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A note on the generalized $3n+1$ problem

by

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Introduction. We will study here some features of the generalized Collatz problem, i.e., given two natural numbers d, m with $m > d \geq 2$ and $\gcd(d,m) = 1$, let R_d be a complete system of non-zero residues modulo d and $\varphi : \mathbb{N} \to R_d$ the canonical projection of $\mathbb{N}$ in R_d. Then we define the Hasse function $H : \mathbb{N} \to \mathbb{N}$ by [1]

$$H(x) = \begin{cases} x/d & \text{if } x \equiv_d 0, \\ (mx - \varphi(mx))/d & \text{otherwise}, \end{cases} \tag{1}$$

and we investigate the dynamics of the orbits of x by H.

We will consider here the case $m < d^{d/(d-1)}$. An old conjecture states that, in this situation, for all $x \in \mathbb{N}$ the orbit of x is bounded.

We remember that if $d = 2$ and $R_d = \{0, -1\}$ then we have the classical Collatz problem, also called the *Syracuse problem*, or $3n+1$ *problem*. In this case we call H the *Collatz function* and denote it by T.

A very good recent review of the state of art in this problem can be found in Chapter 1 of Wirsching's book [Wir98]. We will present here only a brief discussion of some questions related to our work.

Two natural problems arise:

(i) *How "large" can the set of all "different" orbits of H be?*
(ii) *If the conjecture is false, how can an unbounded trajectory of H grow?*

In a classical 1985 paper, Lagarias [Lag85] shows that (for the $3n+1$ case) there exist $c_1 > 0$ and $\eta \in (0,1)$ such that

$$\#\{n \in \mathbb{N} : n \leq x,\ T^k(n) > n,\ \forall k \geq 1\} \leq c_1 x^{1-\eta}.$$

From this result, it is reasonable to claim that *if* there exists an unbounded trajectory for this case then it cannot grow too slowly. In fact, in

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[1] In this note we use $\mathbb{N}$ to denote the set of non-negative integers (including 0) and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Corollary 1 of Section 2, we show that this is true, in the sense of Banach density (for the definitions of density and Banach density of a subset of $\mathbb{N}$ see Section 1).

In another work related to the second question, Korec [Kor94] proved, also for the Collatz function, that the set

$$M_c = \{y \in \mathbb{N} : \exists n \in \mathbb{N},\ T^n(y) < y^c\}$$

has density one for all $c > \log_4 3$.

For the Hasse function H, when $m < d^{d/(d-1)}$, the important result of Heppner [Hep78], which we will state in Section 1, shows that Korec's result is true in this situation for some $c_0 \in (0,1)$. However, unlike Korec's result, we do not have an estimate for c_0 in this case.

As to the first question, Korec and Znám in [KZ87] defined an equivalence relation in $\mathbb{N}$ by

$$a \sim_1 b \quad \text{iff} \quad \text{there are integers } n \text{ and } m \text{ such that } T^n(a) = T^m(b),$$

and showed that a complete set of representatives of $\mathbb{N}/{\sim_1}$ has density zero. Although this was proved for the $3n+1$ context, it is not difficult to extend it to the general situation of the Hasse function H, when $m < d^{d/(d-1)}$.

In our work we shall consider this general situation, i.e., the function H when $m < d^{d/(d-1)}$, and we will improve the result of Korec and Znám; precisely, we consider a stronger relation in $\mathbb{N}$,

$$a \sim b \quad \text{iff} \quad \text{there is an integer } k \text{ such that } H^k(a) = H^k(b),$$

and we prove that a complete set of representatives of $\mathbb{N}/{\sim}$ has density zero. Moreover, we show (Theorem 1) that such a set has Banach density zero.

A direct consequence is that *any orbit* $\mathcal{O}(n)$ *under* H has Banach density zero (Corollary 1). This gives a more precise answer to question (ii) above as we give here a direct measure of the orbits of H.

This paper comprises this introduction and 2 more sections. In Section 1 we shall state the basic definitions and state some fundamental results that we will need later in the text. In Section 2 we will develop the necessary tools to prove Theorem 1.

1. Basic results. Consider, as in the introduction, integers m, d with $m > d \geq 2$. Suppose that $\gcd(m,d) = 1$ and $m < d^{d/(d-1)}$. Let R_d be a complete system of non-zero residues modulo d and $\varphi : \mathbb{N} \to R_d$ the canonical projection of $\mathbb{N}$ in R_d.

We will study the dynamics induced in the set $\mathbb{N}^*$ of positive integers by Hasse's function $H : \mathbb{N}^* \to \mathbb{N}^*$ defined by (1).

Since we are interested in studying "how large some subsets of $\mathbb{N}$ are" (or "how small they are"), we introduce the concept of Banach density of a subset of $\mathbb{N}$. First, consider the simpler (and more usual) concept of density.

Definition 1. A subset $B \subset \mathbb{N}$ has *density* μ if

$$\lim_{n\to\infty} \frac{\#(B\cap\{1,\ldots,n\})}{n} = \mu.$$

When this limit exists it will be denoted by $\varrho(B)$. Although this concept is very "natural", we will use in this article a more subtle concept, which gives a more uniform measure of the "size" of B.

