ACTA ARITHMETICA
XC.1 (1999)

Rapidly convergent series representations for $\zeta(2n+1)$ and their χ-analogue

by

Masanori Katsurada (Kagoshima)

1. Introduction. Let $s=\sigma+it$ be a complex variable. The Riemann zeta-function $\zeta(s)$ is defined by

$$\zeta(s)=\sum_{m=1}^{\infty} m^{-s} \qquad (\operatorname{Re} s=\sigma>1),$$

and its meromorphic continuation over the whole s-plane, whose only singularity is a simple pole at $s=1$ with residue unity.

For specific values of $\zeta(s)$ at positive even integers, the formula

$$\zeta(2n)=(-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n} \qquad (n=1,2,\ldots), \tag{1.1}$$

due to Euler, is classically known. Here B_n $(n\ge 0)$ is the Bernoulli number defined by the Taylor series expansion

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n \qquad (|z|<2\pi).$$

Evaluations in closed form like (1.1), however, for the values of $\zeta(s)$ at positive odd integers have been unknown up to the present time.

It is the purpose of this paper to study rapidly convergent series representations for the values of $\zeta(s)$ at positive odd integers. We shall prove certain transformation formulae for the power series including the values of $\zeta(s)$ at positive *even* integers in their coefficients (see Theorems 1 and 2 below). A particular case of each of these formulae implies the previously known rapidly convergent series representations for the values of $\zeta(s)$ at

---

1991 *Mathematics Subject Classification*: Primary 11M06; Secondary 11M35.

Key words and phrases: Riemann zeta-function, Dirichlet L-function, Mellin–Barnes integral, series representation.

The author was supported in part by Grant-in-Aid for Scientific Research (No. 90224485), the Ministry of Education, Science, Sports and Culture in Japan.

positive odd integers. (One is classic, and the other is recently found.) A χ-analogue of our transformation formulae will also be given (Theorem 3).

It was found by Euler in 1772 (see [Ay, p. 1080, Section 7]) that $\zeta(3)$ has an infinite series representation

$$\zeta(3)=\frac{1}{7}\pi^2\biggl\{1-4\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}\biggr\}. \tag{1.2}$$

This formula was rediscovered by Ramaswami [Ra] and (more recently) by Ewell [Ew1]. The basic frame of Ewell's proof of (1.2) was due to Boo Rim Choe [Ch], who gave an elementary derivation of (1.1) for $n=1$. Euler's formula (1.2) was in fact reproduced by Srivastava [Sr1, p. 7, (2.23)] from the work of Ramaswami [Ra]. Inspired by Ewell's rediscovery of (1.2) and by his subsequent result [Ew2], Yue and Williams [YW] established a generalization of (1.2), which, though complicated, gives an exact series representation for $\zeta(2n+1)$ with any nonnegative integer n. The formula of Yue and Williams was considerably simplified by Cvijović and Klinowski [CK, Theorem A], who proved

$$\begin{aligned}\zeta(2n+1)&=(-1)^n\frac{(2\pi)^{2n}}{n(2^{2n+1}-1)}\\&\quad\times\biggl\{\sum_{k=1}^{n-1}(-1)^{k-1}\frac{k\zeta(2k+1)}{(2n-2k)!\pi^{2k}}+\sum_{k=0}^{\infty}\frac{(2k)!\zeta(2k)}{(2n+2k)!2^{2k}}\biggr\}\end{aligned} \tag{1.3}$$

for any positive integer n, where the finite sum on the right-hand side is to be regarded as null if $n=1$. Since $\zeta(0)=-1/2$, we see that (1.3) reduces to (1.2) when $n=1$.

Srivastava [Sr2] (see also [Sr3]) recently found the existence of certain families of rapidly convergent series representations for $\zeta(2n+1)$. Cvijović and Klinowski's formula (1.3) belongs to one of these families, while another family includes classical Wilton's [Wi] formula

$$\begin{aligned}\zeta(2n+1)=(-1)^{n-1}\pi^{2n}\biggl\{&\frac{1}{(2n+1)!}\biggl(\sum_{m=1}^{2n+1}\frac{1}{m}-\log\pi\biggr)\\&+\sum_{k=1}^{n-1}(-1)^k\frac{\zeta(2k+1)}{(2n-2k+1)!\pi^{2k}}\\&+2\sum_{k=1}^{\infty}\frac{(2k-1)!\zeta(2k)}{(2n+2k+1)!2^{2k}}\biggr\}.\end{aligned} \tag{1.4}$$

From the observation of various series representations for $\zeta(2n+1)$ appearing in [Sr2], we may say that Cvijović and Klinowski's formula (1.3) is one of the formulae that have the simplest form among these families.

