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On approximation of real numbers by real algebraic numbers

by

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1. Introduction. In this paper we consider several related problems of the theory of Diophantine approximation. The following notation will be used. We denote by $\#S$ the number of elements in a finite set S. The Lebesgue measure of a measurable set $S \subset \mathbb{R}$ is denoted by $|S|$. The set $S \subset \mathbb{R}$ has full measure means that $|\mathbb{R} \setminus S| = 0$. Throughout the paper, Ψ denotes a monotonic sequence of positive numbers. We denote by P_n the set of integral polynomials of degree $\le n$. The set of real algebraic numbers of degree n is denoted by $\mathbb{A}_n$. Given a polynomial P, $H(P)$ denotes the height of P. Given an algebraic number α, $H(\alpha)$ denotes the height of α. We use the Vinogradov symbol $\ll$, which means "$\le$ up to a constant multiplier". We begin with a short review.

In 1924 Khinchin proved a remarkable result on the approximation of real numbers by rationals [9]. According to his theorem, for almost all $x \in \mathbb{R}$ the inequality

$$|qx - p| < \Psi(q)$$

has at most finitely or infinitely many solutions $p, q \in \mathbb{Z}$ according as the sum $\sum_{q=1}^{\infty} \Psi(q)$ converges or diverges.

In 1932 K. Mahler [13] introduced a classification of real numbers and showed [12] that almost all numbers are S-numbers. In fact, he proved that $w_n(x) \le 4n$ for almost all $x \in \mathbb{R}$, where $w_n(x)$ is defined to be the supremum of the set of real numbers w for which the inequality

$$|P(x)| < H(P)^{-w} \tag{1}$$

has infinitely many solutions $P \in P_n$. At the same time Mahler conjectured that $w_n(x) = n$ for almost all $x \in \mathbb{R}$. In 1964 Mahler's conjecture was completely proved by V. Sprindžuk [15–17].

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There have been many efforts to make the error term more precise on the right hand side of inequality (1). The case $n=2$ has been individually considered by Cassels, Kubilius and Bernik. In 1966 A. Baker [1] proved that for almost all $x\in\mathbb{R}$ the inequality

$$|P(x)|<\Psi^n(H(P)) \tag{2}$$

has at most finitely many solutions $P\in P_n$ if $\sum_{h=1}^{\infty}\Psi(h)<\infty$. At the same time Baker conjectured that a stronger result had to be true. Regarding this V. Bernik proved in 1989 [6] that for almost all $x\in\mathbb{R}$ the inequality

$$|P(x)|<H(P)^{-n+1}\Psi(H(P)) \tag{3}$$

has at most finitely many solutions $P\in P_n$ if $\sum_{h=1}^{\infty}\Psi(h)<\infty$. There were some grounds to suppose that the convergence condition in Bernik's theorem could not be omitted. In this paper we confirm this by proving the following theorem.

THEOREM 1. *Let Ψ be a decreasing sequence of positive numbers such that $\sum_{h=1}^{\infty}\Psi(h)=\infty$. Then for almost all $x\in\mathbb{R}$ the inequality* (3) *has infinitely many solutions $P\in P_n$.*

It should be noted that there is an analogous problem for polynomials of complex variables. One should expect that for almost all $z\in\mathbb{C}$ the inequality $|P(z)|<H(P)^{-(n-2)/2}\Psi^{1/2}(H(P))$ has at most finitely or infinitely many solutions in integral polynomials of degree $\le n$ according as $\sum_{h=1}^{\infty}\Psi(h)$ converges or diverges. The methods of this paper and those of [6] with necessary modifications can probably be applied for solving the problem. But the question remains open for both the convergence and the divergence case.

The ideas of this paper can also be generalized to Diophantine approximation of points of smooth manifolds. Consider the solubility problem for the inequality

$$|a_nx_n+\ldots+a_1x_1+a_0|<H^{-n+1}\Psi(H), \tag{4}$$

in $(a_0,\ldots,a_n)\in\mathbb{Z}^{n+1}$, where $H=\max\{|a_0|,\ldots,|a_n|\}$ and the points $\mathbf{x}=(x_1,\ldots,x_n)$ lie on a manifold M. If $M=\mathbb{R}^n$, it has been shown by Groshev (see [18, pp. 28–33]) that a so-called Khinchin-type theorem is available. This means that for almost all $\mathbf{x}\in M$ the inequality (4) has at most finitely or infinitely many solutions according as $\sum_{h=1}^{\infty}\Psi(h)$ converges or diverges.

There have been many attempts to prove Khinchin-type theorems for manifolds of dimension $<n$ embedded in $\mathbb{R}^n$ satisfying various arithmetic, analytic and(or) geometric conditions. In particular, one is available when a manifold of dimension at least $\max\{2,n/2\}$ satisfies a curvature condition that for surfaces in $\mathbb{R}^3$ corresponds to the Gaussian curvature being positive almost everywhere [8]. V. Bernik proved a Khinchin-type theorem for a

manifold being a topological product of at least four 3-smooth curves in $\mathbb{R}^2$ with non-vanishing curvature almost everywhere [7]. A Khinchin-type theorem for inhomogeneous approximation by values of 2-degree integral polynomials has been obtained in [5].

Very recently, D. Y. Kleinbock and G. A. Margulis [10] have obtained a strong extremality result for general non-degenerate $C^{(l)}$-manifolds of dimension $d < n$ in $\mathbb{R}^n$. In addition they generalized the theorem of Baker (see (2)) to these manifolds; more precisely, they proved that the inequality (4) has infinitely many solutions almost nowhere provided that the decreasing sequence $h\Psi(h)$, $h = 1, 2, \ldots$, satisfies $\sum_{h=1}^{\infty} h^{-1}(h\Psi(h))^{1/(dl)} < \infty$. Generalizations to simultaneous approximation can also be considered.

