_\mathfrak{p}}^G \Delta G_\mathfrak{p} \to \Delta G$ is surjective. The kernels of the vertical arrows in (D) fit into

$$\begin{array}{ccccc} \Delta S' & \rightarrowtail & B & \twoheadrightarrow & L \\ \downarrow \varphi & & \downarrow & & \| \\ E_{S'} & \rightarrowtail & A & \twoheadrightarrow & L \end{array}$$

and φ takes $\mathfrak{p} - \mathfrak{q}_0$ to $a_{(\mathfrak{p})} N_{K/K^{G_\mathfrak{p}}} \beta_{(\mathfrak{p})} / a_{(\mathfrak{q}_0)}$ which is the principal idèle $\alpha_{(\mathfrak{p})}$.

We now compare this with the kernels of the vertical maps in

$$\begin{array}{ccccccccc} & \bigoplus_{S'_*} \operatorname{ind}_{G_\mathfrak{p}}^G \mathbb{Z} & \rightarrowtail & & \bigoplus_{S'_*} \operatorname{ind}_{G_\mathfrak{p}}^G \mathbb{Z}G_\mathfrak{p} & \twoheadrightarrow & & \bigoplus_{S'_*} \operatorname{ind}_{G_\mathfrak{p}}^G \Delta G_\mathfrak{p} & \\ \swarrow & | & & \swarrow & | & & \swarrow & & \\ \mathbb{Z}S' & & \rightarrowtail & \mathbb{Z}S' \otimes \mathbb{Z}G & & \twoheadrightarrow & \mathbb{Z}S' \otimes \Delta G & & \downarrow \\ \searrow & \downarrow & & \searrow & \downarrow & & \searrow & & \\ & \mathbb{Z} & \rightarrowtail & & \mathbb{Z}G & \twoheadrightarrow & & & \Delta G \end{array}$$

with outer southwest arrows $g \otimes_{G_\mathfrak{p}} 1 \mapsto g\mathfrak{p}$ and middle one $g \otimes_{G_\mathfrak{p}} x \mapsto g\mathfrak{p} \otimes gxy_\mathfrak{p}$ where $y_\mathfrak{p} = 1 + g_0 + \ldots + g_0^{[G:G_\mathfrak{p}]-1}$.

This is

$$\begin{array}{ccccc} \Delta S' & \rightarrowtail & B & \twoheadrightarrow & L \\ \| & & \downarrow & & \downarrow \\ \Delta S' & \rightarrowtail & \Delta S' \otimes \mathbb{Z}G & \twoheadrightarrow & \Delta S' \otimes \Delta G \end{array}$$

and we regard the two vertical isomorphisms as identifications. Then

$$\text{(T)} \qquad \begin{array}{ccccc} \Delta S' & \rightarrowtail & \Delta S' \otimes \mathbb{Z}G & \twoheadrightarrow & \Delta S' \otimes \Delta G \\ \downarrow \varphi & & \downarrow & & \| \\ E_{S'} & \rightarrowtail & A & \twoheadrightarrow & \Delta S' \otimes \Delta G \end{array}$$

and the bottom row is the $\tau_{S'}$ of the proposition.

Remark. 1. There always exist such φ which are injective. We omit the proof.

2. If $G_\mathfrak{p} = 1$, then the only restriction on $\alpha_\mathfrak{p}$ is to belong to $E_{S'}$.

If the φ in Proposition 3.2 is injective, we can build the Ω_φ as in [GRW1,2] and express the Lifted Root Number Conjecture in terms of a conjectural representing homomorphism for the finite cohomologically trivial module $\operatorname{coker} \varphi$.

This is carried out next. Observe that $\operatorname{coker} \varphi$ then coincides with the cokernel of the middle map in diagram (T), in which $\Delta S' \otimes \mathbb{Z}G$ and A are cohomologically trivial, so it is so itself as well.

The map φ induces $\widetilde{\varphi} : B \xrightarrow{\beta} L \oplus \Delta S' \xrightarrow{1 \oplus \varphi} L \oplus E_{S'} \xrightarrow{\alpha} A$. Now B and L are just abbreviations for $\Delta S' \otimes \mathbb{Z}G$ and $\Delta S' \otimes \Delta G$. The auxiliary maps β

and α can be any maps resulting from commuting diagrams

$$\begin{array}{ccccc} L & \rightarrowtail & B & \twoheadrightarrow & \Delta S' \\ \downarrow{\scriptstyle |G|} & & \downarrow{\scriptstyle \beta} & & \| \\ L & \rightarrowtail & L\oplus \Delta S' & \twoheadrightarrow & \Delta S' \end{array} \qquad \begin{array}{ccccc} E_{S'} & \rightarrowtail & L\oplus E_{S'} & \twoheadrightarrow & L \\ \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle |G|} \\ E_{S'} & \rightarrowtail & A & \twoheadrightarrow & L \end{array}$$

Ω_φ is defined as the element $[\operatorname{coker}\widetilde{\varphi}] - 2\partial(L,|G|)$ in the Grothendieck group $K_0T(\mathbb{Z}G)$ of finite cohomologically trivial $\mathbb{Z}G$-modules (see [GRW1 or GRW2]).

Analogously we obtain a map $\widetilde{1} : B \xrightarrow{\beta_1} L\oplus \Delta S' \xrightarrow{1\oplus 1} L\oplus \Delta S \xrightarrow{\alpha_1} B$ and define

$$\mho_1 = [\operatorname{coker}\widetilde{1}] - 2\partial(L,|G|),$$

i.e., we have replaced $\varphi : \Delta S' \to E_{S'}$ by the identity map $1 : \Delta S' = \Delta S'$ and the Tate sequence $E_{S'} \rightarrowtail A \to B \twoheadrightarrow \Delta S'$ by $\Delta S' \rightarrowtail \Delta S'\otimes \mathbb{Z}G \to \Delta S'\otimes \mathbb{Z}G \twoheadrightarrow \Delta S'$, which, as before, is $\mathbb{Z} \overset{\widehat{G}}{\rightarrowtail} \mathbb{Z}G \xrightarrow{g_0-1} \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$ tensored with $\Delta S'$.

Lemma 3.3. $\Omega_\varphi - \mho_1 = [\operatorname{coker}\varphi]$.

This follows from the commutativity of a diagram

$$\begin{array}{ccccccc} B & \xrightarrow{\beta_1} & L\oplus \Delta S' & \xrightarrow{1\oplus 1} & L\oplus \Delta S' & \xrightarrow{\alpha_1} & B \\ \| & & \| & & \downarrow{\scriptstyle 1\oplus\varphi} & & \downarrow{\scriptstyle \varphi_0} \\ B & \xrightarrow{\beta} & L\oplus \Delta S' & \xrightarrow{1\oplus\varphi} & L\oplus E_{S'} & \xrightarrow{\alpha} & A \end{array}$$

with suitably chosen β_1, α_1, β, α, and in which φ_0 is the middle map of diagram (T). For it implies

$$\begin{array}{ccccc} B & \overset{\widetilde{1}}{\rightarrowtail} & B & \twoheadrightarrow & \operatorname{coker}\widetilde{1} \\ \| & & \downarrow{\scriptstyle \varphi_0} & & \downarrow \\ B & \overset{\widetilde{\varphi}}{\rightarrowtail} & A & \twoheadrightarrow & \operatorname{coker}\widetilde{\varphi} \end{array}$$

so the snake lemma proves the assertion because $\operatorname{coker}\varphi_0 = \operatorname{coker}\varphi$.

In order to see the above claimed commutativity we now define particular maps $\beta = \beta_1 : B \to L\oplus \Delta S', \alpha : L\oplus E_{S'} \to A$ and $\alpha_1 : L\oplus \Delta S' \to B$. To this end, we label, as shown, our maps in the diagrams

$$\begin{array}{ccccc} \Delta S' & \overset{\mu_1}{\rightarrowtail} & B & \overset{\mu_2}{\twoheadrightarrow} & L \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_0} & & \| \\ E_{S'} & \overset{\mu_3}{\rightarrowtail} & A & \overset{\mu_4}{\twoheadrightarrow} & L \end{array} \qquad \begin{array}{ccccc} \Delta S' & \overset{\mu_1}{\rightarrowtail} & B & \overset{\mu_2}{\twoheadrightarrow} & L \\ \downarrow{\scriptstyle 1} & & \downarrow & & \| \\ \Delta S' & \overset{\mu_1}{\rightarrowtail} & B & \overset{\mu_2}{\twoheadrightarrow} & L \end{array}$$

and in the right end of the Tate sequence $L \overset{\rho_1}{\rightarrowtail} B \overset{\rho_2}{\twoheadrightarrow} \Delta S'$.

Choose $\mathbb{Z}$-maps μ_2', ρ_1' with $\mu_2\mu_2' = \mathrm{id}_L = \rho_1'\rho_1$ and build the G-maps $\widetilde{\mu}_2 = \widehat{G}\mu_2',\ \widetilde{\rho}_1 = \widehat{G}\rho_1'$.

The left diagram then gives $\mu_4\varphi_0\mu_2' = \mathrm{id}_L$. We set $\widetilde{\mu}_4 = \varphi_0\widetilde{\mu}_2$ and

$$\beta(b) = (\widetilde{\rho}_1(b), \rho_2(b)), \quad \alpha(y,e) = \widetilde{\mu}_4(y) + \mu_3(e), \quad \alpha_1(y,d) = \widetilde{\mu}_2(y) + \mu_1(d)$$

for $b \in B,\ y \in L,\ e \in E_{S'},\ d \in \Delta S'$. Then

$$\varphi_0\alpha_1(y,d) = \varphi_0\widetilde{\mu}_2(y) + \varphi_0\mu_1(d) = \widetilde{\mu}_4(y) + \mu_3\varphi(d) = \alpha(1\oplus\varphi)(y,d).$$

Passing to the Hom description of $K_0T(\mathbb{Z}G)$ (see Appendix A in [GRW1]), we now have

LEMMA 3.4. $\mho_1$ *is represented by*

$$a_{S'}(\chi) = |G|^{(\chi,\theta)} \prod_{\psi\neq 1} (\psi(g_0)-1)^{-(\chi\psi^{-1},\theta)} \ (^{10})$$

with ψ running through the irreducible characters of G and θ denoting the character of $\Delta S'$.