It is in fact possible to show that (1.3) is a particular case of a general transformation formula:

THEOREM 1. *Let n be a positive integer, and x a real variable with $|x| \le 1$. Then*

$$(1.5)\qquad n\zeta(2n+1) - n\sum_{l=1}^{\infty}\frac{\cos(2\pi lx)}{l^{2n+1}} - \pi x\sum_{l=1}^{\infty}\frac{\sin(2\pi lx)}{l^{2n}}$$
$$= (-1)^n(2\pi x)^{2n}\biggl\{\sum_{k=1}^{n-1}(-1)^{k-1}\frac{k\zeta(2k+1)}{(2n-2k)!(2\pi x)^{2k}} + \sum_{k=0}^{\infty}\frac{(2k)!\zeta(2k)}{(2n+2k)!}x^{2k}\biggr\}.$$

REMARK 1. Since

$$\sum_{l=1}^{\infty}\frac{(-1)^l}{l^{2n+1}} = (2^{-2n}-1)\zeta(2n+1),$$

we see that the case $x = 1/2$ of Theorem 1 implies (1.3).

REMARK 2. The formula (1.5) contains the variable x, so that (1.5) can be differentiated and integrated, as a function of x. In this way new identities can be derived. This also holds true for the formulae (1.6)–(1.8) in the following theorems.

For the proof of Theorem 1 we treat the infinite sum on the right-hand side of (1.5), based on a Mellin transform technique (see (2.1) and (2.2) below). This technique has the advantage over heuristic treatments, particularly for the infinite sums of the type mentioned above. Studies on certain power series and asymptotic series associated with the Riemann zeta and allied zeta-functions, based on this technique, were recently made by the author (see [Ka1]–[Ka3]). The same technique also yields another transformation formula, which includes Wilton's formula (1.4) as a particular case.

THEOREM 2. *Let n be a positive integer, and x a real variable with $|x| \le 1$. Then*

$$(1.6)\qquad \zeta(2n+1) + \frac{1}{2\pi x}\sum_{l=1}^{\infty}\frac{\sin(2\pi lx)}{l^{2n+2}}$$
$$= (-1)^{n-1}(2\pi x)^{2n}\biggl\{\frac{1}{(2n+1)!}\biggl(\sum_{m=1}^{2n+1}\frac{1}{m} - \log 2\pi x\biggr)$$
$$+ \sum_{k=1}^{n-1}(-1)^k\frac{\zeta(2k+1)}{(2n-2k+1)!(2\pi x)^{2k}} + 2\sum_{k=1}^{\infty}\frac{(2k-1)!\zeta(2k)}{(2n+2k+1)!}x^{2k}\biggr\}.$$

REMARK. The formula which has a similar nature to (1.6) was proved in a quite different way by Ewell [Ew3, Theorem 1]. His formula yields a determinant expression of $\zeta(2n+1)$, from which he derived exact infinite series representations for $\zeta(2n+1)$ with $n=1,2$ and 3.

Furthermore, the proof of Theorem 1 suggests that a χ-analogue of (1.5) exists. Let q be a positive integer, and χ a Dirichlet character of modulus q. We denote by $L(s,\chi)$ the Dirichlet L-function attached to χ, and $\tau(\chi)$ Gauss' sum defined by

$$\tau(\chi)=\sum_{a=1}^{q}\chi(a)e^{2\pi ia/q}.$$

THEOREM 3. *Let n be a positive integer, and x a real variable with $|x|\le 1$. For any primitive character χ of modulus q, we have the following formulae.*

(i) *If χ is an even character* (*i.e.*, $\chi(-1)=1$), *then*

$$\begin{aligned}(1.7)\qquad nL(2n+1,\chi)-n\sum_{l=1}^{\infty}\frac{\chi(l)\cos(2\pi lx/q)}{l^{2n+1}}-\pi x\sum_{l=1}^{\infty}\frac{\chi(l)\sin(2\pi lx/q)}{l^{2n}}\\ =(-1)^n\left(\frac{2\pi x}{q}\right)^{2n}\Bigg\{\sum_{k=1}^{n-1}(-1)^{k-1}\frac{kL(2k+1,\chi)}{(2n-2k)!(2\pi x/q)^{2k}}\\ +\frac{\tau(\chi)}{q}\sum_{k=0}^{\infty}\frac{(2k)!L(2k,\overline{\chi})}{(2n+2k)!}x^{2k}\Bigg\}.\end{aligned}$$