In view of the existence of correlations between the approximation of zero by values of integral polynomials and approximation of real numbers by algebraic numbers, we are interested in the solubility of the inequality

$$|x - \alpha| < H(\alpha)^{-n}\Psi(H(\alpha)) \tag{5}$$

in real algebraic numbers of degree n. It should be noted that there is a classification of Koksma for real numbers [11] based on the solubility of the inequality

$$|x - \alpha| < H(\alpha)^{-w^*-1}. \tag{6}$$

The error term in (6) is a particular case of that of (5). Koksma considered the value $w_n^*(x)$, which is defined to be the supremum of the set of real numbers w^* such that the inequality (6) has infinitely many solutions in real algebraic numbers of degree $\le n$, where $n \in \mathbb{N}$. It can be shown by the theorem of Sprindžuk [17], a result of Wirsing [19] and the lemma of Borel–Cantelli that $w_n^*(x) = n$ for almost all real x. We are interested in the measure of the set

$$A_n(\Psi) = \{x \in \mathbb{R} : \text{inequality (5) holds for infinitely many } \alpha \in \mathbb{A}_n\}.$$

It was expected that it would essentially depend on the behaviour of the $\sum_{h=1}^{\infty} \Psi(h)$ as in the polynomial case above. We prove

THEOREM 2. *Let Ψ be a decreasing sequence of positive numbers. Then*

$$|A_n(\Psi)| = \begin{cases} 0 & \text{if } \sum_{h=1}^{\infty} \Psi(h) < \infty, \\ \text{full} & \text{if } \sum_{h=1}^{\infty} \Psi(h) = \infty. \end{cases}$$

The proof of Theorem 2 is based on the distribution of real algebraic numbers. We use the concept of regular systems introduced by A. Baker and W. Schmidt [2].

DEFINITION 1. Let Γ be a countable set of real numbers and $N : \Gamma \to \mathbb{R}$ be a positive function. The pair (Γ, N) is called a *regular system* if there exists a constant $C_1 = C_1(\Gamma, N) > 0$ such that for any interval I there exists a sufficiently large number $T_0 = T_0(\Gamma, N, I) > 0$ such that for any integer $T \geq T_0$ there are $\alpha_1, \ldots, \alpha_t$ in $\Gamma \cap I$ such that

$$N(\alpha_i) \leq T \quad (1 \leq i \leq t), \tag{7}$$

$$|\alpha_i - \alpha_j| \geq T^{-1} \quad (1 \leq i < j \leq t), \tag{8}$$

$$t \geq C_1 |I| T. \tag{9}$$

For the set of algebraic numbers the function N normally depends on the height of the corresponding algebraic number. A. Baker and W. Schmidt have found that the set of real algebraic numbers of degree $\leq n$ together with the function $N(\alpha) = H(\alpha)^{n+1}(\ln H(\alpha))^{-3n^2}$ is a regular system. For $n = 2$ this has been generalized by R. Baker [3] to the set of zeros of functions of a general form. Also, it has been shown in [4] that the set of quadratic irrationals on the interval $[0, 1]$ together with the function $N(\alpha) = H(\alpha)^3$ is a regular system.

In this paper we extend this to the set of real algebraic numbers of any degree.

THEOREM 3. *The set* $\mathbb{A}_n$ *together with the function*

$$N(\alpha) = (H(\alpha)/(1 + |\alpha|)^n)^{n+1}$$

is a regular system.

2. Effective measure bounds. Throughout this section, n denotes an integer ≥ 2, Q a natural number, ε a positive number, and I an interval of the form $[a, b)$ embedded in $[-1/2, 1/2)$. Given n and Q, we define $P_n(Q) = \{P \in P_n : H(P) \leq Q\}$. Given I, Q, ε and $P \in P_n(Q)$, we denote by $\sigma(P)$ the set consisting of $x \in I$ satisfying

$$|P(x)| \leq \varepsilon, \quad |P'(x)| \geq 2|I|^{-1}. \tag{10}$$

Given n, I, Q and ε, let $B_{n,I}(Q, \varepsilon)$ denote the union of $\sigma(P)$ over all $P \in P_n(Q)$. The aim of this section is to obtain an upper bound for $|B_{n,I}(Q, \varepsilon)|$. We use the following

LEMMA 1. *Given* n, $I = [a, b)$, Q *and* ε *such that* $\varepsilon < (16Q)^{-1}$, *define*

$$I''_\varepsilon = [a, a + \varepsilon] \cup [b - \varepsilon, b) \quad \text{and} \quad I'_\varepsilon = I \setminus I''_\varepsilon.$$

Then for any $P \in P_n(Q)$ *such that* $\sigma(P) \cap I'_\varepsilon \neq \emptyset$, *for any* $x_0 \in \sigma(P) \cap I'_\varepsilon$ *there exists* $\alpha \in I$ *such that* $P(\alpha) = 0$, $|P'(\alpha)| > |P'(x_0)|/2$ *and*

$$|x_0 - \alpha| < 2\varepsilon |P'(\alpha)|^{-1}. \tag{11}$$

Proof. Let $P(x) = a_n x^n + \ldots + a_0 \in P_n(Q)$ satisfy $\sigma(P) \cap I'_\varepsilon \neq \emptyset$. Fix $x_0 \in \sigma(P) \cap I'_\varepsilon$. Given x satisfying $|x - x_0| \leq \varepsilon$, we readily verify that $|x| \leq 1/2$. By Lagrange's formula, we have $P'(x) = P'(x_0) + P''(x_1)(x - x_0)$, where x_1 is a point between x and x_0. Using $|x| \le 1/2$, it is easy to obtain the estimate

$$|P''(x_1)| = |n(n-1)a_n x_1^{n-2} + \ldots + 2a_2| < 16\max\{|a_0|, \ldots, |a_n|\} \le 16Q.$$