The proof starts out from the two diagrams

$$\begin{array}{ccccc} \mathbb{Z} & \rightarrowtail & \Delta G\oplus\mathbb{Z} & \twoheadrightarrow & \Delta G \\ \| & & \downarrow \alpha_0 & & \downarrow |G| \\ \mathbb{Z} & \rightarrowtail & \mathbb{Z}G & \twoheadrightarrow & \Delta G \end{array} \qquad \begin{array}{ccccc} \Delta G & \rightarrowtail & \mathbb{Z}G & \twoheadrightarrow & \mathbb{Z} \\ \downarrow |G| & & \downarrow \beta_0 & & \| \\ \Delta G & \to & \Delta G\oplus\mathbb{Z} & \twoheadrightarrow & \mathbb{Z} \end{array}$$

in which

$$\alpha_0(d,z) = d\sum_{i=1}^{|G|-1} ig_0^i + z\widehat{G} \quad (d\in\Delta G, z\in\mathbb{Z}) \quad \text{and} \quad \beta_0(1) = (|G|-\widehat{G}, 1).$$

The identity

$$(g_0-1)\sum_{i=1}^{|G|-1} ig_0^i = |G| - \widehat{G}$$

shows the commutativity.

Now,

$$(\alpha_0\beta_0)(1) = (|G|-\widehat{G})\sum_{i=1}^{|G|-1} ig_0^i + \widehat{G} = x, \quad \text{say.}$$

Before proceeding, we note that

$$\psi\left(\frac{x}{|G|^2}\right) = \begin{cases} 1/|G|, & \psi = 1, \\ 1/(\psi(g_0)-1), & \psi \neq 1. \end{cases}$$

The following computations (including notation) are based on Appendix A in [GRW1]. Tensor the diagrams with $\Delta S'$. Then we have

$$\begin{aligned} \mho_1 &= [\Delta S'\otimes\mathbb{Z}G/\Delta S'\otimes\mathbb{Z}G\cdot x] - 2\partial(\Delta S'\otimes\Delta G, |G|) \\ &= \partial(\Delta S'\otimes\mathbb{Z}G, x) - 2\partial(\Delta S'\otimes\mathbb{Z}G, |G|) + 2\partial(\Delta S', |G|) \\ &= \partial(\Delta S'\otimes\mathbb{Z}G, x/|G|^2) + 2\partial(\Delta S', |G|). \end{aligned}$$

(^{10}) (χ_1,χ_2) denotes the scalar product of the characters χ_1, χ_2 of G.

The first term has representing homomorphism

$$\begin{aligned}\chi &\mapsto \det(x/|G|^2 \mid \mathrm{Hom}_{FG}(V_\chi, F\otimes(\Delta S'\otimes \mathbb{Z}G)))\\ &\doteq \det(x/|G|^2 \mid \mathrm{Hom}_{FG}(V_\chi\otimes(F\otimes\Delta S')^\vee, FG))\\ &= \det(x/|G|^2 \mid V_\chi\otimes(F\otimes\Delta S')^\vee)\end{aligned}$$

by Lemma A.1 in [GRW1]. The equality $\doteq$ holds because of the isomorphism

$$\mathrm{Hom}_F(V, W\otimes Z)\simeq \mathrm{Hom}_F(V\otimes W^\vee, Z),\qquad t\mapsto [v\otimes\omega\mapsto(\widetilde{\omega}t)v],$$

where $\omega\in W^\vee=\mathrm{Hom}_F(W,F)$ induces $\widetilde{\omega}: W\otimes Z\to Z, w\otimes z\mapsto \omega(w)\cdot z$. This isomorphism respects the G-structure and composition by a G-endomorphism of Z.

Now,

$$\begin{aligned}\det(x/|G|^2 \mid V_\chi\otimes(F\otimes\Delta S')^\vee) &= \prod_\psi \det(x/|G|^2\mid V_\psi)^{(\chi\theta^\vee,\psi)}\\ &= \prod_\psi \psi\Big(\frac{x}{|G|^2}\Big)^{(\chi,\theta\psi)}\\ &= |G|^{-(\chi,\theta)}\prod_{\psi\neq 1}(\psi(g_0)-1)^{-(\chi\psi^\vee,\theta)}.\end{aligned}$$

The second term is represented by

$$\chi\mapsto \det(|G|\mid \mathrm{Hom}_{FG}(V_\chi, F\otimes\Delta S'))^2 = |G|^{2(\chi,\theta)}.$$

Multiplying the two gives the result.

Corollary. *The Lifted Root Number Conjecture holds for K/k if, and only if, $[\mathbb{Z}_l\otimes\operatorname{coker}\varphi]$ is represented by $\chi\mapsto A_\varphi^{(l)}(\check{\chi})/a_{S'}^{(l)}(\chi)$ for all (finite) primes l.*

The Lifted Root Number Conjecture asserts that $\chi\mapsto A_\varphi(\check{\chi})$ represents Ω_φ, which by Lemmas 3.3 and 3.4 is equivalent to $\chi\mapsto A_\varphi(\check{\chi})/a_{S'}(\chi)$ representing $[\operatorname{coker}\varphi]$. This is then restated one prime l at a time by considering the idèlic component above l in the representing homomorphisms [GRW1, Appendix A].

4. Adapting S to the local nature of the Lifted Root Number Conjecture. In the previous section we required S' to be sufficiently large in order to have the Tate class $\tau_{S'}\in \mathrm{Ext}^1_G(\Delta S'\otimes\Delta G, E_{S'})$ at our disposal. In this section we restrict K to be absolutely abelian and real, but work with a finite G-set S of primes of K containing the set S_∞ of infinite primes as well as all ramified primes of the extension K/k and just enough primes to generate the l-part of cl_K for the given prime l.

Let n denote the conductor of K, so $K \subset \mathbb{Q}(\zeta_n)^+$, and let ∞ be a distinguished infinite prime of K. We use the letter σ to denote automorphisms of $K/\mathbb{Q}$, so each infinite prime of K is some ∞^σ.

Recall from the introduction the Ramachandra number

$$\xi_K = \prod_{1\neq d\|n} N_{\mathbb{Q}(\zeta_n)/K}(1-\zeta_n^{n/d}),$$

with $d \,\|\, n$ meaning $d \mid n$ & $(d, n/d) = 1$, and define an S-unit α_∞ in $K^\times$ by

$$\xi_K^{g_0-1} = \alpha_\infty^{(g_0-1)^{h+1}}$$

with some $h \geq 0$ (as in Lemma 2.2). Moreover, define $\varphi_\infty : \Delta S_\infty \to E_K$ by $\varphi_\infty(\infty^\sigma - \infty) = \alpha_\infty^{\sigma-1}$. The comparison of the notation here and in [RW, §10] is done by means of the dictionary below.

[RW]	is here
ξ_K	ξ_K^2
φ_∞	$2(g_0-1)^h\varphi_\infty$

Having thus taken care of all infinite primes of K we get from elements $\alpha_{\mathfrak{p}}$ ($\mathfrak{p} \in S_*$ (11), $\mathfrak{p} \notin S_\infty$, $\mathfrak{p} \neq \mathfrak{q}_0$), as appearing in Proposition 3.2 with S' replaced by S, a map $\varphi : \Delta S \to E_S$ making the left square of the diagram

$$\begin{array}{ccccc} \Delta S_\infty & \rightarrowtail & \Delta S & \twoheadrightarrow & \mathbb{Z}S_f \\ \downarrow{\scriptstyle \varphi_\infty} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \widetilde{\varphi}} \\ E_K & \rightarrowtail & E_S & \twoheadrightarrow & E_S/E_K \end{array}$$

commute by sending $\mathfrak{p} - \mathfrak{q}_0$ to $\alpha_{\mathfrak{p}}$ and $\infty - \mathfrak{q}_0$ to α_∞. In the diagram, $S_f = S \setminus S_\infty, \Delta S \to \mathbb{Z}S_f$ is given by

$$\mathfrak{p}' - \infty \mapsto \begin{cases} \mathfrak{p}', & \mathfrak{p}' \text{ finite}, \\ 0, & \mathfrak{p}' \text{ infinite}, \end{cases}$$

and the right vertical map $\widetilde{\varphi}$ is the induced one, whence

$$\widetilde{\varphi}(\mathfrak{p}) = \varphi(\mathfrak{p} - \infty)E_K = \varphi(\mathfrak{p} - \mathfrak{q}_0 + \mathfrak{q}_0 - \infty)E_K = \alpha_{\mathfrak{p}}/\alpha_\infty \cdot E_K$$

for $\mathfrak{p} \in S_*, \mathfrak{p} \notin S_\infty, \mathfrak{p} \neq \mathfrak{q}_0$. Similarly, $\widetilde{\varphi}(\mathfrak{q}_0) = \alpha_\infty^{-1}E_K$.

We define the Dirichlet map λ as in [GRW1]: $\lambda_S : \mathbb{C} \otimes E_S \to \mathbb{C} \otimes \Delta S$ sends $u \in E_S$ to $-\sum_{\mathfrak{p}\in S} \log |u|_{\mathfrak{p}}\mathfrak{p}$ (12). Recall that, for a character χ of G,

$$A_\varphi(\check{\chi}) = \frac{\det(\lambda_S \circ \varphi \mid \mathrm{Hom}_{\mathbb{C}G}(V_\chi, \mathbb{C} \otimes \Delta S))}{c_S(\check{\chi})}$$

is the Tate–Stark number [Ta, p. 27]. We compute it by exploiting our

(11) The $*$ indicates again that S is replaced by a set S_* of G-representatives.