(ii) *If χ is an odd character* (*i.e.*, $\chi(-1)=-1$), *then*

$$\begin{aligned}(1.8)\qquad L(2n,\chi)-\sum_{l=1}^{\infty}\frac{\chi(l)\cos(2\pi lx/q)}{l^{2n}}\\ =(-1)^n\left(\frac{2\pi x}{q}\right)^{2n-1}\Bigg\{\sum_{k=1}^{n-1}(-1)^{k}\frac{L(2k,\chi)}{(2n-2k)!(2\pi x/q)^{2k-1}}\\ +2i\frac{\tau(\chi)}{q}\sum_{k=0}^{\infty}\frac{(2k)!L(2k+1,\overline{\chi})}{(2n+2k)!}x^{2k+1}\Bigg\}.\end{aligned}$$

REMARK. The shape of the left-hand side of (1.8) shows that this formula is rather a χ-analogue of (1.6).

The author would like to express his sincere gratitude to Professor H. M. Srivastava for kindly sending the paper [Sr2]. He would also like to thank the referee for valuable comments on the earlier version of the present paper.

We prove Theorem 1 in the next section. Theorem 2 is shown in Section 3. The last section is devoted to the proof of Theorem 3.

2. Proof of Theorem 1. Let n be a fixed positive integer, x a real variable, and set

$$I(x) = \frac{1}{4i} \int_{(\sigma_0)} \cot(\pi s/2)\zeta(s) \frac{x^s}{(s+1)(s+2)\dots(s+2n)}\, ds \qquad (|x| \le 1), \tag{2.1}$$

where σ_0 is a constant satisfying $-1/2 < \sigma_0 < 0$, and (σ_0) denotes the vertical straight line from $\sigma_0 - i\infty$ to $\sigma_0 + i\infty$. The integral in (2.1) converges absolutely, because the integrand is $O(|t|^{1/2-\sigma_0-2n+\varepsilon})$ as $t \to \pm\infty$, with an arbitrary small $\varepsilon > 0$. This immediately follows from the vertical estimate $\zeta(s) = O(|t|^{1/2-\sigma+\varepsilon})$ for $\sigma < 0$ (cf. [Ti, p. 95, Chapter V]).

We start the proof of Theorem 1 with the observation that

$$I(x) = -\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)\dots(2k+2n)} x^{2k} \qquad (|x| \le 1). \tag{2.2}$$

This can be shown by moving the path (σ_0) of the integral in (2.1) to the right, and collecting the residues of the poles at $s = 2k$ $(k = 0, 1, 2, \dots)$, because the order of the integrand is $O\{(K + |t|)^{-2n-1}|x|^K\}$ as $t \to \pm\infty$, on the line $\sigma = 2K + 1$ $(K = 1, 2, \dots)$.

We next transform the integral in (2.1) by applying the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s) \tag{2.3}$$

(cf. [Iv, p. 9, Chapter 1, (1.24)]), where $\Gamma(s)$ denotes the gamma function. Substituting this into the integral in (2.1) and changing the variable s into $1 - s$, we obtain

$$I(x) = \frac{1}{2i} x \int_{(\sigma_1)} \sin(\pi s/2) F(s) \zeta(s) (2\pi x)^{-s}\, ds, \tag{2.4}$$

where $\sigma_1 = 1 - \sigma_0$ and

$$F(s) = \frac{\Gamma(s)}{(s-2)(s-3)\dots(s-2n-1)}.$$

Note that σ_1 satisfies $1 < \sigma_1 < 3/2$. Since $\zeta(s) = \sum_{l=1}^{\infty} l^{-s}$ converges absolutely for $\sigma = \sigma_1$, it follows from (2.4) that

$$I(x) = \frac{1}{2}\pi i x \sum_{l=1}^{\infty} \{f(2\pi i l x) - f(-2\pi i l x)\}, \tag{2.5}$$