Definition 2. The *Banach density* of a subset $B \subset \mathbb{N}$ is

$$\limsup_{n\to\infty}\left(\max_{a\in\mathbb{N}^*} \frac{\#(B\cap\{a,\ldots,a+n-1\})}{n}\right).$$

The Banach density of B will be denoted by $\varrho_b(B)$.

Of course, the Banach density of B always exists and if $\varrho(B)$ and $\varrho_b(B)$ exist then $\varrho(B) \le \varrho_b(B)$. Therefore, in order to show that B is "small" the information $\varrho_b(B) = 0$ is more significant than $\varrho(B) = 0$.

We now start the study of the dynamics of H.

The following function $\ell : \mathbb{N}\times\mathbb{N}^* \to \mathbb{N}$ will play an important role in this note:

$$\ell(n,k) = \#\{0 \le s \le k-1 : H^s(n) \equiv 0 \pmod d\}. \tag{2}$$

Lemma 1. *If n, k and r are positive integers then*

$$H^k(n+rd^k) = H^k(n) + rm^{k-\ell(n,k)}.$$

P r o o f. We proceed by induction in k. The case $k = 0$ is obvious. Assume the result for $k-1$. Then

$$\begin{aligned} H^k(n+sd^k) &= H(H^{k-1}(n+dsd^{k-1})) \\ &= H(H^{k-1}(n) + dsm^{k-1-\ell(n,k-1)}). \end{aligned} \tag{3}$$

Now we note that $H^{k-1}(n) \equiv H^{k-1}(n) + dsm^{k-1-\ell(n,k-1)} \pmod d$, so we have:

(i) If $H^{k-1}(n) \equiv 0 \pmod d$ then $H^k(n) = H^{k-1}(n)/d$ and $\ell(n,k) = \ell(n,k-1)+1$, and, by the definition of H,

$$H^k(n+sd^k) = \frac{H^{k-1}(n)}{d} + sm^{k-1-\ell(n,k-1)} = H^k(n) + sm^{k-\ell(n,k)}.$$

(ii) If $H^{k-1}(n) \not\equiv 0 \pmod d$ then $\ell(n,k) = \ell(n,k-1)$ and a simple calculation shows that

$$H^k(n+sd^k) = H^k(n) + sm^{k-\ell(n,k)}. \blacksquare$$

As a direct consequence we have

Lemma 2. *If $H^k(n) = H^k(r)$ and $\ell(n,k) = \ell(r,k)$ then for all s*

$$H^k(n+sd^k) = H^k(r+sd^k).$$

Now we state an important result of Heppner.

Proposition 1 (Heppner). *Let* m, d, R_d *and* H *be as above, with* $m < d^{d/(d-1)}$. *There exist* $\delta_1 = \delta_1(m,d)$ *and* $\delta_2 = \delta_2(m,d)$ *in* $(0,1)$ *such that, if* $N(k) = \lfloor \log_d(k) \rfloor$ *and* $g(k) = \#\{n \le k : H^{N(k)}(n) \ge nk^{-\delta_1}\}$, *then* $g(k)$ *is* $O(k^{\delta_2})$.

The reader can find the proof of this proposition in [Hep78].

We will use this result on several occasions in this paper, the first time to obtain

Proposition 2. *Let* B *be a subset of* $\{1, \ldots, k\}$ *such that* $\#B > k^{1-\delta_1} + g(k)$ *where* δ_1 *and* g *are given by Heppner's result. Then there are* r_1 *and* r_2 *in* B, $r_1 \neq r_2$, *such that* $H^{\lfloor \log_d(k) \rfloor}(r_1) = H^{\lfloor \log_d(k) \rfloor}(r_2)$.

Proof. By Proposition 1, there is $B_1 \subset B$ such that $\#B_1 > k^{1-\delta_1}$ and

$$H^{\lfloor \log_d(k) \rfloor}(s) < sk^{-\delta_1} \le k^{1-\delta_1}, \quad \forall s \in B_1.$$

Then, it follows from the pigeonhole principle that there are r_1 and r_2 in B_1 with $r_1 \neq r_2$ and $H^{\lfloor \log_d(k) \rfloor}(r_1) = H^{\lfloor \log_d(k) \rfloor}(r_2)$. ∎

Note that if A is a subset of $\mathbb{N}$ which does not have zero Banach density then there is a $k \in \mathbb{N}$ such that, for all $x \in \mathbb{N}^*$, $\#(A \cap \{x, \ldots, x+k-1\}) > k^{1-\delta_1} + g(k)$, because $g(k)$ is $O(k^{\delta_2})$ and δ_1 and δ_2 lay in $(0,1)$.

We will use this observation in the next section.

2. Main results

Lemma 3 (Fundamental Lemma). *Let* A *be a subset of* $\mathbb{N}^*$ *and let* x *and* k *in* $\mathbb{N}^*$ *be such that*

$$\#(A \cap \{x, x+1, \ldots, x+k-1\}) > 2(\lfloor \log_d(k) \rfloor + 1)(k^{1-\delta_1} + g(k)) \tag{4}$$

where δ_1 *and* $g(k)$ *are given by Heppner's result* (*Proposition* 1). *Then there exist* $r_1 \neq r_2$ *in* $A \cap \{x, x+1, \ldots, x+k-1\}$ *such that* $H^{\lfloor \log_d(k) \rfloor}(r_1) = H^{\lfloor \log_d(k) \rfloor}(r_2)$.