(12) Observe that this is $-\lambda_S$ in [RW].

diagram above which induces

$$
(4.1)\qquad
\begin{array}{ccccc}
\mathbb{C}\otimes\Delta S_\infty & \rightarrowtail & \mathbb{C}\otimes\Delta S & \twoheadrightarrow & \mathbb{C}S_f\\
\downarrow \varphi_\infty & & \downarrow \varphi & & \downarrow \widetilde{\varphi}\\
\mathbb{C}\otimes E_K & \rightarrowtail & \mathbb{C}\otimes E_S & \twoheadrightarrow & \mathbb{C}\otimes E_S/E_K\\
\downarrow \lambda_\infty & & \downarrow \lambda_S & & \downarrow \widetilde{\lambda}\\
\mathbb{C}\otimes\Delta S_\infty & \rightarrowtail & \mathbb{C}\otimes\Delta S & \twoheadrightarrow & \mathbb{C}S_f
\end{array}
$$

This implies the factorization

$$
(4.2)\qquad A_\varphi(\check{\chi}) = \frac{\det(\lambda_\infty\varphi_\infty \mid \mathrm{Hom}_{\mathbb{C}G}(V_\chi, \mathbb{C}\otimes\Delta S_\infty))}{c_{S_\infty}(\check{\chi})}\cdot \det{}_f(\chi)\cdot\frac{c_{S_\infty}(\check{\chi})}{c_S(\check{\chi})},
$$

with $\det_f(\chi)$ short for $\det(\widetilde{\lambda}\widetilde{\varphi} \mid \mathrm{Hom}_{\mathbb{C}G}(V_\chi, \mathbb{C}S_f))$.

The first and third factor have been studied in [RW].

To compute the middle one we use the non-zero elements in $\{e_\chi\mathfrak{p} \mid \mathfrak{p}\in S_*,\ \mathfrak{p}\notin S_\infty\}$ as a basis of $\mathrm{Hom}_{\mathbb{C}G}(V_\chi,\mathbb{C}S_f) = e_\chi\mathbb{C}S_f$. Here e_χ is the primitive idempotent corresponding to χ. We have

$$
(4.3)\qquad \widetilde{\lambda}\widetilde{\varphi}(e_\chi\mathfrak{p}) = -\sum_{\mathfrak{p}'\in S_f}\log\left|\frac{\alpha_\mathfrak{p}}{\alpha_\infty}\right|_{\mathfrak{p}'} e_\chi\mathfrak{p}'
$$

and will evaluate such determinants in Section 5 by applying the

Lemma 4.1. *If S_{f*} is a set of G-representatives in S_f, and if $(\alpha) = \prod_{\mathfrak{p}\in S_{f*}}\mathfrak{p}^{x_\mathfrak{p}}$ with $x_\mathfrak{p}\in\mathbb{Z}G$, then*

$$
-\sum_{\mathfrak{p}\in S_f}\log|\alpha|_\mathfrak{p} e_\chi\mathfrak{p} = \sum_{\mathfrak{p}\in S_{f*}}\chi(x_\mathfrak{p})\log(N\mathfrak{p})e_\chi\mathfrak{p}.
$$

For the proof pick a $\mathfrak{p}\in S_{f*}$ and consider the orbit sum

$$
-\sum_{\mathfrak{p}'\in G\cdot\mathfrak{p}}\log|\alpha|_{\mathfrak{p}'}e_\chi\mathfrak{p}' = -\frac{1}{|G_\mathfrak{p}|}\sum_{g\in G}\log|\alpha^{g^{-1}}|_\mathfrak{p}\chi(g)e_\chi\mathfrak{p}.
$$

With $\kappa:\mathbb{C}G\to\mathbb{C}$ denoting the $\mathbb{C}$-linear map taking g to 1 or 0 according as $g\in G_\mathfrak{p}$ or $g\notin G_\mathfrak{p}$ we deduce from $v_\mathfrak{p}(\alpha^{g^{-1}}) = \kappa(g^{-1}x_\mathfrak{p})$ that the above orbit sum equals

$$
\begin{aligned}
-\frac{1}{|G_\mathfrak{p}|}&\sum_{g\in G}(-\kappa(g^{-1}x_\mathfrak{p})\log(N\mathfrak{p}))\chi(g)e_\chi\mathfrak{p}\\
&= \frac{\log(N\mathfrak{p})}{|G_\mathfrak{p}|}\kappa\Big(\sum_{g\in G}\chi(g)g^{-1}x_\mathfrak{p}\Big)e_\chi\mathfrak{p} = \frac{\log(N\mathfrak{p})}{|G_\mathfrak{p}|}\kappa(|G|e_\chi x_\mathfrak{p})e_\chi\mathfrak{p}
\end{aligned}
$$

$$= \frac{\log(N\mathfrak{p})}{|G_\mathfrak{p}|}\kappa(\chi(x_\mathfrak{p})|G|e_\chi)e_\chi\mathfrak{p}$$
$$= \frac{\log(N\mathfrak{p})}{|G_\mathfrak{p}|}\chi(x_\mathfrak{p})\Big(\sum_{g\in G_\mathfrak{p}}\chi(g)\Big)e_\chi\mathfrak{p}$$
$$= \frac{\log(N\mathfrak{p})}{|G_\mathfrak{p}|}\chi(x_\mathfrak{p})|G_\mathfrak{p}|e_\chi\mathfrak{p},$$

since $e_\chi\mathfrak{p} = 0$ whenever $\operatorname{res}^G_{G_\mathfrak{p}}\chi$ is non-trivial.

LEMMA 4.2. *Even though S does not satisfy the hypothesis of Proposition* 3.2, *the Corollary at the end of Section* 3 *remains true for S with respect to the given prime l.*

To see this, let S' be a finite G-set containing S and G-orbits of split primes $\mathfrak{r}$ (over k) so that the S'-class group of K vanishes. The existence of an S' follows from the Chebotarev density theorem.

Assume now that we have a map $\varphi : \Delta S \rightarrowtail E_S$ satisfying $\varphi(\mathfrak{p}-\mathfrak{q}_0) = \alpha_\mathfrak{p}$ for $\mathfrak{p} \in S_*,\ \mathfrak{p} \neq \mathfrak{q}_0$, where the $\alpha_\mathfrak{p}$ are as above ($\mathfrak{p} = \infty$ included). With h' denoting the l-prime part of $|cl_K|$, we pick generators $\alpha_\mathfrak{r} \in K$ of the principal ideals $\mathfrak{r}^{h'}$ and extend φ to $\varphi' : \Delta S' \to E_{S'}$ by mapping $\mathfrak{r} - \mathfrak{q}_0$ to $\alpha_\mathfrak{r}$. Then φ' is a map as described in Proposition 3.2, because the $\mathfrak{r}$ split over k and since the $\alpha_\mathfrak{r}$ are S'-units.

We first compare $\operatorname{coker}\varphi$ and $\operatorname{coker}\varphi'$. To do so build

$$\begin{array}{ccccc}
\Delta S & \rightarrowtail & \Delta S' & \twoheadrightarrow & \mathbb{Z}[S'\setminus S] \\
\downarrow \varphi & & \downarrow \varphi' & & \downarrow \\
E_S & \rightarrowtail & E_{S'} & \twoheadrightarrow & E_{S'}/E_S \\
 & & & & \downarrow \\
 & & & & \mathbb{Z}[S'\setminus S]
\end{array}$$

analogous to the earlier diagram (with S_∞, S instead of S, S') and with $E_{S'}/E_S \to \mathbb{Z}[S'\setminus S]$ taking $u \in E_{S'}$ to $\sum_\mathfrak{r} v_\mathfrak{r}(u)\mathfrak{r}$. The composite right vertical map is then multiplication by $h' : \mathfrak{r} \mapsto \alpha_\mathfrak{r} \mapsto h'\mathfrak{r}$. Hence $\mathbb{Z}_l \otimes \operatorname{coker}\varphi = \mathbb{Z}_l \otimes \operatorname{coker}\varphi'$.

We next turn to the numerators of the A-numbers. We tensor the above diagram with $\mathbb{C}$ and get a diagram similar to (4.1) with maps φ, φ', φ'' and λ, λ', λ'', say. Thus

$$\det(\lambda\varphi) = \det(\lambda'\varphi')/\det(\lambda''\varphi'') = \det(\lambda'\varphi')/\prod_\mathfrak{r}(h'\log N\mathfrak{r})$$

by $\lambda''\varphi''(\mathfrak{r}) = h'(\log N\mathfrak{r})\mathfrak{r}$.

Now,

$$\frac{A_{\varphi'}(\check\chi)}{A_\varphi(\check\chi)} = \frac{c_S(\check\chi)}{c_{S'}(\check\chi)}\cdot\prod_\mathfrak{r} h'\log(N\mathfrak{r}) = \prod_\mathfrak{r} h'$$

by [RW, Lemma 7 with $G_\mathfrak{r} = I_\mathfrak{r} = 1,\ f_\mathfrak{r} = 1$].

Finally we look at $a_{S'}(\chi)/a_S(\chi)$. As the character of $\mathbb{Z}[S'\setminus S]$ is a multiple of the regular character ϱ, we get from Lemma 3.4, for irreducible χ,

$$\frac{a_{S'}(\chi)}{a_S(\chi)} = \prod_{\mathfrak{r}} \Big(|G|^{(\chi,\varrho)} \prod_{\psi\neq 1} (\psi(g_0)-1)^{-(\chi\psi^{-1},\varrho)}\Big) = \prod_{\mathfrak{r}} (-1)^{|G|-1},$$

because the multiplicity in ϱ of every irreducible character is 1.

Putting things together, we see that $A_{\varphi'}(\chi)/a_{S'}(\chi)$ and $A_{\varphi}(\check{\chi})/a_S(\chi)$ differ by a constant $b\in\mathbb{Z}_l^\times$ which is independent of the irreducible character χ. Since $\chi\mapsto b^{\chi(1)}\in \mathrm{Det}(\mathbb{Z}_lG)$ represents the trivial element in $K_0T(\mathbb{Z}_lG)$, the lemma is proved.

5. Calculation of A_φ. We go back to our initial situation in which K is as in the introduction.

With a choice of $\mathfrak{q}_0,\ldots,\mathfrak{q}_{r-1}$ as in Lemma 2.3, suppose that $\mathfrak{a}_i$, $1\le i\le r-1$, are ideals supported in $\{\mathfrak{p}_1,\ldots,\mathfrak{p}_r\}$ such that $\mathfrak{a}_i\cdot\mathfrak{q}_i^{(g_0-1)^{h_i-1}}$ are principal. Let $\alpha_{\mathfrak{q}_i}$ be the corresponding generators.