where

$$f(z) = \frac{1}{2\pi i} \int_{(\sigma_1)} F(s) z^{-s}\, ds. \tag{2.6}$$

This integral converges absolutely for $|\arg z| \le \pi/2$, since the order of the integrand is $O\{|t|^{\sigma_1-1/2-2n}e^{-(\pi/2-|\arg z|)|t|}\}$ as $t \to \pm\infty$ (cf. [Iv, p. 492, Appendix, (A.34)]), and so that the interchange of the order of summation and integration is justified by the fact that $f(\pm 2\pi ilx) = O(l^{-\sigma_1})$ for $l = 1, 2, \ldots$ The identity

$$\frac{1}{(s-2)(s-3)\ldots(s-2n-1)} = \frac{1}{(s-1)\ldots(s-2n)} + \frac{2n}{(s-1)\ldots(s-2n-1)}$$

and the functional equation $\Gamma(z+1) = z\Gamma(z)$ show that

$$F(s) = \Gamma(s-2n) + 2n\Gamma(s-2n-1). \tag{2.7}$$

To evaluate the integral in (2.6), we need

LEMMA. *Let σ_1 be a constant satisfying $1 < \sigma_1 < 3/2$. For any integer $k \ge 2$ and any complex z with $|\arg z| \le \pi/2$, we have*

$$\frac{1}{2\pi i}\int_{(\sigma_1)} \Gamma(s-k)z^{-s}\,ds = z^{-k}\Big\{e^{-z} - \sum_{h=0}^{k-2}\frac{(-z)^h}{h!}\Big\}. \tag{2.8}$$

Proof. Suppose first that $|\arg z| < \pi/2$. Then changing the variable s into $s+k$, we see that the left-hand side of (2.8) is equal to

$$\frac{1}{2\pi i}z^{-k}\int_{(\sigma_1-k)} \Gamma(s)z^{-s}\,ds.$$

We move the path $(\sigma_1 - k)$ of this integral to the left, noting that $1-k < \sigma_1 - k < 3/2 - k$ $(< 2-k)$. Collecting the residues of the poles at $s = -h$ $(h = k-1, k, k+1, \ldots)$, we find that the left-hand side of (2.8) is further modified as $z^{-k}\sum_{h=k-1}^{\infty}(-z)^h/h!$. This proves the lemma for $|\arg z| < \pi/2$. The remaining case follows from the continuity of the integral in (2.8), since the order of the integrand is $O\{|t|^{\sigma_1-k-1/2}e^{-(\pi/2-|\arg z|)|t|}\}$ for $|\arg z| \le \pi/2$ as $t \to \pm\infty$. ■

It follows from (2.6), (2.7) and the Lemma that

$$\begin{aligned} f(2\pi ilx) - f(-2\pi ilx) \\ = &-4n(2\pi ilx)^{-2n-1} + 4n(2\pi ilx)^{-2n-1}\cos(2\pi lx) \\ &- 2i(2\pi ilx)^{-2n}\sin(2\pi lx) - 4\sum_{k=1}^{n-1}\frac{k}{(2n-2k)!}(2\pi ilx)^{-2k-1}. \end{aligned}$$

Substituting this into (2.5), we obtain

$$
\begin{aligned}
I(x) = & -n(2\pi ix)^{-2n}\zeta(2n+1) + n(2\pi ix)^{-2n}\sum_{l=1}^{\infty}\frac{\cos(2\pi lx)}{l^{2n+1}} \\
& + \pi x(2\pi ix)^{-2n}\sum_{l=1}^{\infty}\frac{\sin(2\pi lx)}{l^{2n}} - \sum_{k=1}^{n-1}\frac{k\zeta(2k+1)}{(2n-2k)!}(2\pi ix)^{-2k},
\end{aligned}
$$

which together with (2.2) completes the proof of Theorem 1. ∎

3. Proof of Theorem 2. In this section we prove Theorem 2. The skeleton of the proof is the same as that of Theorem 1, so the details will be omitted. Throughout the following sections, σ_0 and σ_1 are constants satisfying $-1/2 < \sigma_0 < 0$ and $1 < \sigma_1\ (= 1-\sigma_0) < 3/2$ respectively.

