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On limits of PV k-tuples

by

David Garth (Manhattan, KS)

1. Introduction and definitions. Recall that an algebraic integer $\alpha > 1$ is a *Pisot number* if all of its conjugates lie in the open unit disk. Let S_1 be the set of all Pisot numbers. In 1944 Salem proved the surprising result that the set S_1 is closed [4]. Let S_2 denote the set of algebraic integers which are greater than 1 in absolute value and have exactly one other conjugate outside the unit circle, with the rest inside. S_2 can be written as a disjoint union of S_2', the set of real numbers in S_2, and S_2'', the set of complex numbers in S_2. In 1950, J. B. Kelley proved that the set $S_1 \cup S_2''$ is closed [3]. In [1], D. G. Cantor studied certain k-tuples of algebraic integers which generalize these ideas. Specifically, let $(\alpha_1, \ldots, \alpha_k)$ be a k-tuple of distinct algebraic integers, each with absolute value strictly greater than 1, and let $P(z)$ be the monic polynomial of least degree with integer coefficients which $\alpha_1, \ldots, \alpha_k$ all satisfy. If the remaining roots of $P(z)$ lie in the open unit disk, then $(\alpha_1, \ldots, \alpha_k)$ is called a *PV k-tuple*, and $P(z)$ is its *defining polynomial*. Moreover, if $P(z)$ is irreducible, then $(\alpha_1, \ldots, \alpha_k)$ is said to be an *irreducible PV k-tuple*.

Two PV k-tuples are said to be *equal* if they are equal as sets. Suppose that $(\alpha_1^{(n)}, \ldots, \alpha_k^{(n)})$ is a sequence of PV k-tuples. Suppose also that each of these k-tuples can be ordered in such a way that $\lim_{n\to\infty} \alpha_i^{(n)} = \alpha_i$, and that $\alpha_i^{(n)} \neq \alpha_i$ for $1 \le i \le k$ and for $n \ge 1$. Then $(\alpha_1^{(n)}, \ldots, \alpha_k^{(n)})$ is said to have a *limit*, and this limit is the j-tuple $(\alpha_1, \ldots, \alpha_j)$ of distinct elements among the α_i's. This definition was given in [1]. The restriction that $\alpha_i^{(n)} \neq \alpha_i$ for $1 \le i \le k$ and $n \ge 1$ is made so that each α_i in the limit j-tuple is actually a limit point of the sequence $\{\alpha_i^{(n)}\}$.

A great deal of work has been done on the study of limit points of the set S_1, and to a lesser extent S_2. Several of the results have analogs for the PV k-tuples. For instance, it was shown in [2] that totally real Pisot numbers are limit points of S_1. Theorem 5.8 of [1] shows that if $(\alpha_1, \ldots, \alpha_k)$ is an

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irreducible, totally real PV k-tuple, that is, a PV k-tuple whose defining polynomial has all real roots, then $(\alpha_1, \ldots, \alpha_k)$ is a limit of PV k-tuples. It was also shown in [2] that if $\alpha \in S_1$, then α^n is a limit point of S_1 for every $n \geq 2$. Virtually the same proof can be used to prove that if $(\alpha_1, \ldots, \alpha_k)$ is an irreducible PV k-tuple, then $(\alpha_1^n, \ldots, \alpha_k^n)$ is a limit of PV k-tuples for every $n \geq 2$. It is conjectured that the limit of a sequence of PV k-tuples is a PV j-tuple, where $j \leq k$. However, this question remains open.

In this note we will be concerned with PV k-tuples whose defining polynomials are reciprocal. Recall that a polynomial $P(z)$ of degree d is reciprocal if $P(z) = z^d P(1/z)$. We will say that a PV k-tuple is *reciprocal* if its defining polynomial is reciprocal. Vijayaraghavan showed that every reciprocal quadratic unit is a limit point of S_1 [7]. Samet showed that every reciprocal biquadratic unit in S_2, that is, every reciprocal unit of degree 4 in S_2, is a limit point of S_2 ([5] and [6]). A natural question to ask is whether reciprocal PV k-tuples are limits of PV k-tuples. While we cannot answer this question in general, we will prove the following theorem, which gives conditions under which it is true.