Lemma 5.1. *The $r\times r$ matrix*

$$\begin{pmatrix} -v_{\mathfrak{p}_1}(\alpha_\infty) & \ldots & -v_{\mathfrak{p}_r}(\alpha_\infty)\\ -v_{\mathfrak{p}_1}(\alpha_{\mathfrak{q}_1}) & \ldots & -v_{\mathfrak{p}_r}(\alpha_{\mathfrak{q}_1})\\ \vdots & \ldots & \vdots\\ -v_{\mathfrak{p}_1}(\alpha_{\mathfrak{q}_{r-1}}) & \ldots & -v_{\mathfrak{p}_r}(\alpha_{\mathfrak{q}_{r-1}}) \end{pmatrix}$$

is non-singular modulo l. Moreover, its determinant c' satisfies $-v_{\mathfrak{p}_j}(\alpha_\infty)\equiv c'B_j \bmod l$ for $1\le j\le r$. In other words, $c'\equiv\widetilde{c} \bmod l$.

In fact, $(\alpha_{\mathfrak{q}_i}) = \mathfrak{q}_i^{(g_0-1)^{h_i-1}}\prod_{s=1}^r \mathfrak{p}_s^{v_{\mathfrak{p}_s}(\alpha_{\mathfrak{q}_i})}$ and (1.2) imply

$$\sum_{s=1}^r (-v_{\mathfrak{p}_s}(\alpha_{\mathfrak{q}_i}))b_{js} \equiv \delta_{ij} \bmod l,\qquad 1\le i,j\le r-1.$$

Since the B_j and $v_{\mathfrak{p}_j}(\alpha_\infty)$ are proportional modulo l, we also have

$$\sum_{s=1}^r (-v_{\mathfrak{p}_j}(\alpha_\infty))b_{js}\equiv 0 \bmod l \qquad \text{for } 1\le j\le r-1.$$

With $\left[\begin{smallmatrix} z\\ b_{ij}\end{smallmatrix}\right]$ denoting the matrix

$$\begin{pmatrix} z_1 & \ldots & z_r\\ & b_{ij} & \end{pmatrix}_{\substack{1\le j\le r\\ 1\le i\le r-1}},$$

where $(z_1,\ldots,z_r)$ is any row vector, we see that the matrix in the statement of the lemma times the transpose of $\left[{z \atop b_{ij}}\right]$ is

$$\begin{pmatrix} \sum_{j=1}^r (-v_{\mathfrak{p}_j}(\alpha_\infty))z_j & 0 & \dots & 0 \\ * & 1 & & \\ \vdots & & & \\ * & & & 1 \end{pmatrix}.$$

Taking determinants yields $c' \cdot (\sum_{j=1}^r z_j B_j) \equiv \sum_{j=1}^r (-v_{\mathfrak{p}_j}(\alpha_\infty))z_j \bmod l$ by the definition of the B_j. The second assertion of Lemma 5.1 follows by varying z and then the first assertion from Lemma 2.2.

In the vacuous case $r = 1$ we see that we had to set $B_1 = 1$.

Corollary. *For each* $1 \le j \le r$, p_j *is in the* $\mathbb{Z}_l G$*-span of the* q_i *and the* $\alpha_{\mathfrak{q}_i}$, α_∞.

For $\alpha_{\mathfrak{q}_i}^{\widehat{G}} = q_i^{\delta_i} p_1^{v_{\mathfrak{p}_1}(\alpha_{\mathfrak{q}_i})} \dots p_r^{v_{\mathfrak{p}_r}(\alpha_{\mathfrak{q}_i})}$ in $\mathbb{Z}_l \otimes E_S$, with $\delta_i = 0$ or 1 according as $h_i > 1$ or $h_i = 1$, and similarly for α_∞.

We use the $\alpha_{\mathfrak{q}_i}$ and α_∞ together with an integer matrix (x_{ij}) congruent modulo l to the inverse of the matrix (c_{ij}) of Lemma 2.3 in order to define a G-map $\varphi : \Delta S \to E_S$, where S is the G-set generated by $\{\infty, \mathfrak{p}_1, \dots, \mathfrak{p}_r, \mathfrak{q}_0, \dots, \mathfrak{q}_{r-1}\}$. Here, ∞ is the infinite prime defined by $K \subset \mathbb{Q}(\zeta_n) \hookrightarrow \mathbb{C}$, $\zeta_n \mapsto e^{2\pi i/n}$.

ΔS is spanned over $\mathbb{Z}G$ by

$$\mathfrak{p}_j - \mathfrak{q}_0,\ 1 \le j \le r, \qquad \mathfrak{q}_i - \mathfrak{q}_0,\ 1 \le i \le r-1, \qquad \infty - \mathfrak{q}_0.$$

G acts trivially on the $\mathfrak{p}_j - \mathfrak{q}_0$; the other generators are free over $\mathbb{Z}G$. The map φ is defined by

$$\mathfrak{p}_j - \mathfrak{q}_0 \mapsto \prod_{i=0}^{r-1} q_i^{x_{ji}} =: \alpha_{\mathfrak{p}_j}, \qquad \mathfrak{q}_i - \mathfrak{q}_0 \mapsto \alpha_{\mathfrak{q}_i}, \qquad \infty - \mathfrak{q}_0 \mapsto \alpha_\infty.$$

Lemma 5.2. φ *satisfies the conditions of Proposition* 3.2 *and induces an* $\mathbb{Z}_l G$*-isomorphism* $\mathbb{Z}_l \otimes \Delta S \to \mathbb{Z}_l \otimes E_S$.

Proof. Because $\mathfrak{q}_1, \dots, \mathfrak{q}_{r-1}, \infty$ are split over $\mathbb{Q}$ there is no condition on the $\alpha_{\mathfrak{q}_i}$ and on α_∞ in Proposition 3.2. Concerning the $\alpha_{\mathfrak{p}_j}$ we have

$$(\alpha_{\mathfrak{p}_j}, K_{\mathfrak{p}'_j}/\mathbb{Q}_{p'_j}) = \prod_{i=0}^{r-1} (\mathfrak{q}_i, K_{\mathfrak{p}'_j}/\mathbb{Q}_{p'_j})^{x_{ji}} = g_0^{\sum_i x_{ji} c_{ij'}} = g_0^{\delta_{jj'}}.$$

Moreover, $(\alpha_{\mathfrak{p}_j}, K_{\mathfrak{p}}/\mathbb{Q}_p) = 1$ for all $p \ne q_0, p_1, \dots, p_r$, since either p splits or $\alpha_{\mathfrak{p}_j}$ is a unit in the unramified extension $K_{\mathfrak{p}}/\mathbb{Q}_p$.

To see that φ is an l-adic isomorphism we show first that $q_0, q_1, \dots, q_{r-1}$, $\alpha_\infty, \alpha_{\mathfrak{q}_1}, \dots, \alpha_{\mathfrak{q}_{r-1}}$ generate $\mathbb{Z}_l \otimes E_S$ as $\mathbb{Z}_l G$-module. Pick $u \in E_S$. Then, by definition of S,

$$(u) = \mathfrak{p}_1^{b_1} \dots \mathfrak{p}_r^{b_r} \mathfrak{q}_0^{c_0} \dots \mathfrak{q}_{r-1}^{c_{r-1}}$$

with $b_j \in \mathbb{Z}$ and $c_i \in \mathbb{Z}G$. Because q_0 is inert and $q_0 \in E_S$, we may assume $c_0 = 0$. Reading then the above equation in cl_K shows $\mathfrak{C}_1^{c_1} \ldots \mathfrak{C}_{r-1}^{c_{r-1}} \in cl_K^G$ and so $c_i = c_i'(g_0 - 1)^{h_i - 1} + c_i''\widehat{G}$ with $c_i' \in \mathbb{Z}G$ and $c_i'' \in \mathbb{Z}$.

It follows from the definition of the $\alpha_{\mathfrak{q}_i}$ that

$$\Big(u \prod_{i=1}^{r-1} \alpha_{\mathfrak{q}_i}^{-c_i'}\Big) = \mathfrak{p}_1^{b_1'} \ldots \mathfrak{p}_r^{b_r'} \prod_{i=1}^{r-1} q_i^{c_i''}.$$

So $\mathfrak{p}_1^{b_1'} \ldots \mathfrak{p}_r^{b_r'}$ is principal, whence $(\mathfrak{p}_1^{b_1'} \ldots \mathfrak{p}_r^{b_r'})^N = (\alpha_\infty^d a)$ with $d \in \mathbb{Z}$, a a product of the p_j and with a suitable $N \not\equiv 0 \bmod l$. Therefore u^N and $\alpha_\infty^d a \prod_{i=1}^{r-1}(\alpha_{q_i}^{Nc_i'} q_i^{Nc_i''})$ differ by a unit. Read in $\mathbb{Z}_l \otimes E_S$, this unit becomes a $\mathbb{Z}_l G$-power of $\alpha_\infty^{g_0 - 1}$ by Lemma 2.2. Because of the above corollary and since $l \nmid N$, our generation claim is proved.

The non-degeneracy of the matrix (x_{ij}) modulo l then implies that φ is surjective modulo l, so l-adically as well. This finishes the proof by Dirichlet's unit theorem.