We begin the proof with the integral

$$
J(x) = \frac{1}{4i}\int_{(\sigma_0)} \cot(\pi s/2)\zeta(s)\frac{x^s}{s(s+1)\dots(s+2n+1)}\,ds \qquad (|x|\le 1). \tag{3.1}
$$

Noting that $\zeta(0) = -1/2$, $\zeta'(0) = -(1/2)\log 2\pi$, and

$$
\begin{aligned}
\frac{1}{s(s+1)\dots(s+2n+1)} &= \frac{\Gamma(s)}{\Gamma(s+2n+2)} \\
&= \frac{s^{-1}}{\Gamma(2n+2)}\cdot\frac{1+\psi(1)s+O(s^2)}{1+\psi(2n+2)s+O(s^2)}
\end{aligned}
$$

with $\psi(s) = (\Gamma'/\Gamma)(s)$, we see that the residue of the pole at $s=0$ of the integrand in (3.1) is

$$
\frac{1}{\pi(2n+1)!}(\psi(2n+2)-\psi(1)-\log 2\pi x).
$$

Then moving the path of integration in (3.1) to the right, collecting the residues of the poles at $s = 2k$ ($k = 0,1,2,\dots$), and using $\psi(z+1) = \psi(z)+1/z$, we get

$$
\begin{aligned}
J(x) = & -\frac{1}{2(2n+1)!}\left(\sum_{m=1}^{2n+1}\frac{1}{m} - \log 2\pi x\right) \\
& - \sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k)(2k+1)\dots(2k+2n+1)}x^{2k} \qquad (|x|\le 1).
\end{aligned} \tag{3.2}
$$

On the other hand, we substitute (2.3) into the integral in (3.1), then change the variable s into $1-s$, and obtain

$$
J(x) = \frac{1}{2i}x\int_{(\sigma_1)} \sin(\pi s/2)G(s)\zeta(s)(2\pi x)^{-s}\,ds, \tag{3.3}
$$

where

$$G(s)=\frac{\Gamma(s)}{(s-1)(s-2)\dots(s-2n-2)}=\Gamma(s-2n-2). \tag{3.4}$$

Remark. In comparison with (2.7), the gamma factor (3.4) does not split in this case; the evaluation of $J(x)$ becomes simpler than that of $I(x)$ in the preceding case.

Substituting the representation $\zeta(s)=\sum_{l=1}^{\infty} l^{-s}$ into the integral in (3.3) and changing the order of summation and integration, we obtain

$$J(x)=\frac{1}{2}\pi ix\sum_{l=1}^{\infty}\{g(2\pi ilx)-g(-2\pi ilx)\}, \tag{3.5}$$

where

$$g(z)=\frac{1}{2\pi i}\int_{(\sigma_1)} G(s)z^{-s}\,ds$$

for $|\arg z|\le\pi/2$. Hence by the Lemma and (3.5),

$$J(x)=\frac{1}{2}(2\pi ix)^{-2n}\zeta(2n+1)+\pi x(2\pi ix)^{-2n-2}\sum_{l=1}^{\infty}\frac{\sin(2\pi lx)}{l^{2n+2}}$$
$$+\frac{1}{2}\sum_{k=1}^{n-1}\frac{\zeta(2k+1)}{(2n-2k+1)!}(2\pi ix)^{-2k}.$$

Together with (3.2) this establishes Theorem 2. ∎

4. Proof of Theorem 3. We first treat the even character case (i) of Theorem 3. In this case the functional equation is of the form

$$L(s,\overline{\chi})=2\tau(\chi)^{-1}(2\pi/q)^{s-1}\sin(\pi s/2)\Gamma(1-s)L(1-s,\chi) \tag{4.1}$$

(cf. [Wa, p. 29, Chapter 4]). This suggests considering the integral

$$K(x)=\frac{1}{4i}\int_{(\sigma_0)}\cot(\pi s/2)L(s,\overline{\chi})\frac{x^s}{(s+1)(s+2)\dots(s+2n)}\,ds \qquad (|x|\le 1), \tag{4.2}$$

as an initial setting. We first move the path (σ_0) to the right, passing over the poles at $s=2k$ $(k=0,1,2,\dots)$ of the integrand, and obtain

$$K(x)=-\sum_{k=0}^{\infty}\frac{L(2k,\overline{\chi})}{(2k+1)(2k+2)\dots(2k+2n)}x^{2k} \qquad (|x|\le 1). \tag{4.3}$$

Next substituting (4.1) into the integral in (4.2), and then changing the variable s into $1-s$, we get

$$K(x) = \frac{1}{2i} x\tau(\chi)^{-1} \int_{(\sigma_1)} \sin(\pi s/2) F(s) L(s,\chi)(2\pi x/q)^{-s}\, ds.$$