THEOREM 1. *Let $\vartheta = (\alpha_1, \ldots, \alpha_k)$ be an irreducible PV k-tuple, with defining polynomial $P(z)$. Suppose $P(z)$ is reciprocal. If $|\alpha_i| > (3 + \sqrt{5})/2$ for each complex α_i in ϑ, then ϑ is a limit of PV k-tuples.*

We can say more if we restrict our attention to reciprocal PV 3-tuples. It will be convenient to define S_3 as the set of algebraic integers which are greater than 1 in absolute value and have exactly two other conjugates outside the unit circle, with the rest inside. Notice that S_3 also divides naturally into 2 subsets. Let S_3' be the set of elements of S_3 whose minimal polynomials have 3 real zeros outside the unit circle, and S_3'' the set of elements of S_3 whose minimal polynomials have one real and two complex zeros outside the unit circle. Thus, $S_3 = S_3' \cup S_3''$. Notice that if $\alpha \in S_3''$ with minimal polynomial $P(z)$, then by considering $P(-z)$ we may assume without loss of generality that the real zero of $P(z)$ outside the unit circle is positive. If $\alpha \in S_3$ has a reciprocal minimal polynomial, then α is necessarily a triquadratic unit, that is, a unit of degree 6. We will prove the following two theorems.

THEOREM 2. *The reciprocal triquadratic units in S_3' are limit points of S_3'.*

THEOREM 3. *Suppose $\alpha \in S_3''$, with minimal polynomial*

$$P(z) = z^6 + az^5 + bz^4 + cz^3 + bz^2 + az + 1. \tag{1.1}$$

If either (i) $a \geq 0$, (ii) $a < 0$ *and* $b \leq 4a - 9$, *or* (iii) $a < 0$ *and* $b \geq a^2/3 + 3$ *hold, then α is a limit point of S_3''.*

We were unable to determine if Theorem 3 holds for all $a < 0$. However, we conjecture that it does. Notice that for each $a < 0$, there are only finitely many values of b for which the conclusion of Theorem 3 may not hold. It will follow from Lemmas 2.2 and 2.3 below that if $a < 0$ and b are fixed in (1.1), then there is at most one value of c for which α may not be a limit point of S_3''. Based on this fact, we have performed numerical calculations for several values of $a < 0$ and have been unable to find a counterexample.

2. Preliminary lemmas. Our main tool will be the following theorem of Cantor [1], which gives us a necessary and sufficient condition for an irreducible PV k-tuple to be a limit of PV k-tuples.

CANTOR'S THEOREM. *Let $\vartheta = (\alpha_1, \ldots, \alpha_k)$ be an irreducible PV k-tuple with defining polynomial $P(z)$. Then ϑ is a limit of PV k-tuples if and only if there exists a polynomial $A(z)$, with integer coefficients, such that $|A(z)| \le |P(z)|$ whenever $|z| = 1$, with equality at only a finite number of points.*

In order to show that a certain number $\alpha \in S_3$ is a limit point of S_3, our strategy will be to use Cantor's Theorem applied to the irreducible PV 3-tuple to which α belongs. Specifically, let $\alpha \in S_3$, with minimal polynomial $P(z)$. Let $\alpha = \alpha_1, \alpha_2, \alpha_3$ be the conjugates of α outside the unit circle. Thus, $\vartheta = (\alpha_1, \alpha_2, \alpha_3)$ is an irreducible PV 3-tuple with defining polynomial $P(z)$. Suppose we can find a polynomial $A(z)$ with integer coefficients which satisfies the conditions of Cantor's Theorem. We then get a sequence of PV 3-tuples $\vartheta_n = (\alpha_1^{(n)}, \alpha_2^{(n)}, \alpha_3^{(n)})$ which approaches ϑ. This implies in particular that α is a limit point of the sequence $\{\alpha_1^{(n)}\}$. If $\alpha_1^{(n)}$ is in S_3 for infinitely many n, it then follows that α is a limit point of S_3. Cantor's Theorem does not guarantee this, however, since ϑ_n need not be irreducible for infinitely many n. We need the following lemma which shows that in the case of PV 3-tuples, the ϑ_n are irreducible for n large enough.