We now turn to the computation of $A_\varphi(\check{\chi})$. Because of (4.2) this amounts to computing the three quantities

$$\frac{\det(\lambda_\infty \varphi_\infty \mid \mathrm{Hom}_G(V_\chi, \Delta S_\infty))}{c_{S_\infty}(\check{\chi})}, \quad \det{}_f(\chi) \quad \text{and} \quad \frac{c_{S_\infty}(\check{\chi})}{c_S(\check{\chi})},$$

where φ is decomposed according to the diagram

$$\begin{array}{ccccc} \Delta S_\infty & \rightarrowtail & \Delta S & \twoheadrightarrow & \mathbb{Z}S_f \\ \downarrow \varphi_\infty & & \downarrow \varphi & & \downarrow \widetilde{\varphi} \\ E_K & \rightarrowtail & E_S & \twoheadrightarrow & E_S/E_K \end{array}$$

with $\varphi_\infty(\infty^g - \infty) = \alpha_\infty^{g-1}$ $(g \in G)$, and where $\det_f(\chi)$ is the determinant of the map $e_\chi \mathbb{C} S_f \to e_\chi \mathbb{C} S_f$ taking:

- $e_\chi \mathfrak{p}_j$ $(1 \le j \le r)$ to

$$-\sum_{\mathfrak{p} \in S_f} \Big(\log \Big|\frac{\alpha_{\mathfrak{p}_j}}{\alpha_\infty}\Big|_{\mathfrak{p}}\Big) e_\chi \mathfrak{p} = -\sum_{j=1}^{r} v_{\mathfrak{p}_j}(\alpha_\infty)(\log p_j) e_\chi \mathfrak{p}_j + \sum_{i=0}^{r-1} \chi(\widehat{G}) x_{ji} (\log q_i) e_\chi \mathfrak{q}_i$$

by Lemma 4.1, since $\alpha_{\mathfrak{p}_j} = \prod_{i=0}^{r-1} q_i^{x_{ji}}$, $q_i = \mathfrak{q}_i^{\widehat{G}}$ $(1 \le i \le r-1)$, $N(\mathfrak{q}_0) = q_0^l$ and $e_\chi \mathfrak{q}_0 = 0$ if $\chi \ne 1$, and since α_∞ is supported in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$;

- $e_\chi \mathfrak{q}_0$ to

$$-\sum_{\mathfrak{p} \in S_f} (\log |\alpha_\infty^{-1}|_{\mathfrak{p}}) e_\chi \mathfrak{p} = -\sum_{j=1}^{r} v_{\mathfrak{p}_j}(\alpha_\infty)(\log p_j) e_\chi \mathfrak{p}_j;$$

- $e_\chi \mathfrak{q}_i$ $(1 \le i \le r-1)$ to

$$\sum_{j=1}^{r} v_{\mathfrak{p}_j}(\alpha_{\mathfrak{q}_i}\alpha_\infty^{-1})(\log p_j)e_\chi \mathfrak{p}_j + (\chi(g_0)-1)^{h_i-1}(\log q_i)e_\chi \mathfrak{q}_i,$$

since $(\alpha_{\mathfrak{q}_i}) = \mathfrak{q}_i^{(g_0-1)^{h_i-1}}\mathfrak{a}_i$.

Regarding the first quantity we have

$$\frac{\det_\chi(\lambda_\infty\varphi_\infty)}{c_{S_\infty}(\check{\chi})} = \begin{cases} -2 & \text{if } \chi = 1, \\ 2/(\chi(g_0)-1)^h & \text{if } \chi \ne 1. \end{cases} \tag{5.1}$$

This follows from $e_1 \Delta S_\infty = 0, c_{S_\infty}(1) = -1/2$ and from Proposition 12 of [RW] adjusted appropriately (see §4); in particular, the λ_∞ there is here $-\lambda_\infty$.

The quantity $c_{S_\infty}(\check{\chi})/c_S(\check{\chi})$ is given by [RW, Lemma 7] on observing

$$\check{V}^{G_{\mathfrak{p}_j}} = 0 = \check{V}^{I_{\mathfrak{p}_j}} \ (1 \le j \le r), \quad \check{V}^{G_{\mathfrak{q}_i}} = \check{V} = \check{V}^{I_{\mathfrak{q}_i}} \ (1 \le i \le r-1),$$
$$\check{V}^{G_{\mathfrak{q}_0}} = 0, \quad \check{V}^{I_{\mathfrak{q}_0}} = \check{V},$$

where $\check{V}$ is a $\mathbb{C}G$-module affording the character $\check{\chi} \ne 1$, and where $I_\mathfrak{p}$, for a prime $\mathfrak{p}$, is the inertia group of $\mathfrak{p}$. Thus

$$\frac{c_{S_\infty}(\check{\chi})}{c_S(\check{\chi})} \tag{5.2}$$
$$= \begin{cases} \prod_{j=1}^{r}(\log p_j)^{-1}\prod_{i=0}^{r-1}(\log q_i)^{-1} & \text{for } \chi = 1, \\ \Big(\Big(\prod_{i=1}^{r-1}\log q_i\Big)(1-\check{\chi}(g_0))\Big)^{-1} = \dfrac{\chi(g_0)}{\chi(g_0)-1}\prod_{i=1}^{r-1}(\log q_i)^{-1} & \text{for } \chi \ne 1. \end{cases}$$

We are left with computing $\det_f(\chi)$. Assume first $\chi \ne 1$. Then $e_\chi \mathfrak{p}_j = 0 = e_\chi \mathfrak{q}_0, 1 \le j \le r$, and our map $e_\chi \mathbb{C}S_f \to e_\chi \mathbb{C}S_f$ is diagonal with diagonal entries $(\chi(g_0)-1)^{h_i-1}(\log q_i)$, so

$$\det_f(\chi) = (\chi(g_0)-1)^{\sum_{i=1}^{r-1}h_i-(r-1)}\prod_{i=1}^{r-1}(\log q_i).$$

Assume next that $\chi = 1$. We label the row and columns of the matrix of $e_1\mathbb{C}S_f \to e_1\mathbb{C}S_f$ by $e_1\mathfrak{p}_j$ $(1 \le j \le r), e_1\mathfrak{q}_i$ $(0 \le i \le r-1)$ starting at the top and on the left, respectively. Then we view it as having the form $\left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$ with $r \times r$ matrices $A_{\nu\mu}$. Subtracting row $e_1\mathfrak{q}_0$ from all the other rows changes A_{11} into the zero matrix but does not affect A_{12} as follows from the description of the map $e_\chi \mathbb{C}S_f \to e_\chi \mathbb{C}S_f$. However, A_{21} becomes

$$A'_{21} = \begin{pmatrix} -v_{\mathfrak{p}_1}(\alpha_\infty)\log p_1 & \dots & -v_{\mathfrak{p}_r}(\alpha_\infty)\log p_r \\ v_{\mathfrak{p}_1}(\alpha_{\mathfrak{q}_1})\log p_1 & \dots & v_{\mathfrak{p}_r}(\alpha_{\mathfrak{q}_1})\log p_r \\ \vdots & & \vdots \\ v_{\mathfrak{p}_1}(\alpha_{\mathfrak{q}_{r-1}})\log p_1 & \dots & v_{\mathfrak{p}_r}(\alpha_{\mathfrak{q}_{r-1}})\log p_r \end{pmatrix}.$$

Hence

$$\begin{aligned} \det{}_f(1) &= (-1)^r \det A_{12} \det A'_{21} \\ &= (-1)^r \det A_{12} \cdot \Big(\prod_{j=1}^{r} \log p_j\Big) \cdot (-1) \cdot \det(v_{\mathfrak{p}_j}(\alpha_{\mathfrak{q}_i})), \end{aligned}$$

where in accordance with our usual numbering we better read α_∞ as $\alpha_{\mathfrak{q}_0}$.

As for $\det A_{12}$, we remember that

$$A_{12} = \begin{pmatrix} lx_{10}\log q_0 & \dots & lx_{1,r-1}\log q_{r-1} \\ \vdots & & \vdots \\ lx_{r0}\log q_0 & \dots & lx_{r,r-1}\log q_{r-1} \end{pmatrix},$$

so

$$\det A_{12} = l^r \Big(\prod_{i=0}^{r-1} \log q_i\Big) \det(x_{ji}).$$

Taking everything into account we arrive at

(5.3) $\det_f(\chi)$

$$= \begin{cases} (-1)l^r \Big(\prod_{j=1}^{r} \log p_j\Big)\Big(\prod_{i=0}^{r-1} \log q_i\Big) \det(-v_{\mathfrak{p}_j}(\alpha_{\mathfrak{q}_i})) \det(x_{ji}) & \text{if } \chi = 1, \\ \Big(\prod_{i=1}^{r-1} \log q_i\Big)(\chi(g_0)-1)^{\sum_{i=1}^{r-1} h_i - (r-1)} & \text{if } \chi \neq 1. \end{cases}$$

(5.1)–(5.3) together yield

$A_\varphi(\check{\chi})$

$$= \begin{cases} 2l^r \det(-v_{\mathfrak{p}_j}(\alpha_{\mathfrak{q}_i})) \det(x_{ji}) & \text{if } \chi = 1, \\ \dfrac{2}{(\chi(g_0)-1)^h} \cdot \dfrac{\chi(g_0)}{\chi(g_0)-1} (\chi(g_0)-1)^{\sum_{i=1}^{r-1} h_i} / (\chi(g_0)-1)^{r-1} & \text{if } \chi \neq 1. \end{cases}$$

Using the decomposition $\mathbb{C} \otimes \Delta S = \mathbb{C}^r \oplus (\mathbb{C}G)^r$ we quickly evaluate the representing homomorphism $a(\chi)$ from Lemma 3.4:

$$a(\chi)=\begin{cases} l^{2r}\prod_{\psi\neq 1}(\psi(g_0)-1)^{-(\psi^{-1},\Delta S)}=\dfrac{l^{2r}}{\prod_{\psi\neq 1}(\psi(g_0)-1)^r}=l^r, & \chi=1,\\ l^r(\chi(g_0)-1)^{-2r}\prod_{\psi\neq 1,\chi}(\psi(g_0)-1)^r=\dfrac{l^r(\chi(g_0)-1)^{-r}}{\prod_{\psi\neq 1}(\psi(g_0)-1)^r} & \\ \qquad =(\chi(g_0)-1)^{-r}, & \chi\neq 1.\end{cases}$$