Then

$$|P''(x_1)(x - x_0)| \le 16Q\varepsilon < 1 \le |I|^{-1} \le |P'(x_0)|/2.$$

Hence, for any x satisfying $|x - x_0| \le \varepsilon$, we have

$$|P'(x)| \ge |P'(x_0)| - |P''(x_1)(x - x_0)| > |P'(x_0)|/2.$$

By Lagrange's formula we have $P(x) = P(x_0) + P'(x_2)(x - x_0)$, where x_2 is between x and x_0. As shown above, $|P'(x_2)| > |P'(x_0)|/2$. Further, if we let $x = x_0 \pm \varepsilon$ then $|P'(x_2)(x-x_0)| > \varepsilon|P'(x_0)|/2 \ge \varepsilon$. Moreover, the expression $P'(x_2)(x - x_0)$ has different signs at $x_0 - \varepsilon$ and $x_0 + \varepsilon$. Since $|P(x_0)| \le \varepsilon$, we conclude that $P(x) = P(x_0) + P'(x_2)(x - x_0)$ also has different signs at $x_0 \pm \varepsilon$. Therefore, there exists $\alpha \in [x_0 - \varepsilon, x_0 + \varepsilon] \subset I$ satisfying $P(\alpha) = 0$. As shown above, $|P'(\alpha)| \ge |P'(x_0)|/2$. Next, by Taylor's formula, we write

$$P(x_0) = \big(P'(\alpha) + \tfrac{1}{2}P''(x_3)(x_0 - \alpha)\big)(x_0 - \alpha).$$

Using the estimate $\left|\frac{1}{2}P''(x_3)(x_0 - \alpha)\right| \le |P'(x_0)|/4$, we get

$$\left|P'(\alpha) + \tfrac{1}{2}P''(x_3)(x_0 - \alpha)\right| \ge |P'(\alpha)|/2.$$

This inequality and $|P(x_0)| \le \varepsilon$ yield (11). The proof is complete.

PROPOSITION 1. *Given n and $I = [a, b)$, define $Q_1 = \max\{(2^{n-1}|I|)^{-1/n}, 4n^2\}$. Then $|B_{n,I}(Q, \varepsilon)| \le n2^{n+2}\varepsilon Q^n|I|$ for any $Q > Q_1$ and any $\varepsilon < n^{-1}2^{-n-2}Q^{-n}$.*

Proof. Since the sets of solutions of the systems (10) defined by a polynomial P and the polynomial $-P$ coincide, without loss of generality, we consider the polynomials of $P_n(Q)$ with the coefficient of x^n being non-negative. Given $P \in P_n(Q)$ and a real number α such that $P'(\alpha) \ne 0$, $\sigma(P, \alpha)$ denotes the interval $\{x \in I : |x - \alpha| < 2\varepsilon|P'(\alpha)|^{-1}\}$. Let I'_ε and I''_ε be defined as in Lemma 1. For every $P \in P_n(Q)$, we define

$$Z_I(P) = \{\alpha \in I : P(\alpha) = 0 \text{ and } |P'(\alpha)| \ge |I|^{-1}\}.$$

By Lemma 1, for any $P \in P_n(Q)$ we have

$$\sigma(P) \cap I'_\varepsilon \subset \bigcup_{\alpha \in Z_I(P)} \sigma(P, \alpha). \tag{12}$$

Fix integers $a_1, \ldots, a_n$ such that $|a_i| \le Q$ $(i = 1, \ldots, n)$ and $a_n \ge 0$. Set $R(x) = a_n x^n + \ldots + a_1 x$ and $P_n(Q, R) = \{P \in P_n(Q) : P - R \in \mathbb{Z}\}$. There exists a collection of pairwise non-intersecting intervals $[w_{i-1}, w_i) \subset I$ $(i = 1, \ldots, s)$ which cover I and the derivative R' is monotonic and of constant sign on each $[w_{i-1}, w_i)$. It is clear that s can be chosen such that $1 \le s \le 2n-2$. Order the set $Z_{I,R} = \bigcup_{P\in P_n(Q,R)} Z_I(P)$ as

$$Z_{I,R} = \{\alpha_1^{(1)}, \dots, \alpha_1^{(k_1)}, \alpha_2^{(1)}, \dots, \alpha_2^{(k_2)}, \dots, \alpha_s^{(1)}, \dots, \alpha_s^{(k_s)}\}.$$

Here $k_i = \#(Z_{I,R} \cap [w_{i-1}, w_i))$ and $Z_{I,R} \cap [w_{i-1}, w_i) = \{\alpha_i^{(1)}, \dots, \alpha_i^{(k_i)}\}$, where $\alpha_i^{(j)} < \alpha_i^{(j+1)}$. Given $P \in P_n(Q,R)$, by the identity $P' \equiv R'$, we have $\sigma(P,\alpha) = \sigma(R,\alpha)$ for any $\alpha \in Z_I(P)$. Using (12), we get

$$\Big(\bigcup_{P\in P_n(Q,R)} \sigma(P)\Big) \cap I'_\varepsilon \subset \Big(\bigcup_{P\in P_n(Q,R)} \bigcup_{\alpha\in Z_I(P)} \sigma(R,\alpha)\Big) = \bigcup_{i=1}^{s} \bigcup_{j=1}^{k_i} \sigma(R, \alpha_i^{(j)}). \tag{13}$$