LEMMA 2.1. *Let $\vartheta_n = (\alpha_1^{(n)}, \alpha_2^{(n)}, \alpha_3^{(n)})$ be a sequence of PV 3-tuples, with defining polynomials $P_n(z)$. Let $\vartheta = (\alpha_1, \alpha_2, \alpha_3)$ be an irreducible PV 3-tuple with defining polynomial $P(z)$. If ϑ is the limit of the sequence, then for n large enough, ϑ_n is an irreducible PV 3-tuple.*

Proof. Assume that there exists a subsequence $P_{n_k}(z)$ of polynomials, each of which factors into two polynomials, say $f_{n_k}(z)$ and $g_{n_k}(z)$, both of positive degree. Note that each $f_{n_k}(z)$ and $g_{n_k}(z)$ is monic with integer coefficients, by Gauss's Lemma. Also note that for each k, one of the two polynomials, say f_{n_k}, possesses exactly 1 root outside the unit circle, and the other, g_{n_k}, has exactly two roots outside the unit circle. If this were not the case, then one of the polynomials would be forced to have all of its roots strictly inside the unit circle. However, this is impossible for a monic polynomial with integer coefficients, since the product of the roots must be an integer.

Thus, each $f_{n_k}(z)$ has a Pisot number as one of its roots. Therefore, for each k, $(\alpha_1^{(n_k)}, \alpha_2^{(n_k)}, \alpha_3^{(n_k)})$ contains a Pisot number. Thus, there is an i such that $\alpha_i^{(n_k)}$ contains a subsequence of Pisot numbers. However, this subsequence of Pisot numbers approaches a root of $P(z)$. Since the Pisot numbers are closed, this root must also be a Pisot number. This is a contradiction, since $P(z)$ is the minimal polynomial of an S_3 number. Therefore $P_n(z)$ is irreducible for n large enough. ∎

The following two lemmas will be used in the proof of Theorem 3. The first is a technical lemma. The second illustrates how Cantor's Theorem will be used to show that an element of S_3'' is a limit point of S_3''.

LEMMA 2.2. *Let* $\alpha \in S_3''$, *with minimal polynomial*

$$P(z) = z^6 + az^5 + bz^4 + cz^3 + bz^2 + az + 1.$$

Let $x = z + 1/z$, *and define*

$$Q(x) = z^{-3}P(z) = x^3 + ax^2 + (b-3)x + c - 2a.$$

Then $Q(x) < 0$ *whenever* $x \le 2$. *Also*, $c < -2|a|$. *Moreover*, *if* $a = 0$, *then* $c < -2$. *Therefore* $c < -2$ *always holds.*

Proof. Since $P(z)$ has exactly two positive roots, and no roots on the unit circle, it follows from its definition that $Q(x)$ has one real root in the interval $(2, \infty)$, and two complex roots. Since $Q(x) \to \infty$ as $x \to \infty$ and $Q(x) \to -\infty$ as $x \to -\infty$, it is easy to see that

$$Q(x) < 0 \quad \text{for } x \le 2.$$

In particular, this gives the following three inequalities:

$$c - 2a = Q(0) < 0, \tag{2.1}$$

$$8 + 2a + 2(b-3) + c = Q(2) < 0, \tag{2.2}$$

$$-8 + 2a - 2(b-3) + c = Q(-2) < 0. \tag{2.3}$$

Adding (2.2) and (2.3) gives

$$2a + c < 0. \tag{2.4}$$

By (2.1) and (2.4), it follows that

$$c < -2|a|. \tag{2.5}$$

If $a \ne 0$, it follows from (2.5) that $c < -2$. Suppose $a = 0$. By (2.5), we have $c < 0$. Then, since $Q(x)$ has 1 real and 2 complex roots, its discriminant is negative, so

$$-4(b-3)^3 < 27c^2. \tag{2.6}$$

Suppose $c = -1$. Then (2.6) gives $-4(b-3)^3 < 27$, which implies that $b > 1$. But (2.2) implies