Therefore,

$$\frac{A_\varphi(\check\chi)}{a(\chi)}=\begin{cases}2\det(-v_{\mathfrak{p}_j}(\alpha_{\mathfrak{q}_i}))\det(x_{ji}) & \text{if } \chi=1,\\ \dfrac{2}{(\chi(g_0)-1)^h}(\chi(g_0)-1)^{\sum_{i=1}^{r-1}h_i}\cdot\chi(g_0) & \text{if } \chi\neq 1.\end{cases}$$

As has been pointed out at the end of Section 3, $\chi\mapsto A_\varphi^{(l)}(\check\chi)/a^{(l)}(\chi)$ is to represent $\mathbb{Z}_l\otimes\operatorname{coker}\varphi=0$ by the Lifted Root Number Conjecture. The main result in [RW] confirms this modulo $DT(\mathbb{Z}_lG)$, i.e., all $A_\varphi(\check\chi)/a(\chi)$ generate the same ideal in $\mathbb{Z}_l[\zeta_l]$. In particular,

$$(\natural)\qquad \sum_{i=1}^{r-1}h_i=h,$$

since $\det(-v_{\mathfrak{p}_j}(\alpha_{\mathfrak{q}_i}))\cdot\det(x_{ji})\not\equiv 0 \bmod l$ by Lemmas 2.3 and 5.1. Consequently, the Lifted Root Number Conjecture amounts to

$$A_\varphi(\check\chi)/a(\chi)\equiv A_\varphi(1)/a(1) \bmod \chi(g_0)-1$$

(for all $\chi\neq 1$), by [GRW1, Proposition 8(iii)], that is

$$(\mathrm{LC})\qquad \det(-v_{\mathfrak{p}_j}(\alpha_{\mathfrak{q}_i}))\equiv\det(c_{ij}) \bmod l$$

by Lemma 2.3. This is indeed conjecture (C), by Lemma 5.1.

We quickly dispose of the case $r=1$, whence $h=0$ by ($\natural$). First of all, $c_{01}=-1$ by the reciprocity law. In fact, $(q_0,K_{\mathfrak{p}_1}/\mathbb{Q}_{p_1})=g_0^{c_{01}}$, $(q_0,K_{\mathfrak{q}_0}/\mathbb{Q}_{q_0})=g_0$, and $(q_0,K_{\mathfrak{p}}/\mathbb{Q}_p)=1$ for all other primes p, since these are unramified. Moreover, K is the subfield of $\mathbb{Q}(\zeta_{p_1})$ of degree l over $\mathbb{Q}$ and so ξ_K is a prime element for $\mathfrak{p}_1$. By Lemma 2.2 we can take $\alpha_\infty=\xi_K$. Hence, (C) and (LC) both say $-1\equiv -1 \bmod l$.

From now on, we assume that $r\geq 2$.

We close this section with an observation concerning the $\mathfrak{p}_j$-value of an element $\alpha\in K^\times$ and the congruence class of α^{g_0-1} modulo $\mathfrak{p}_j$. To do so, we first define

$$m_j=\frac{p_j-1}{l},\qquad 1\leq j\leq r,$$

and recall that $(q_0,K_{\mathfrak{p}_j}/\mathbb{Q}_{p_j})=g_0^{c_{0j}}$.

LEMMA 5.3. *The $\mathfrak{p}_j$-value a_j of an element $\alpha \in K^\times$ is determined modulo l by the congruence*

$$\alpha^{c_{0j}(g_0-1)} \equiv q_0^{-m_j a_j} \bmod \mathfrak{p}_j.$$

For the proof we view all occurring quantities as elements of the completion $K_{\mathfrak{p}_j}$, which is a totally, tamely ramified extension of $\mathbb{Q}_{p_j}$. Thus $K_{\mathfrak{p}_j} = \mathbb{Q}_{p_j}(\sqrt[l]{p_j v_j})$ with a unit $v_j \in \mathbb{Q}_{p_j}$. Abbreviate $\pi_j = \sqrt[l]{p_j v_j}$, so π_j is a prime element in $K_{\mathfrak{p}_j}$, and write $\alpha = \pi_j^{a_j} \cdot v$ with a unit v. Then

$$\begin{aligned}(\alpha^{g_0-1})^{c_{0j}} &\equiv (\alpha^{g_0-1})^{1+g_0+\ldots+g_0^{c_{0j}-1}} = \alpha^{g_0^{c_{0j}}-1} = (\pi_j^{a_j} v)^{g_0^{c_{0j}}-1} \equiv (\pi_j^{a_j})^{g_0^{c_{0j}}-1} \\ &= \pi_j^{a_j((q_0, K_{\mathfrak{p}_j}/\mathbb{Q}_{p_j})-1)} \overset{(1)}{=} (p_j v_j, q_0)^{a_j}_{\mathbb{Q}_{p_j},l} = (p_j, q_0)^{a_j}_{\mathbb{Q}_{p_j},l} (v_j, q_0)^{a_j}_{\mathbb{Q}_{p_j},l} \\ &\overset{(2)}{\equiv} q_0^{-m_j a_j} \cdot 1,\end{aligned}$$

with the equality (1) and congruence (2) coming from [Se, pp. 215–217].

6. Euler systems. The purpose of this section is to recall some basic properties of Euler systems. The general reference is [Ru].

Let $K \subset \mathbb{Q}(\zeta_n)^+$, with n denoting the conductor of K, and let Q abbreviate squarefree products of rational primes q splitting in K. For each such q we fix a generator σ_q of $\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$ which whenever convenient is also regarded as a generator of the Galois group G_q of the extension $K(\zeta_q)/K$. With this notation we have

$$(6.1)\qquad D_q := \sum_{i=1}^{q-1} i\sigma_q^i \in \mathbb{Z}[G_q] \quad \text{satisfying} \quad (\sigma_q - 1)D_q = q - 1 - \widehat{\sigma}_q.$$

Set $D_Q = \prod_{q|Q} D_q \in \mathbb{Z}[G_Q]$ where $G_Q = \mathrm{Gal}(K(\zeta_Q)/K)$ is identified with $\prod_{q|Q} G_q$ in the usual way.

Our Euler system is $Q \mapsto \xi_Q = N_{\mathbb{Q}(\zeta_n,\zeta_Q)/K(\zeta_Q)}(1-\zeta_n\zeta_Q)$. It satisfies ES 1–4 in [Ru]. We now fix an odd prime $l \nmid n$ and a high power L of it and use the notation $\mathfrak{b}_1 =_L \mathfrak{b}_2$ for ideals $\mathfrak{b}_1, \mathfrak{b}_2$ of K in order to indicate that $\mathfrak{b}_1\mathfrak{b}_2^{-1}$ is an Lth power of an ideal in K. In what follows the primes q not only split in K but also satisfy $q \equiv 1 \bmod L$.

Assume that $\alpha \in K^\times$ is prime to q. Then there is a $\beta \in K(\zeta_q)^\times$ such that $\alpha \equiv \beta^{1-\sigma_q} \bmod \mathfrak{Q}$ for all prime ideals $\mathfrak{Q} \mid q$ of $K(\zeta_q)$. We define the ideal $\varphi_q(\alpha)$ by

$$(6.2)\qquad \varphi_q(\alpha) =_L \prod_{\mathfrak{q}|q} \mathfrak{q}^{v_{\mathfrak{q}}(\beta)}\ (^{13}).$$

So φ_q is a $\mathrm{Gal}(K/\mathbb{Q})$-homomorphism taking values in the ideals of K sup-

[13] $v_{\mathfrak{q}}(\beta)$ is the $\mathfrak{Q}$-value of β for the unique prime $\mathfrak{Q} \mid \mathfrak{q}$ in $K(\zeta_q)$.

ported in q, modulo Lth powers of ideals; it is in fact the precise analogue of the φ_q in the situation of [Ru].

The following holds:

(6.3) $\xi_Q^{D_Q(\sigma-1)}$ is an Lth power in $K(\zeta_Q)$ for all $\sigma \in G_Q$.

(6.4) $\sigma \mapsto \sqrt[L]{\xi_Q^{D_Q(\sigma-1)}}$ is a split 1-cocycle, so yields a unique $\kappa_Q \in K^\times/K^{\times L}$ with $\kappa_Q \equiv \xi_Q^{D_Q} \bmod K(\zeta_Q)^{\times L}$.

As in [Ru], we set $\kappa_1 = \xi_1$.

(6.5) The q-part $(\kappa_Q)_q$ in the principal ideal (κ_Q) is

$$=_L \begin{cases} (1) & \text{if } q \nmid Q, \\ \varphi_q(\kappa_{Q/q}) & \text{if } q \mid Q. \end{cases}$$

(6.6) Set $G = \mathrm{Gal}(K/\mathbb{Q})$. Let W be a finite G-submodule of $K^\times/K^{\times L}$ and $\psi : W \to \mathbb{Z}/L[G]$ a G-homomorphism. Then, to a given ideal class $\mathfrak{C}$ of K, there exist a unit $u \in (\mathbb{Z}/L)^\times$ and infinitely many primes $\mathfrak{q}$ of K such that $\mathfrak{q} \in \mathfrak{C}$, the rational prime $q \in \mathfrak{q}$ splits in K and is $\equiv 1 \bmod L$, $(w)_q =_L 1$ and $\varphi_q(w) =_L \mathfrak{q}^{u\psi(w)}$ for all $w \in W$.

The following lemma concentrates on the unit u in (6.6). Again, $G = \mathrm{Gal}(K/\mathbb{Q})$.