Proof. Put $\beta = d^{\lfloor \log_d(k) \rfloor}$. Let $z_1 \in \mathbb{N}^*$ be such that $z_1\beta < x \le (z_1+1)\beta$. Then $y \in \{x, \ldots, x+k-1\}$ clearly implies that either $y - z_1\beta$ or $y - (z_1+1)\beta$ belongs to $\{1, \ldots, k\}$.

Therefore, it follows from (4) and the pigeonhole principle that we can choose $z \in \{z_1, z_1+1\}$ such that if

$$B = B(k,z) = \{1 \le s \le k : \exists q \in A,\ q - zd^{\lfloor \log_d(k) \rfloor} = s\}$$

then

$$\#B > (\lfloor \log_d(k) \rfloor + 1)(k^{1-\delta_1} + g(k)).$$

Since $\ell(\cdot, \lfloor \log_d(k) \rfloor) \in \{0, \ldots, \lfloor \log_d(k) \rfloor\}$, we can apply once again the pigeonhole principle to find a subset B_1 of B with strictly more than $k^{1-\delta_1} + g(k)$ elements such that if u and v are in B_1 then

$$\ell(u, \lfloor \log_d(k) \rfloor) = \ell(v, \lfloor \log_d(k) \rfloor).$$

Now, apply Proposition 2 in order to obtain $s_1 \neq s_2$ in B_1 such that $H^{\lfloor \log_d(k) \rfloor}(s_1) = H^{\lfloor \log_d(k) \rfloor}(s_2)$. Then, since $\ell(s_1, \lfloor \log_d(k) \rfloor) = \ell(s_2, \lfloor \log_d(k) \rfloor)$, it follows from Lemma 2 that

$$H^{\lfloor \log_d(k) \rfloor}(s_1 + z\beta) = H^{\lfloor \log_d(k) \rfloor}(s_2 + z\beta).$$

By the definition of B it is obvious that $r_i = s_i + z\beta \in A$ for $i = 1, 2$, and this concludes the demonstration. ∎

Now we are ready to state and prove our main result.

Consider in $\mathbb{N}^*$ the equivalence relation

$$a \sim b \Leftrightarrow \exists k \in \mathbb{N},\ H^k(a) = H^k(b). \tag{5}$$

Let $\mathcal{P}$ be a complete set of representatives of $\mathbb{N}^*/\sim$.

It seems natural to consider $\mathcal{P}$ as a set of all the different orbits of H. Now we show that this set is "small".

Theorem 1. *The Banach density of $\mathcal{P}$ is zero.*

Proof. It is obvious that if u_1 and u_2 are distinct elements of $\mathcal{P}$ then $H^k(u_1) \neq H^k(u_2)$ for all $k \in \mathbb{N}$. Thus, by the Fundamental Lemma, for all a and k in $\mathbb{N}^*$, we have

$$\#(\mathcal{P} \cap \{a, \ldots, a+k-1\}) \leq 2(\lfloor \log_d(k) \rfloor + 1)(k^{1-\delta_1} + g(k)). \tag{6}$$

Since, by Proposition 1, $g(k)$ is $O(k^{\delta_2})$ and δ_1 and δ_2 belong to $(0,1)$ the result follows when we take the limit $k \to \infty$ in (6). ∎

An important, now trivial, consequence is

Corollary 1. *The Banach density of the orbit $\mathcal{O}(n)$ under H is zero.*

Proof. If $\mathcal{O}(n)$ is finite the result is obvious. If $\mathcal{O}(n)$ is infinite then, for all u_1 and u_2 in $\mathcal{O}(n)$, with $u_1 \neq u_2$, and for all $k \in \mathbb{N}$, $H^k(u_1) \neq H^k(u_2)$ (otherwise, $\mathcal{O}(n)$ would be periodic). Then we can choose a complete set of representatives $\mathcal{P}$ of $\mathbb{N}^*/\sim$ such that $\mathcal{O}(n) \subset \mathcal{P}$. Since $\varrho_b(\mathcal{P}) = 0$ the result follows. ∎

References

[Hep78] E. Heppner, *Eine Bemerkung zum Hasse–Syracuse Algorithmus*, Arch. Math. (Basel) 31 (1978), 317–320.

[Kor94] I. Korec, *A density estimate for the $3x+1$ problem*, Math. Slovaca 44 (1994), 85–89.

[KZ87] I. Korec and Š. Znám, *A note on the* $3x+1$ *problem*, Amer. Math. Monthly 94 (1987), 771–772.

[Lag85] J. Lagarias, *The* $3x+1$ *problem and its generalizations*, ibid. 92 (1985), 1–23.

[Wir98] G. Wirsching, *The Dynamical System Generated by the* $3n+1$ *Function*, Lecture Notes in Math. 1681, Springer, 1998.

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