$$0 > 8 + 2(b-3) - 1 > 7 + 2(1-3) = 3 > 0,$$

a contradiction. Thus $c \neq -1$. Suppose $c = -2$. Then (2.6) gives $-4(b-3)^3 < 108$, which implies that $b > 0$. However, by (2.2), we would then have

$$0 > 8 + 2(b-3) - 2 > 6 + 2(0-3) = 0,$$

another contradiction, and so $c \neq -2$. Therefore, $c < -2$. ∎

Lemma 2.3. *Let $\alpha \in S_3''$, with minimal polynomial*

$$P(z) = z^6 + az^5 + bz^4 + cz^3 + bz^2 + az + 1.$$

Let $x = z + 1/z$, and define

$$Q(x) = z^{-3}P(z) = x^3 + ax^2 + (b-3)x + c - 2a.$$

If

$$Q(x) < -1/2 \quad \textit{for all } x \in [-2,2] \tag{2.7}$$

then α is a limit point of S_3''.

Proof. Let $\vartheta = (\alpha, \alpha_2, \alpha_3)$ be the irreducible PV 3-tuple to which α belongs. Let $A(z) = z^6 + az^5 + bz^4 + (c+1)z^3 + bz^2 + az + 1$. Let $x = z + 1/z$, and let $R(x) = z^{-3}A(z) = Q(x) + 1$. For $|z| = 1$, if $Q(x) \le -1$ then

$$|A(z)| = |R(x)| = |Q(x) + 1| < |Q(x)| = |P(z)|,$$

and if $-1 < Q(x) < -1/2$, then

$$|A(z)| = |R(x)| = Q(x) + 1 < 1/2 < -Q(x) = |Q(x)| = |P(z)|.$$

Thus, by Cantor's Theorem, ϑ is the limit of a sequence $\vartheta_n = (\alpha_1^{(n)}, \alpha_2^{(n)}, \alpha_3^{(n)})$ of PV 3-tuples. By Lemma 2.1 these PV 3-tuples are irreducible for n large enough. Also, since ϑ contains one real and two complex numbers, and since $\alpha_i^{(n)} \to \alpha_i$ as $n \to \infty$ for $1 \le i \le 3$, it follows that ϑ_n contains one real and two complex numbers for n large enough. Therefore α is a limit point of S_3''. ∎

3. Proof of Theorem 1. This theorem is actually a simple corollary of Theorem 5.8 of [1], but we will give the full details for the convenience of the reader. Let $\beta_1, \ldots, \beta_r$ be the positive real roots of $P(z)$, and let $\varrho_1, \ldots, \varrho_s$ be the negative real roots. Let $\gamma_1, \ldots, \gamma_t, 1/\gamma_1, \ldots, 1/\gamma_t$ be the complex roots, where $|\gamma_i| > 1$ for $1 \le i \le t$. Then, for $|z| = 1$, we have

$$\frac{|z-\beta_i|}{|z-1|} \ge \frac{1+\beta_i}{2} > \sqrt{\beta_i} \quad \text{and} \quad \frac{|z-\varrho_i|}{|z+1|} \ge \frac{1+\varrho_i}{2} > \sqrt{|\varrho_i|}.$$

Also, if $|z| = 1$, we have

$$|z-\gamma_i| \cdot \left|z - \frac{1}{\gamma_i}\right| \ge (|\gamma_i| - 1)\left(1 - \frac{1}{|\gamma_i|}\right) = \frac{(|\gamma_i|-1)^2}{|\gamma_i|}.$$

Since $|\gamma_i| > (3+\sqrt{5})/2$, it follows that $|\gamma_i|^2 - 3|\gamma_i| + 1 > 0$, and hence $(|\gamma_i| - 1)^2 > |\gamma_i|$. Therefore, we have

$$\frac{|P(z)|}{|z-1|^r|z+1|^s} = \prod_{i=1}^{r} \frac{|z-\beta_i|}{|z-1|} \prod_{i=1}^{s} \frac{|z-\varrho_i|}{|z+1|} \prod_{i=1}^{t} |z-\gamma_i| \cdot \left|z - \frac{1}{\gamma_i}\right| > \prod_{i=1}^{r} \sqrt{\beta_i} \prod_{i=1}^{s} \sqrt{\varrho_i} = 1.$$

Thus, $|P(z)| > |z-1|^r|z+1|^s$ whenever $|z| = 1$. Therefore Theorem 1 follows from Cantor's Theorem, with $A(z) = (z-1)^r(z+1)^s$.