LEMMA 6.1. *Let q_0 be a rational prime which is unramified in K. Assume that we are given a triple L, W, ψ as in* (6.6) *with $q_0 \in W$, $\psi(q_0) = \widehat{G}$ and that in accordance with these data and a given ideal class $\mathfrak{C}$ in K a prime $\mathfrak{q}$ has been picked. Then*

(i) *$(q_0, \mathbb{Q}(\zeta_q)/\mathbb{Q})$ generates* $\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$ *modulo Lth powers.*

(ii) *If the restriction of σ_q to $\mathbb{Q}(\zeta_q)$ and $(q_0, \mathbb{Q}(\zeta_q)/\mathbb{Q})^{-1}$ differ by an Lth power, then the corresponding φ_q satisfies $\varphi_q(w) =_L \mathfrak{q}^{\psi(w)}$ for all $w \in W$.*

(iii) *Assuming σ_q chosen as in* (ii) *and $\psi(a) = t\widehat{G}$ for a rational $a \in W$, the automorphisms $\sigma_q^t|_{\mathbb{Q}(\zeta_q)}$ and $(a, \mathbb{Q}(\zeta_q)/\mathbb{Q})^{-1}$ differ by an Lth power.*

Proof. For the chosen generator σ_q of G_q define $s \in \mathbb{Z}/(q-1)$ by

$$(q_0, \mathbb{Q}(\zeta_q)/\mathbb{Q})^{-1} = \sigma_q^s|_{\mathbb{Q}(\zeta_q)}.$$

Then $\varphi_q(q_0) =_L \mathfrak{q}^{u\psi(q_0)} = \mathfrak{q}^{u\widehat{G}} = (q^u)$ and so, by (6.2),

$$\prod_{\text{all } \mathfrak{q} \mid q} \mathfrak{q}^{v_\mathfrak{q}(\beta)} =_L (q^u) \quad \text{for any } \beta \in K(\zeta_q)^\times$$

with $q_0 \equiv \beta^{1-\sigma_q} \bmod \mathfrak{Q}$ for all $\mathfrak{Q} \mid q$. Now,

$$(1-\zeta_q)^{(q_0, \mathbb{Q}(\zeta_q)/\mathbb{Q})-1} = \frac{1-\zeta_q^{q_0}}{1-\zeta_q} = 1 + \zeta_q + \ldots + \zeta_q^{q_0-1} \equiv q_0 \bmod (1-\zeta_q)$$

implies

$$[(1-\zeta_q)^s]^{1-\sigma_q} \equiv (1-\zeta_q)^{(1-\sigma_q)(1+\sigma_q+\ldots+\sigma_q^{s-1})} = (1-\zeta_q)^{1-\sigma_q^s}$$
$$= [(1-\zeta_q)^{(q_0,\mathbb{Q}(\zeta_q)/\mathbb{Q})-1}]^{(q_0,\mathbb{Q}(\zeta_q)/\mathbb{Q})^{-1}} \equiv q_0 \bmod (1-\zeta_q).$$

Hence we may take $\beta = (1-\zeta_q)^s$ and obtain

$$(q^s) = \prod_{\text{all } \mathfrak{q}|q} \mathfrak{q}^{s\cdot v_{\mathfrak{q}}(1-\zeta_q)} =_L (q^u)$$

and then $s \equiv u \bmod L$. Thus s is a unit $\bmod L$ and (i) is proved.

Letting $s' \in (\mathbb{Z}/(q-1))^\times$ satisfy $s' \equiv s \bmod L$, we arrive at a new generator $\sigma'_q = \sigma_q^{s'}$ of G_q and at a corresponding φ'_q with $\varphi'_q(\alpha^{s'}) = \varphi_q(\alpha)$ for all $\alpha \in K^\times$ prime to q. Indeed, $\beta^{1-\sigma_q} \equiv \alpha \bmod \mathfrak{Q}$ implies $\beta^{1-\sigma'_q} = (\beta^{1-\sigma_q})^{1+\sigma_q+\ldots+\sigma_q^{s'-1}} \equiv (\beta^{1-\sigma_q})^{s'} \equiv \alpha^{s'} \bmod \mathfrak{Q}$. Thus, for $w \in W$, $\varphi'_q(w)^{s'} =_L \mathfrak{q}^{u\psi(w)} =_L \mathfrak{q}^{s'\psi(w)}$, which gives (ii).

In order to see (iii) we go back to the beginning of our proof and replace q_0 by a, s by t' and assume $u = 1$. Then $\varphi_q(a) =_L \mathfrak{q}^{t\widehat{G}} = (q^t)$ and we conclude that $t' \equiv t \bmod L$. Observe that (iii) implies that, modulo q, a and q_0^t only differ by an Lth power.

7. The case $r = 2$. In this section we turn to the special case

$$K \subset \mathbb{Q}(\zeta_n), \qquad n = p_1p_2, \qquad [K:\mathbb{Q}] = l \nmid n$$

and prove conjecture (LC) as stated in Section 5 (and thus, at the same time, (C)). The notation is the one of the previous sections with

- q_0 chosen to simultaneously generate the Sylow l-subgroups of the multiplicative residue groups modulo p_1 and p_2, and to be inert in K ([14]),
- L a power of l which is greater than the power of l in $p_1 - 1$ and $p_2 - 1$.

Set $W = \langle \alpha_\infty, q_0, p_1, p_2\rangle K^{\times Ll^h}$ and let $\psi : W \to \mathbb{Z}/Ll^h[G]$ be the G-map assigning α_∞ to 1, q_0 to $\widehat{G}$ and p_j to $t_j\widehat{G}$ ($j = 1,2$) where the integers t_j are chosen to satisfy

$$t_1 v_{\mathfrak{p}_1}(\alpha_\infty) + t_2 v_{\mathfrak{p}_2}(\alpha_\infty) \equiv 1 \bmod Ll^h.$$

This congruence ensures that ψ respects every relation $\alpha_\infty^x q_0^{z_0} p_1^{z_1} p_2^{z_2} \in K^{\times Ll^h}$ with $x \in \mathbb{Z}G$ and integers z_0, z_1, z_2. In fact, first of all $\alpha_\infty^{(g_0-1)x}$ is an Ll^hth power in $K^\times$ and, being a unit, therefore in E_K. Read in $\mathbb{Z}_l \otimes E_K$, the proof of Lemma 2.2 yields an $x_1 \in \mathbb{Z}_l G$ such that $\alpha_\infty^{(g_0-1)x} = \alpha_\infty^{(g_0-1)x_1 Ll^h}$, i.e.,

([14]) The compositum $\widehat{K}$ of the extensions $K_j \subset \mathbb{Q}(\zeta_j)$ of degree l over $\mathbb{Q}$ contains a field K_0 of degree l which is different from K_1, K, K_2 as $l \neq 2$, and we choose q_0 so that the Frobenius automorphism at q_0 generates $\mathrm{Gal}(\widehat{K}/K_0)$.

$x - x_1 Ll^h = x_2 \widehat{G}, x_2 \in \mathbb{Z}_l G$. It follows that

$$\alpha_\infty^{\widehat{G}x_2} q_0^{z_0} p_1^{z_1} p_2^{z_2} = p_1^{x_2 v_{\mathfrak{p}_1}(\alpha_\infty)+z_1} p_2^{x_2 v_{\mathfrak{p}_2}(\alpha_\infty)+z_2} q_0^{z_0} \in K^{\times Ll^h} \cap \mathbb{Q}^\times = \mathbb{Q}^{\times Ll^h},$$

so $Ll^h \mid z_0,\ Ll^h \mid x_2 v_{\mathfrak{p}_j}(\alpha_\infty) + z_j$ and the above relation is a consequence of

$$\alpha_\infty^{\widehat{G}} p_1^{-v_{\mathfrak{p}_1}(\alpha_\infty)} p_2^{-v_{\mathfrak{p}_2}(\alpha_\infty)} \in K^{\times Ll^h},$$

which is respected by ψ.

In accordance with the data Ll^h, W and ψ we employ (6.6) and choose a prime $\mathfrak{q}_1 \in \mathfrak{C}_1$ so that $N_{K/\mathbb{Q}}\mathfrak{q}_1 = q_1 \equiv 1 \bmod Ll^h$ splits in K and $(\kappa_{q_1}) =_{Ll^h} \varphi_{q_1}(\kappa_1) =_{Ll^h} \mathfrak{q}_1^{\psi(\kappa_1)}$, by (6.5) and Lemma 6.1 with an appropriate choice of σ_{q_1}.

Now, $\kappa_1 = \xi_1 = \alpha_\infty^{(g_0-1)^h}$. This is seen as follows. First, $\xi_1^{g_0-1} = \xi_K^{g_0-1}$ because

$$\xi_K = \xi_1 \cdot N_{\mathbb{Q}(\zeta_n)/K}((1-\zeta_n^{p_1})(1-\zeta_n^{p_2}))$$

with

$$\begin{aligned} N_{\mathbb{Q}(\zeta_n)/K}(1-\zeta_n^{p_j}) &= N_{\mathbb{Q}(\zeta_n)/K}(1-\zeta_{p_{j'}}) \\ &= N_{K(\zeta_{p_{j'}})/K} N_{\mathbb{Q}(\zeta_n)/K(\zeta_{p_{j'}})}(1-\zeta_{p_{j'}}) = p_{j'}^{m_j} \end{aligned}$$

where $\{j, j'\} = \{1, 2\}$ and $m_j = (p_j - 1)/l$. Second, by Lemma 2.2, $\xi_1^{g_0-1} = \alpha_\infty^{(g_0-1)^{h+1}}$, which gives the assertion as ξ_1 and $\alpha_\infty^{(g_0-1)^h}$ both have norm 1.

We therefore get $(\kappa_{q_1}) = \mathfrak{q}_1^{(g_0-1)^h} \cdot \mathfrak{r}^{Ll^h}$ for some ideal $\mathfrak{r}$. Since $\mathfrak{r}^{Ll^h}$ is principal, $\mathfrak{r}^{l^h}$ is principal, $\mathfrak{r}^{l^h} = (\rho)$ say. Replacing κ_{q_1} by $\kappa_{q_1}\rho^{-L}$ we may assume

$$(\kappa_{q_1}) = \mathfrak{q}_1^{(g_0-1)^h}.$$

It is important to note that this κ_{q_1} is obtained as a splitting of a cocycle as in (6.4), although with L and no longer with Ll^h.

The element κ_{q_1} has norm 1 in $\mathbb{Q}$ (after multiplying it with $-1 = (-1)^L$ if necessary), so there exists an $\alpha_{\mathfrak{q}_1} \in K^\times$ such that $\alpha_{\mathfrak{q}_1}^{g_0-1} = \kappa_{q_1}$. Then $(\alpha_{\mathfrak{q}_1}) = \mathfrak{q}_1^{(g_0-1)^{h-1}}\mathfrak{a}_1$ with $\mathfrak{a}_1$ (after multiplying $\alpha_{\mathfrak{q}_1}$ by a rational number) supported on $\mathfrak{p}_1, \mathfrak{p}_2$. The notation $\alpha_{\mathfrak{q}_1}$ can now be interpreted in the sense of Section 5.