Fix an index i $(1 \le i \le s)$. If $k_i \ge 2$ then we can consider two consecutive roots $\alpha_i^{(j)}$ and $\alpha_i^{(j+1)}$ of two polynomials $R + a_0^{i,j}$ and $R + a_0^{i,j+1}$ respectively. For convenience we assume that R' is increasing and positive on $[w_{i-1}, w_i)$. Then R is monotonic on $[w_{i-1}, w_i)$, and $a_0^{i,j} \ne a_0^{i,j+1}$. It follows that $|a_0^{i,j} - a_0^{i,j+1}| \ge 1$. Using Lagrange's formula and the monotonicity of R', we get

$$\begin{aligned}1 &\le |a_0^{i,j} - a_0^{i,j+1}| = |R(\alpha_i^{(j)}) - R(\alpha_i^{(j+1)})| \\ &= |R'(\widetilde{\alpha}_i^{(j)})| \cdot |\alpha_i^{(j)} - \alpha_i^{(j+1)}| \le |R'(\alpha_i^{(j+1)})| \cdot |\alpha_i^{(j)} - \alpha_i^{(j+1)}|,\end{aligned}$$

where $\widetilde{\alpha}_i^{(j)}$ is between $\alpha_i^{(j)}$ and $\alpha_i^{(j+1)}$. This implies $|R'(\alpha_i^{(j+1)})|^{-1} \le \alpha_i^{(j+1)} - \alpha_i^{(j)}$, whence we readily get

$$\sum_{j=1}^{k_i-1} |R'(\alpha_i^{(j+1)})|^{-1} \le \sum_{j=1}^{k_i-1} (\alpha_i^{(j+1)} - \alpha_i^{(j)}) = \alpha_i^{(k_i)} - \alpha_i^{(1)} \le w_i - w_{i-1}.$$

The last inequality and $|R'(\alpha_i^{(1)})| \ge |I|^{-1}$ yield

$$\sum_{j=1}^{k_i} |R'(\alpha_i^{(j)})|^{-1} \le w_i - w_{i-1} + |I|. \tag{14}$$

This method can be applied to all situations, i.e. when the behaviour of R' differs from the above, giving (14). This estimate also remains true when $k_i = 1$, and certainly when $k_i = 0$. Summing (14) over all i, we find

$$\sum_{i=1}^{s} \sum_{j=1}^{k_i} |R'(\alpha_i^{(j)})|^{-1} \le \sum_{i=1}^{s} (w_i - w_{i-1} + |I|) \le (2n-1)|I|. \tag{15}$$

The obvious estimate $|\sigma(R,\alpha)| \le 4\varepsilon |R'(\alpha)|^{-1}$ together with (13) and (15) gives

$$\Big|\bigcup_{P\in P_n(Q,R)} \sigma(P)\cap I'_\varepsilon\Big| \le 4\varepsilon(2n-1)|I|. \tag{16}$$

We notice that

$$B_{n,I}(Q,\varepsilon) = \bigcup_R \Big(\bigcup_{P\in P_n(Q,R)} \sigma(P)\Big). \tag{17}$$

Since the number of different polynomials R is at most $(Q+1)(2Q+1)^{n-1}$, using (16) and (17), we conclude that

$$|B_{n,I}(Q,\varepsilon)\cap I'_\varepsilon| \le 4\varepsilon(2n-1)|I|(Q+1)(2Q+1)^{n-1}.$$

Now we make the following transformations:

$$\begin{aligned}|B_{n,I}(Q,\varepsilon)| &\le 4\varepsilon(2n-1)|I|(Q+1)(2Q+1)^{n-1} + 2\varepsilon\\ &\le n2^{n+2}\varepsilon Q^n|I|((1-(2n)^{-1})(1+Q^{-1})^n + 2^{-n-1}n^{-1}Q^{-n}|I|^{-1})\\ &\le n2^{n+2}\varepsilon Q^n|I|\\ &\quad\times((1-(4n)^{-1})(1+Q^{-1})^n + 2^{-n-1}n^{-1}Q^{-n}|I|^{-1} - (4n)^{-1}).\end{aligned}$$

The inequality $Q > (2^{n-1}|I|)^{-1/n}$ gives $2^{-n-1}n^{-1}Q^{-n}|I|^{-1} - (4n)^{-1} < 0$. The inequality $Q > 4n^2$ implies $(1-(4n)^{-1})(1+Q^{-1})^n < 1$. Then we get the required estimate and the proof is complete.

3. Distribution of real algebraic numbers. This section is devoted to the study of the distribution of real algebraic numbers. To prove Theorem 3 we need the following

PROPOSITION 2. *Let I be a finite interval. Then for almost all $x\in I$ the system*

$$|P(x)| < H(P)^{-n}, \qquad |P'(x)| < 2|I|^{-1} \tag{18}$$

has at most finitely many solutions $P\in P_n$.

This follows from Propositions 1–3 of [6], where a more general statement is proved.

Now we proceed to prove Theorem 3. First of all, note that it is sufficient to show that the required distribution holds for any interval of length ≤ 1. Fix an interval $I\subset[-1/2,1/2)$ and $Q\in\mathbb{N}$. Let $\varepsilon_Q = n^{-1}2^{-n-5}Q^{-n}$. We now define five relatively small subsets of I.

1. The first is $B_1(I,Q) = B_{n,I}(Q,\varepsilon_Q)$. By Proposition 1, we have $|B_1(I,Q)| \le |I|/8$ whenever $Q > Q_1$.

2. Given $P \in P_n$, define $\sigma_2(P)$ to consist of all solutions of system (18), and set

$$B_2(I,Q) = \bigcup_{\substack{P\in P_n\\ H(P)>Q}} \sigma_2(P).$$

By Proposition 2, we have $|B_2(I,Q)| \to 0$ as $Q \to \infty$. Therefore, there exists a sufficiently large number Q_2 such that $|B_2(I,Q_2)| \le |I|/16$.

3. For any non-zero $P \in P_n(Q_2)$ define $\sigma_3(P,Q) = \{x \in I : |P(x)| < \varepsilon_Q\}$. Let $B_3(I,Q_2,Q)$ be the union of $\sigma_3(P,Q)$ over all $P \in P_n(Q_2) \setminus \{0\}$. Since Q_2 depends on I and n only, the number of different intervals $\sigma_3(P,Q)$ is bounded by a constant independent of Q. Moreover, $|\sigma_3(P,Q)| \to 0$ as $Q \to \infty$. Now, it is not difficult to see that there exists $Q_3 > 0$ such that for any $Q > Q_3$ we have $|B_3(I,Q_2,Q)| \le |I|/16$. Note that the constant Q_3 can be explicitly calculated.