4. Proofs of Theorems 2 and 3. Let $\alpha \in S_3$ be a reciprocal triquadratic unit with minimal polynomial $P(z)$. Let $\alpha = \alpha_1, \alpha_2, \alpha_3$ be the conjugates of α which lie outside the unit circle. Then $(\alpha_1, \alpha_2, \alpha_3)$ is an irreducible PV 3-tuple.

If $\alpha \in S_3'$, then all the roots of $P(z)$ are real. Since totally real PV k-tuples are limits of PV k-tuples, $(\alpha_1, \alpha_2, \alpha_3)$ is the limit of a sequence $(\alpha_1^{(n)}, \alpha_2^{(n)}, \alpha_3^{(n)})$ of PV 3-tuples. By Lemma 2.1, $(\alpha_1^{(n)}, \alpha_2^{(n)}, \alpha_3^{(n)})$ is irreducible for n large enough. It is clear from the definition of a limit of PV k-tuples that, for n large enough, $\alpha_i^{(n)}$ is real for $1 \le i \le 3$. Thus, α is a limit point of S_3'. This proves Theorem 2.

Suppose now that $\alpha \in S_3''$. Let $x = z + 1/z$, and define

$$Q(x) = z^{-3}P(z) = x^3 + ax^2 + (b-3)x + c - 2a.$$

To prove case (i) of Theorem 3, we will consider the following six cases separately:

1. $a \ge 0$ and $b \ge 0$.
2. $a \ge 2$ and $b < 0$.
3. $a = 0$ and $b \le -9$.
4. $a = 1$ and $b \le -5$.
5. $a = 0$ and $-9 \le b \le -1$.
6. $a = 1$ and $-5 \le b \le -1$.

For the first four cases we will show that (2.7) holds. It will then follow from Lemma 2.3 that α is a limit point of S_3''. The last two cases will require a little more care.

Suppose that case 1 holds. By (2.2) we have, for $x \in [-2, 2]$,

$$\begin{aligned} Q(x) &\le x^3 + ax^2 + (b-3)x - 8 - 2a - 2(b-3) - 2a - 1 \\ &= (x^3 - 3x - 2) + a(x^2 - 4) + b(x-2) - 1 \\ &= (x-2)(x+1)^2 + a(x^2 - 4) + b(x-2) - 1. \end{aligned} \tag{4.1}$$

Since $a \ge 0$ and $b \ge 0$, the maximum value of (4.1) on $[-2, 2]$ is -1, which occurs when $x = 2$. Therefore (2.7) holds for case 1.

We now turn to cases 2, 3, and 4. Notice that

$$Q'(x) = 3x^2 + 2ax + (b-3). \tag{4.2}$$

Thus the critical points of $Q(x)$ occur at

$$x = \frac{-a \pm \sqrt{a^2 - 3(b-3)}}{3}.$$

For a and b satisfying the conditions in cases 2, 3, or 4, we see that $Q(x)$ has two critical points, one in $(-\infty, -2]$ and one in $[-2, \infty)$. Since $Q(x) < 0$ for $x \in (-\infty, 2]$, and since $Q(x) \to \infty$ as $x \to \infty$, it follows that the critical point in $[-2, \infty)$ cannot be a local maximum. Hence, for a, b satisfying the conditions in cases 2, 3, or 4, we see that $Q(x)$ attains its maximum on $[-2, 2]$ at one of the endpoints. Since $Q(-2)$ and $Q(2)$ are both integers, we have, for $x \in [-2, 2]$, $Q(x) \le -1$. Therefore (2.7) holds for cases 2, 3, and 4.