Proposition 7.1. *$\kappa_{q_1} \equiv q_1^{m_j t_{j'}} \bmod \mathfrak{p}_j$, where $j' \neq j$.*

Recall that $m_j = (p_j - 1)/l$. The proof of the proposition is delayed to the end of this section.

Because $(q_1, K_{\mathfrak{p}_j}/\mathbb{Q}_{p_j})^{c_{0j}} = (q_0, K_{\mathfrak{p}_j}/\mathbb{Q}_{p_j})^{c_{1j}}$, $q_1^{c_{0j}}$ and $q_0^{c_{1j}}$ differ by a norm, so by an lth power modulo p_j, and we deduce $q_1^{m_j c_{0j}} \equiv q_0^{m_j c_{1j}} \bmod p_j$. Substituting this in Proposition 7.1 we have

$$\alpha_{\mathfrak{q}_1}^{c_{0j}(g_0-1)} \equiv q_0^{m_j t_{j'} c_{1j}} \bmod \mathfrak{p}_j.$$

We compare this congruence with the one given in Lemma 5.3:

$$\alpha_{\mathfrak{q}_1}^{c_{0j}(g_0-1)} \equiv q_0^{-m_j v_{\mathfrak{p}_j}(\alpha_{\mathfrak{q}_1})} \bmod \mathfrak{p}_j.$$

By the choice of $q_0, q_0^{m_j} \not\equiv 1 \bmod p_j$, whence

$$-v_{\mathfrak{p}_j}(\alpha_{\mathfrak{q}_1}) \equiv t_{j'} c_{1j} \bmod l.$$

The reciprocity law gives

$$\sum_{j=1}^{2} c_{ij} \equiv \begin{cases} -1, & i=0, \\ 0, & i \neq 0, \end{cases}$$

thus

$$-c_{12} \equiv \det \begin{pmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{pmatrix} \equiv c_{11} \bmod l$$

and

$$\begin{aligned} \det \begin{pmatrix} -v_{\mathfrak{p}_1}(\alpha_\infty) & -v_{\mathfrak{p}_2}(\alpha_\infty) \\ -v_{\mathfrak{p}_1}(\alpha_{\mathfrak{q}_1}) & -v_{\mathfrak{p}_2}(\alpha_{\mathfrak{q}_1}) \end{pmatrix} &\equiv \det \begin{pmatrix} -v_{\mathfrak{p}_1}(\alpha_\infty) & -v_{\mathfrak{p}_2}(\alpha_\infty) \\ t_2 c_{11} & t_1 c_{12} \end{pmatrix} \\ &\equiv c_{11} \det \begin{pmatrix} -v_{\mathfrak{p}_1}(\alpha_\infty) & -v_{\mathfrak{p}_2}(\alpha_\infty) \\ t_2 & -t_1 \end{pmatrix} \\ &\equiv c_{11}(t_1 v_{\mathfrak{p}_1}(\alpha_\infty) + t_2 v_{\mathfrak{p}_2}(\alpha_\infty)) \\ &\equiv c_{11} \equiv \det \begin{pmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{pmatrix} \bmod l, \end{aligned}$$

which proves (LC).

Proof (of Proposition 7.1). Fix a generator σ_{q_1} of G_{q_1}, as in Lemma 6.1(ii). Then $\sigma_{q_1}^{\tilde{t}_j}|_{\mathbb{Q}(\zeta_{q_1})} = (p_j, \mathbb{Q}(\zeta_{q_1})/\mathbb{Q})^{-1}$ for a unique $\widetilde{t}_j \bmod (q_1-1)$ and we have $\widetilde{t}_j \equiv t_j \bmod L$ by Lemma 6.1(iii). The Euler system in Section 6 has

$$\begin{aligned} \xi_{q_1} &= N_{\mathbb{Q}(\zeta_n,\zeta_{q_1})/K(\zeta_{q_1})}(1-\zeta_n\zeta_{q_1}) \\ &\equiv N_{\mathbb{Q}(\zeta_n,\zeta_{q_1})/K(\zeta_{q_1})}(1-\zeta_{p_{j'}}\zeta_{q_1}) \bmod (1-\zeta_{p_j}) \\ &= N_{K(\zeta_{p_{j'}},\zeta_{q_1})/K(\zeta_{q_1})} N_{\mathbb{Q}(\zeta_n,\zeta_{q_1})/K(\zeta_{p_{j'}},\zeta_{q_1})}(1-\zeta_{p_{j'}}\zeta_{q_1}) \\ &= N_{K(\zeta_{p_{j'}},\zeta_{q_1})/K(\zeta_{q_1})}(1-\zeta_{p_{j'}}\zeta_{q_1})^{m_j} \\ &= \left(\frac{1-\zeta_{q_1}^{p_{j'}}}{1-\zeta_{q_1}}\right)^{m_j} = (1-\zeta_{q_1})^{m_j((p_{j'},\mathbb{Q}(\zeta_{q_1})/\mathbb{Q})-1)} \\ &= (1-\zeta_{q_1})^{m_j(\sigma_{q_1}^{-\tilde{t}_{j'}}-1)} \end{aligned}$$

and thus

$$\xi_{q_1} \equiv (1-\zeta_{q_1})^{m_j(\sigma_{q_1}-1)(1+\sigma_{q_1}+\ldots+\sigma_{q_1}^{-\tilde{t}_{j'}-1})} \bmod (1-\zeta_{p_j}). \tag{7.1}$$

On applying D_{q_1} this implies, by (6.1),

$$\xi_{q_1}^{D_{q_1}} \equiv (1-\zeta_{q_1})^{m_j(1+\sigma_{q_1}+\ldots+\sigma_{q_1}^{-\tilde t_{j'}-1})(q_1-1-\widehat\sigma_{q_1})} \tag{7.2}$$
$$= (1-\zeta_{q_1})^{m_j(q_1-1)(1+\sigma_{q_1}+\ldots+\sigma_{q_1}^{-\tilde t_{j'}-1})} q_1^{-m_j(1+\sigma_{q_1}+\ldots+\sigma_{q_1}^{-\tilde t_{j'}-1})}$$
$$\equiv (1-\zeta_{q_1})^{m_j(q_1-1)(1+\sigma_{q_1}+\ldots+\sigma_{q_1}^{-\tilde t_{j'}-1})} \cdot q_1^{m_j t_{j'}} \bmod (1-\zeta_{p_j}).$$

By (6.4), with $\beta_{q_1} \in K(\zeta_{q_1})$ satisfying $\beta_{q_1}^{\sigma-1} = \sqrt[L]{\xi_{q_1}^{D_{q_1}(\sigma-1)}}$ we have

$$\kappa_{q_1} = \xi_{q_1}^{D_{q_1}}/\beta_{q_1}^L. \tag{7.3}$$

Because of $(\kappa_{\mathfrak q_1}) = \mathfrak q_1^{(g_0-1)^h}$ and since ξ_{q_1} is a unit, it follows that β_{q_1} is supported in the orbit of $\mathfrak q_1$ (in $K(\zeta_{q_1})$). And we have $\xi_{q_1}^{\widehat\sigma_{q_1}} =_{\mathrm{ES3}} \xi_1^{\mathrm{Frob}(q_1)-1} = 1$, as q_1 splits in K.

Thus $\xi_{q_1}^{D_{q_1}(\sigma_{q_1}-1)} = \xi_{q_1}^{q_1-1-\widehat\sigma_{q_1}} = \xi_{q_1}^{q_1-1}$ and so $\beta_{q_1}^{\sigma_{q_1}-1} = \xi_{q_1}^{(q_1-1)/L}$ which, by (7.1), gives

$$\beta_{q_1}^{\sigma_{q_1}-1} \equiv (1-\zeta_{q_1})^{\frac{q_1-1}{L} m_j(1+\sigma_{q_1}+\ldots+\sigma_{q_1}^{-\tilde t_{j'}-1})(\sigma_{q_1}-1)} \bmod (1-\zeta_{p_j}).$$

Since β_{q_1} and $1-\zeta_{q_1}$ are in $K(\zeta_{q_1})$ this congruence can be read modulo the product $\widetilde{\mathfrak p}_j$ of the primes of $K(\zeta_{q_1})$ above $\mathfrak p_j$, which implies

$$\beta_{q_1} \equiv (1-\zeta_{q_1})^{\frac{q_1-1}{L} m_j(1+\sigma_{q_1}+\ldots+\sigma_{q_1}^{-\tilde t_{j'}-1})} \gamma_{q_1} \bmod \widetilde{\mathfrak p}_j \tag{7.4}$$

with γ_{q_1} in $K(\zeta_{q_1})$ so that $\gamma_{q_1}^{\sigma_{q_1}-1} \equiv 1 \bmod \widetilde{\mathfrak p}_j$. As $\mathfrak p_j$ is unramified in $K(\zeta_{q_1})$ this means that γ_{q_1} may be taken to be in K, hence

$$\gamma_{q_1}^{p_j-1} \equiv 1 \bmod \mathfrak p_j. \tag{7.5}$$

Therefore, by (7.3), (7.2), (7.4),

$$\kappa_{q_1} \equiv q_1^{m_j t_{j'}} \cdot \gamma_{q_1}^{-L} \bmod \mathfrak p_j.$$

Since κ_{q_1} has norm 1, its lth power is congruent to 1 modulo $\mathfrak p_j$, as is the lth power of $q_1^{m_j t_{j'}}$. So $\gamma_{q_1}^{Ll} \equiv 1 \bmod \mathfrak p_j$. By (7.5) and the choice of L thus $\gamma_{q_1}^L \equiv 1 \bmod \mathfrak p_j$. This finishes the proof.

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Institut für Mathematik
Universität Augsburg
D-86135 Augsburg, Germany

Department of Mathematics
University of Alberta
Edmonton T6G 2G1, Canada

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