4. We denote by $B_4(I,Q)$ the union of the intervals $\sigma_4(\alpha,Q) = \{x \in I : |x-\alpha| \le 8\varepsilon_Q Q^{-1}\}$ over all real algebraic numbers of degree $\le n-1$ of height $\le (n2^{4n+2}+1)Q$. Since the number of different intervals in this union is at most $\ll Q^n$ and every interval has length $\ll Q^{-(n+1)}$, there exists a sufficiently large number $Q_4 > 0$ such that $|B_4(I,Q)| \le |I|/8$ for any $Q > Q_4$. The constant Q_4 can be explicitly calculated.

5. Finally, set $B_5(I,Q) = [a, a+\varepsilon_Q] \cup [b-\varepsilon_Q, b]$. Whenever $Q > Q_5 = (|I|n2^{n+1})^{-1/n}$, we have $|B_5(I,Q)| \le |I|/8$.

Now we define

$$B(I,Q) = B_1(I,Q) \cup B_2(I,Q_2) \cup B_3(I,Q_2,Q) \cup B_4(I,Q) \cup B_5(I,Q).$$

According to our calculations above, whenever $Q > \max\{Q_1,\ldots,Q_5\}$ we have the estimate $|B(I,Q)| \le |I|/2$.

Let $x \in I \setminus B(I,Q)$. By Minkowski's linear forms theorem [14, Ch. 2, §3], there exists a non-zero polynomial $P(t) = a_n t^n + \ldots + a_0 \in P_n$ satisfying

$$|P(x)| \le \varepsilon_Q, \quad |P'(x)| \le n2^{4n+2}Q, \quad |a_i| \le Q/8 \ \ (2 \le i \le n). \tag{19}$$

Assume that $|P'(x)| \le Q/2$. Then, using $|x| \le 1/2$ and (19), we find

$$|a_1| \le |P'(x)| + \sum_{i=2}^{n} |ia_i x^{i-1}| \le \frac{Q}{2} + \frac{Q}{8}\sum_{i=2}^{n} i2^{-i+1} \le Q.$$

Next, (19) together with $|x| \le 1/2$ gives

$$|a_0| \le |P(x)| + \sum_{i=1}^{n} |a_i x^i| \le \frac{Q}{2} + Q\sum_{i=1}^{\infty} 2^{-i} \le Q.$$

It follows that $H(P) \le Q$. It is now easy to see that x belongs to one of the sets $B_1(I,Q)$, $B_2(I,Q_2)$ or $B_3(I,Q_2,Q)$, contrary to x being a point of

$I\setminus B(I,Q)$. Hence, whenever $x\in I\setminus B(Q,I)$, there exists a non-zero solution $P\in P_n$ of the system (19) such that

$$|P'(x)|>Q/2. \tag{20}$$

Using (19), it is easy to obtain by the same method as above that

$$H(P)\le(n2^{4n+2}+1)Q. \tag{21}$$

Now we are going to show that there exists a root of P very close to x. To this end we define the constants $Q_6=2^{(3n-3)/(n-1)}+1$ and $Q_7=4|I|^{-1}$ and ensure that $Q>Q_0=\max\{Q_1,\dots,Q_7\}$. In this situation we can apply Lemma 1, and conclude that there exists a real root α of P in I such that

$$|x-\alpha|\le\frac{2|P(x)|}{|P'(x)|/2}\le 8\varepsilon_Q Q^{-1}. \tag{22}$$

By (21), we have $H(\alpha)\le(n2^{4n+2}+1)Q$. Since $x\notin B(I,Q)$, we have $x\notin B_4(I,Q)$. It follows that the degree of α is exactly n.

We choose a maximal collection $\{\alpha_1,\dots,\alpha_t\}\subset I$ consisting of real algebraic numbers with $\deg\alpha_i=n$,

$$H(\alpha_i)\le(n2^{4n+2}+1)Q \tag{23}$$

for all $i\in\{1,\dots,t\}$ and

$$|\alpha_i-\alpha_j|\ge 8\varepsilon_Q Q^{-1}\quad(1\le i<j\le t). \tag{24}$$

As we have proved, for any $x\in I\setminus B(I,Q)$ there exists $\alpha\in\mathbb{A}_n$ satisfying $H(\alpha)\le(n2^{4n+2}+1)Q$ and (22). Since the collection $\{\alpha_1,\dots,\alpha_t\}$ is maximal, there exists α_i in this collection such that $|\alpha-\alpha_i|\le 8\varepsilon_Q Q^{-1}$. Hence, $|x-\alpha_i|\le 16\varepsilon_Q Q^{-1}$ and so

$$I\setminus B(I,Q)\subset\bigcup_{i=1}^{t}\{x\in I:|x-\alpha_i|\le 16\varepsilon_Q Q^{-1}\}.$$

Since $|I\setminus B(I,Q)|\ge|I|/2$ we have $|I|/2\le|I\setminus B(I,Q)|\le t\cdot 32\varepsilon_Q Q^{-1}$ and we get

$$t\ge n2^{n-1}Q^{n+1}|I|. \tag{25}$$

Now, let J be any interval of length ≤ 1. There exists an integer m such that $J_m=(J+m)\cap[-1/2,1/2)$ has measure $\ge|J|/2$. Let $Q_0(J_m)=\max\{Q_1(J_m),\dots,Q_7(J_m)\}$, where $Q_1,\dots,Q_7$ are defined as above. As we have proved, for any $Q>Q_0(J_m)$ there is a collection $\{\alpha_1,\dots,\alpha_t\}\subset\mathbb{A}_n\cap J_m$ satisfying (23)–(25). The numbers $\beta_i=\alpha_i-m\in J$ are algebraic of degree n as well. If $m\ne0$ one readily verifies that $H(\beta_i)\le H(\alpha_i)(1+|m|)^{n+1}/|m|$. Then, using this together with (23) and the obvious inequality $1+|m|\le 2(1+|\beta_i|)$, we get $H(\beta_i)\le(1+|\beta_i|)^n n2^{5n+4}Q$. Writing this in terms of the function N, we get