We now turn our attention to the last two cases. Suppose that a and b satisfy either 5 or 6. Let c_0 be the largest integer for which the polynomial

$$P_{c_0}(z) = z^6 + az^5 + bz^4 + c_0 z^3 + bz^2 + az + 1$$

is the minimal polynomial of an element of S_3''. c_0 exists by Lemma 2.2, and is less than -2. Define $Q_{c_0}(x) = z^{-3} P_{c_0}(z)$. By Lemma 2.2, $Q_{c_0}(x) < 0$ whenever $x \le 2$. If $c < c_0$, then $Q(x) \le Q_{c_0}(x) - 1 < -1$ whenever $x \le 2$, and so by Lemma 2.3, α is a limit point of S_3''. Therefore, we need only consider the case $c = c_0$.

There are only 14 pairs of a and b which satisfy conditions 5 and 6. We need to find the value of c_0 for each pair. This can be accomplished by routine computation. Once these values are found, it is again a routine computation to determine which of the polynomials $Q_{c_0}(x)$ satisfy condition (2.7) of Lemma 2.3. It turns out that of the 14 polynomials, only the polynomials

$$z^6 - 3z^4 - 6z^3 - 3z^2 + 1 \quad \text{and} \quad z^6 - 5z^4 - 9z^3 - 5z^2 + 1$$

yield polynomials $Q_{c_0}(x)$ which do not satisfy this condition. For these two polynomials we appeal to Cantor's Theorem directly. Let

$$A(z) = z^6 + z^5 - z^4 - 3z^3 - z^2 + z + 1.$$

It is not hard to show that for both of the above polynomials, $|A(z)| \le |P(z)|$ whenever $|z| = 1$, with equality for exactly two values of z. We will demonstrate this for the first polynomial only. The proof for the other polynomial is similar. If $P(z) = z^6 - 3z^4 - 6z^3 - 3z^2 + 1$, then for $x = z + 1/z$, $Q(x) = x^3 - 6x - 6$. If $R(x) = z^{-3}A(z)$, then $R(x) = x^3 + x^2 - 4x - 5$. Notice that $R(x) - Q(x) = (x+1)^2$. This difference is positive unless $x = -1$. Thus, for $x \in [-2, 2]$, we have $Q(x) \le R(x)$, with equality only at $x = -1$. Moreover, it is not hard to see that $R(x) < 0$ for $x \in [-2, 2]$. Therefore, if $|z| = 1$, then we have

$$|P(z)| = |Q(x)| \ge |R(x)| = |A(z)|,$$

with equality only if $z = -1/2 \pm \sqrt{3}\,i/2$. Thus, by Cantor's Theorem the irreducible PV 3-tuple to which α belongs is a limit of PV 3-tuples. Using a similar argument as in the end of the proof of Lemma 2.3 we see that α is in fact a limit point of S_3''. This completes the proof of Theorem 3, part (i).

To prove part (ii), suppose that $a < 0$ and $b \le 4a - 9$. Then

$$\sqrt{a^2 - 3(b-3)} \ge \sqrt{a^2 - 12a + 36} = |a - 6| = 6 - a.$$

Hence,

$$\frac{-a - \sqrt{a^2 - 3(b-3)}}{3} \le \frac{-a - (6-a)}{3} = -2.$$

Therefore the local maximum of $Q(x)$ occurs when $x \le -2$. Since there are no other local maximums of $Q(x)$, it follows that the maximum of $Q(x)$ on the interval $[-2, 2]$ occurs at one of the endpoints. Now $Q(-2)$ and $Q(2)$ are negative integers, by Lemma 2.2. Thus, $Q(x)$ satisfies (2.7), and so by Lemma 2.3, α is a limit point of S_3''.

Finally, suppose $a < 0$ and $b \ge a^2/3 + 3$. Then $a^2 \le 3(b-3)$, and so the discriminant of $Q'(x)$ is negative or zero. Thus, $Q(x)$ is increasing, and so it achieves its maximum on the interval $[-2, 2]$ when $x = 2$. Therefore $Q(x) \le Q(2) \le -1$ for all $x \in [-2, 2]$ and it follows from Lemma 2.3 that α is a limit point of S_3''. This completes the proof of the theorem.

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Department of Mathematics
Kansas State University
Manhattan, KS 66506, U.S.A.
E-mail: drgarth@math.ksu.edu

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