$$N(\beta_i) \le (n2^{5n+4})^{n+1}Q^{n+1}. \tag{26}$$

The last inequality also holds when $m = 0$. Further, by (24), (25) and $|J| \ge |J_m|/2$, we have

$$|\beta_i - \beta_j| \ge n^{-1}2^{-n-2}Q^{-n-1} \quad (1 \le i < j \le t), \tag{27}$$

$$t \ge n2^{n-2}Q^{n+1}|J|. \tag{28}$$

Now let $T_0 = (n2^{5n+4})^{n+1}(Q_0(J_m)+1)^{n+1}$. Then for any integer $T > T_0$ the number $Q = [T^{1/(n+1)}/(n2^{5n+4})]$ is greater than $Q_0(J_m)$. As we have shown, there exist $\beta_1, \ldots, \beta_t \in J \cap \mathbb{A}_n$ satisfying (26)–(28). Then we correspondingly have

$$N(\beta_i) \le T \quad (1 \le i \le t),$$
$$|\beta_i - \beta_j| \ge n^{-1}2^{-n-2}(n2^{5n+4})^{n+1}T^{-1} > T^{-1} \quad (1 \le i < j \le t),$$
$$t \ge n2^{n-2}(n2^{5n+5})^{-n-1}T|J| = (n^n2^{5n^2+9n+7})^{-1}T|J|.$$

This completes the proof of Theorem 3 with $C_1 = (n^n2^{5n^2+9n+7})^{-1}$.

4. Proof of Theorem 2. We proceed to prove Theorem 2. For any $\alpha \in \mathbb{A}_n$ we define the interval

$$\sigma(\alpha) = \{x \in \mathbb{R} : |x - \alpha| < H(\alpha)^{-n}\Psi(H(\alpha))\}.$$

It is easy to see that the set $A_n(\Psi)$ consists of all $x \in \mathbb{R}$ belonging to infinitely many intervals $\sigma(\alpha)$.

First we consider the convergence part of Theorem 2. The following calculation is readily verified:

$$\sum_{\alpha\in\mathbb{A}_n}|\sigma(\alpha)| = \sum_{h=1}^{\infty}\sum_{\substack{\alpha\in\mathbb{A}_n\\ H(\alpha)=h}}|\sigma(\alpha)| = \sum_{h=1}^{\infty}\sum_{\substack{\alpha\in\mathbb{A}_n\\ H(\alpha)=h}}2h^{-n}\Psi(h) \ll \sum_{h=1}^{\infty}\Psi(h) < \infty.$$

The Borel–Cantelli Lemma finishes the proof.

Now we proceed to prove the divergence part. We use the following lemmas.

LEMMA 2. *Let A be a measurable set. If there is a positive constant $C_2 < 1$ such that $|A \cap I| \ge C_2|I|$ for any finite interval $I \subset \mathbb{R}$, then A has full measure.*

Proof. Suppose that $|\mathbb{R} \setminus A| > 0$. Then, by the Lebesgue measure density theorem, there exists $x_0 \in \mathbb{R}$ such that for any $0 < \varepsilon < 1$ there is $\delta > 0$ such that $|(\mathbb{R} \setminus A) \cap [x_0 - \delta, x_0 + \delta]| \ge 2\delta(1 - \varepsilon)$. It follows that $|A \cap [x_0 - \delta, x_0 + \delta]| < 2\delta(1 - \varepsilon)$. Putting $\varepsilon = 1 - C_2$, we obtain a contradiction. The proof is finished.

LEMMA 3. *Let $E_i \subset \mathbb{R}$ be a sequence of measurable sets and let the set E consist of the points belonging to infinitely many E_i. If all the sets E_i are totally bounded and $\sum_{i=1}^{\infty} |E_i|$ diverges, then*

$$|E| \ge \limsup_{N\to\infty} \frac{(\sum_{i=1}^{N} |E_i|)^2}{\sum_{i=1}^{N}\sum_{j=1}^{N} |E_i \cap E_j|}.$$

This lemma is proved in [18, Chapter 2, §2].

LEMMA 4. *Let $\{a_i\}_{i=1}^{\infty}$ be a decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} a_i$ diverges. Define $b_i = \min\{a_i; i^{-1}\}$. Then $\{b_i\}_{i=1}^{\infty}$ is also decreasing and $\sum_{i=1}^{\infty} b_i$ diverges.*

Proof. The monotonicity of b_i is readily verified. Further, assume that $\sum_{i=1}^{\infty} b_i$ converges. It follows that

$$a_i > i^{-1} \quad \text{for infinitely many } i. \tag{29}$$

Since b_i is monotonic, for any integer $l > 3$ we have $\sum_{i=[l/2]}^{l} b_i \ge \sum_{i=[l/2]}^{l} b_l \ge lb_l/2$, which implies

$$lb_l \le 2\sum_{i=[l/2]}^{l} b_i. \tag{30}$$

Since we have assumed that $\sum_{i=1}^{\infty} b_i < \infty$, we infer $\sum_{i=[l/2]}^{l} b_i \to 0$ as $l \to \infty$. By (30), we get $lb_l \to 0$ as $l \to \infty$. But according to (29) and the definition of b_i we have $b_i \ge 1$ infinitely often. The derived contradiction tells us that the series $\sum_{i=1}^{\infty} b_i$ must diverge. The proof is complete.

LEMMA 5. *Given a decreasing sequence Ψ of positive numbers such that $\sum_{h=1}^{\infty} \Psi(h)$ converges (diverges), for any number $c > 0$ the series $\sum_{k=0}^{\infty} 2^k \Psi(c2^k)$ converges (diverges).*

Lemma 5 follows from the corresponding property of the integral

$$\int_1^{\infty} 2^x \widetilde{\Psi}(c2^x)\,dx = \frac{1}{c\log 2}\int_1^{\infty} \widetilde{\Psi}(y)\,dy,$$

where $\widetilde{\Psi}$ is monotonic and continuous on $\{x : x \ge 1\}$ and coincides with Ψ on $\mathbb{N}$.

Now we are ready to prove Theorem 2. Fix any finite interval $I \subset \mathbb{R}$ and set $r = (1 + \sup\{|x| : x \in I\})^n$. Define $\Psi_0(h) = \min\{\Psi(h), h^{-1}/2\}$. By Lemma 4, this sequence is monotonic and $\sum_{h=1}^{\infty} \Psi_0(h)$ diverges. Moreover, by the definition we have

$$h^{-n}\Psi_0(h) \le h^{-n-1}/2 \quad \text{for all } h \in \mathbb{N}. \tag{31}$$

Define $\Phi(h) = h\Psi_0(h)/r^{n+1}$. Then Lemma 5 implies

$$\sum_{k=0}^{\infty} \Phi(r2^k) = \infty. \tag{32}$$

By Theorem 3, there exist positive constants $C_1 = C_1(n)$ and $k_0 = k_0(n, I)$ such that for any $k \geq k_0$ there is a collection

$$A_k(I) = \{\alpha_1 < \ldots < \alpha_{t_k}\} \subset \mathbb{A}_n \cap I$$

satisfying the following conditions:

$$H(\alpha) \leq r2^k \quad \text{for all } \alpha \in A_k(I), \tag{33}$$

$$|\alpha - \beta| \geq 2^{-k(n+1)} \quad \text{for any numbers } \alpha, \beta \in A_k(I), \text{ with } \alpha \neq \beta, \tag{34}$$

$$C_1 2^{(n+1)k}|I| \leq t_k \leq 2^{(n+1)k}|I|. \tag{35}$$

These conditions correspond to (7)–(9). Define

$$E_k(\alpha) = \{x \in \mathbb{R} : |x - \alpha| < (r2^k)^{-n}\Psi_0(r2^k)\} \quad (\alpha \in A_k(I)),$$

$$E_k = \bigcup_{\alpha \in A_k(I)} E_k(\alpha).$$

It is easy to verify that

$$|E_k(\alpha)| = 2(r2^k)^{-n}\Psi_0(r2^k) = 2 \cdot 2^{-k(n+1)}\Phi(r2^k). \tag{36}$$

Set

$$E(I) = \bigcap_{N=k_0}^{\infty} \bigcup_{k=N}^{\infty} E_k.$$

Since Ψ is monotonic and $\Psi_0(h) \leq \Psi(h)$ for all $h \in \mathbb{N}$, by (33), we have $E_k(\alpha) \subset \sigma(\alpha)$. It follows that $E(I) \subset A_n(\Psi)$. Since $|E_k(\alpha)| \to 0$ as $k \to \infty$ and $A_k(I) \subset I$, we have $E(I) \subset \bar{I}$, where $\bar{I}$ is the topological closure of I. Then $E(I) \subset A_n(\Psi) \cap \bar{I}$. Since the boundary of I evidently has zero measure, we conclude that

$$|A_n(\Psi) \cap I| \geq |E(I)|. \tag{37}$$

By (31) and (34), $E_k(\alpha) \cap E_k(\beta) = \emptyset$ for all $\alpha, \beta \in A_k(I)$, $\alpha \neq \beta$. Then $|E_k| = t_k \cdot |E_k(\alpha)|$, where $\alpha \in A_k(I)$. By (35) and (36), we have

$$2C_1\Phi(r2^k)|I| \leq |E_k| \leq 2\Phi(r2^k)|I|. \tag{38}$$

It follows that

$$\sum_{k=k_0}^{N} |E_k| \geq 2C_1|I| \sum_{k=k_0}^{N} \Phi(r2^k). \tag{39}$$

Using (32) and (39), we get

$$\sum_{k=k_0}^{\infty} |E_k| = \infty.$$

We proceed to estimate the measures of the intersections. Fix, as we may by (32), a number $N_0 > k_0$ such that

$$\sum_{k=k_0}^{N_0} \Phi(r2^k) > 1. \tag{40}$$

Fix k and l such that $k_0 \le k < l \le N$, where $N > N_0$. For any $\alpha \in A_k(I)$ we have

$$E_l \cap E_k(\alpha) = \bigcup_{\beta \in A_l(I)} E_k(\beta) \cap E_k(\alpha). \tag{41}$$

Given $\alpha \in A_k(I)$, the number of different $\beta \in A_l(I)$ satisfying $E_l(\beta) \cap E_k(\alpha) \ne \emptyset$ is less than

$$2 + |E_k(\alpha)|/2^{-(n+1)l} \overset{(36)}{=} 2 + 2 \cdot 2^{(n+1)(l-k)} \Phi(r2^k).$$

Using (36) and (41), we get

$$\begin{aligned} |E_l \cap E_k(\alpha)| &\le \max_{\beta \in A_l(I)} \{|E_l(\beta)|\}(2 + 2 \cdot 2^{(n+1)(l-k)} \Phi(r2^k)) \\ &\le 4 \cdot 2^{-(n+1)l} \Phi(r2^l)(1 + 2^{(n+1)(l-k)} \Phi(r2^k)). \end{aligned}$$

This is used in the following calculations:

$$\begin{aligned} |E_l \cap E_k| &\le t_k \cdot \max_{\alpha \in A_k(I)} \{|E_l \cap E_k(\alpha)|\} \\ &\le 4t_k 2^{-(n+1)l} \Phi(r2^l)(1 + 2^{(n+1)(l-k)} \Phi(r2^k)) \\ &\overset{(35)}{\le} 4|I|\Phi(r2^l)\Phi(r2^k) + 4|I|2^{-(n+1)(l-k)} \Phi(r2^l). \end{aligned} \tag{42}$$

Since $E_k \cap E_l = E_l \cap E_k$ we have

$$\sum_{l=k_0}^{N} \sum_{k=k_0}^{N} |E_l \cap E_k| = \sum_{k=k_0}^{N} |E_k| + 2 \sum_{l=k_0+1}^{N} \sum_{k=k_0}^{l-1} |E_l \cap E_k|. \tag{43}$$

By (38), we get

$$\sum_{k=k_0}^{N} |E_k| \le 2|I| \sum_{k=k_0}^{N} \Phi(r2^k). \tag{44}$$

The second summand from (43) is estimated with the help of (42):

$$2\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1}|E_l\cap E_k|\le 8|I|\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1}\Phi(r2^l)\Phi(r2^k) + 8|I|\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1}2^{-(n+1)(l-k)}\Phi(r2^l). \tag{45}$$

We transform the last term as follows:

$$\begin{aligned} 8|I|\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1}2^{-(n+1)(l-k)}\Phi(r2^l) \\ = 8|I|\sum_{l=k_0+1}^{N}\Phi(r2^l)\sum_{k=k_0}^{l-1}2^{-(n+1)(l-k)}\le 2|I|\sum_{l=k_0+1}^{N}\Phi(r2^l). \end{aligned}\tag{46}$$

By (40) and (43)–(46), we conclude that

$$\begin{aligned} \sum_{l=k_0}^{N}\sum_{k=k_0}^{N}|E_l\cap E_k| &\le 4|I|\sum_{k=k_0}^{N}\Phi(r2^k)+8|I|\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1}\Phi(r2^l)\Phi(r2^k)\\ &\le 4|I|\Big(\sum_{k=k_0}^{N}\Phi(r2^k)\Big)^2+4|I|\sum_{l=k_0}^{N}\sum_{k=k_0}^{N}\Phi(r2^l)\Phi(r2^k)\\ &= 8|I|\Big(\sum_{k=k_0}^{N}\Phi(r2^k)\Big)^2. \end{aligned}$$

This estimate and (39) give

$$\frac{(\sum_{k=k_0}^{N}|E_k|)^2}{(\sum_{k=k_0}^{N}\sum_{l=k_0}^{N}|E_k\cap E_l|)}\ge\frac{(2C_1|I|)^2(\sum_{k=k_0}^{N}\Phi(r2^k))^2}{8|I|(\sum_{k=k_0}^{N}\Phi(r2^k))^2}=C_1^2|I|/2,$$

for any $N>N_0$. The conditions of Lemma 3 are satisfied. It follows that $|E(I)|\ge C_1^2|I|/2$. By (37), we get $|A(\Psi)\cap I|\ge C_1^2|I|/2$. This holds for any finite interval I. Lemma 2 completes the proof of Theorem 2.

5. Proof of Theorem 1. Now we are ready to give the proof of Theorem 1. Let $P_n(\Psi)$ denote the set of real numbers x satisfying the inequality (3) for infinitely many polynomials $P\in P_n$. Fix a constant $r>0$. Given $d>0$, define $\Psi_d(h)=\Psi(h)/d$. It is clear that $\Psi_d(h)$ is monotonic and $\sum_{h=1}^{\infty}\Psi_d(h)$ diverges. By Theorem 2, the set $A_n(\Psi_d)$ has full measure. It follows that $\mu(A_n(\Psi_d)\cap[-r,r])=2r$. Given $\alpha\in\mathbb{A}_n$, define $\sigma_{r,d}(\alpha)=\{x\in[-r,r]:|x-\alpha|<H(\alpha)^{-n}\Psi_d(H(\alpha))\}$. Then

$$A_n(\Psi_d)\cap[-r,r]=\bigcap_{k=1}^{\infty}\bigcup_{\alpha:H(\alpha)>k}\sigma_{r,d}(\alpha).$$

Given $\alpha \in \mathbb{A}_n$, denote the minimal polynomial of α by P_α. It can be written in the form

$$P_\alpha(t) = (t-\alpha)\sum_{k=1}^{n} \frac{1}{k!} P_\alpha^{(k)}(\alpha)(t-\alpha)^{k-1}.$$

Fix $x \in \sigma_{r,d}(\alpha)$. Since $x \in [-r, r]$ and $|\alpha - x| < H(\alpha)^{-n}\Psi_d(H(\alpha))$, we get

$$\left|\sum_{k=1}^{n} \frac{1}{k!} P_\alpha^{(k)}(\alpha)(x-\alpha)^{k-1}\right| \le C_3 H(\alpha),$$

where $C_3 = C_3(r)$ is a constant. Now we readily get the estimate

$$|P_\alpha(x)| \le C_3 |x-\alpha| H(\alpha).$$

Let $d = C_3$. Then for any $\alpha \in \mathbb{A}_n$ such that $\sigma_{r,C_3}(\alpha) \neq \emptyset$ and any $x \in \sigma_{r,C_3}(\alpha)$ we have

$$\begin{aligned}|P_\alpha(x)| &\le C_3 H(\alpha)|x-\alpha| < C_3 H(\alpha) H(\alpha)^{-n}\Psi_{C_3}(H(\alpha)) \\ &= H(P_\alpha)^{-n+1}\Psi(H(P_\alpha)).\end{aligned}$$

Thus, if $x \in \sigma_{r,C_3}(\alpha)$ then P_α is a solution of (3). It follows that if $x \in A_n(\Psi_{C_3})$ then (3) has infinitely many solutions, and $x \in P_n(\Psi)$. Thus,

$$A_n(\Psi_{C_3}) \cap [-r, r] \subset P_n(\Psi) \cap [-r, r].$$

It follows that $|P_n(\Psi) \cap [-r, r]| = 2r$ for any $r > 0$. This means that $P_n(\Psi)$ has full measure. The proof is complete.

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