Hence, noting that $L(s,\chi) = \sum_{l=1}^{\infty} \chi(l) l^{-s}$ converges absolutely for $\sigma = \sigma_1$, we obtain

$$K(x) = \frac{1}{2}\pi i x\tau(\chi)^{-1} \sum_{l=1}^{\infty} \chi(l) \left\{ f\left(\frac{2\pi i l x}{q}\right) - f\left(-\frac{2\pi i l x}{q}\right)\right\},$$

where $f(z)$ is given by (2.6). The evaluation of $f(2\pi i l x/q) - f(-2\pi i l x/q)$ is the same as in the proof of Theorem 1, so that

$$\begin{aligned} K(x) = &-nq\tau(\chi)^{-1}\left(\frac{2\pi i x}{q}\right)^{-2n} L(2n+1,\chi) \\ &+ nq\tau(\chi)^{-1}\left(\frac{2\pi i x}{q}\right)^{-2n} \sum_{l=1}^{\infty} \frac{\chi(l)\cos(2\pi l x/q)}{l^{2n+1}} \\ &+ \pi x\tau(\chi)^{-1}\left(\frac{2\pi i x}{q}\right)^{-2n} \sum_{l=1}^{\infty} \frac{\chi(l)\sin(2\pi l x/q}{l^{2n}} \\ &- q\tau(\chi)^{-1} \sum_{k=1}^{n-1} \frac{kL(2k+1,\chi)}{(2n-2k)!}\left(\frac{2\pi i x}{q}\right)^{-2k}. \end{aligned}$$

Together with (4.3) this establishes Theorem 3(i).

We proceed to treat the odd character case (ii) of Theorem 3. The functional equation in this case asserts that

$$L(s,\overline{\chi}) = 2i\tau(\chi)^{-1}(2\pi/q)^{s-1}\cos(\pi s/2)\Gamma(1-s)L(1-s,\chi) \tag{4.4}$$

(cf. [Wa, p. 29, Chapter 4]). This suggests considering the integral

$$H(x) = \frac{1}{4i}\int_{(\sigma_0)} \tan(\pi s/2) L(s,\overline{\chi}) \frac{x^s}{s(s+1)\dots(s+2n-1)}\, ds \qquad (|x| \le 1), \tag{4.5}$$

at a starting point. We first move the path of integration in (4.5) to the right, passing over the poles at $s = 2k+1$ ($k = 0,1,2,\dots$) of the integrand, and obtain

$$H(x) = \sum_{k=0}^{\infty} \frac{L(2k+1,\overline{\chi})}{(2k+1)(2k+2)\dots(2k+2n)} x^{2k+1} \qquad (|x| \le 1). \tag{4.6}$$

Next substituting (4.4) into the integral in (4.5), and then changing the variable s into $1-s$, we get

$$H(x)=-\frac{1}{2}x\tau(\chi)^{-1}\int\limits_{(\sigma_1)}\cos(\pi s/2)\Gamma(s-2n)L(s,\chi)\left(\frac{2\pi x}{q}\right)^{-s}ds.$$

This yields

$$H(x)=\frac{1}{2}\pi i x\tau(\chi)^{-1}\sum_{l=1}^{\infty}\chi(l)\left\{h\left(\frac{2\pi ilx}{q}\right)+h\left(-\frac{2\pi ilx}{q}\right)\right\},$$

where

$$h(z)=\frac{1}{2\pi i}\int\limits_{(\sigma_1)}\Gamma(s-2n)z^{-s}\,ds$$

for $|\arg z|\le\pi/2$. The evaluation of $h(2\pi ilx/q)+h(-2\pi ilx/q)$ is performed by the Lemma, and it is seen that

$$\begin{aligned}H(x)= &-\frac{1}{2}q\tau(\chi)^{-1}\left(\frac{2\pi ix}{q}\right)^{-2n+1}L(2n,\chi)\\ &+\frac{1}{2}q\tau(\chi)^{-1}\left(\frac{2\pi ix}{q}\right)^{-2n+1}\sum_{l=1}^{\infty}\frac{\chi(l)\cos(2\pi lx/q)}{l^{2n}}\\ &-\frac{1}{2}q\tau(\chi)^{-1}\sum_{k=1}^{n-1}\frac{L(2k,\chi)}{(2n-2k)!}\left(\frac{2\pi ix}{q}\right)^{-2k+1}.\end{aligned}$$

Together with (4.6) this establishes Theorem 3(ii). The proof of Theorem 3 is therefore complete. ∎

References

[Ay] R. Ayoub, *Euler and the zeta function*, Amer. Math. Monthly 81 (1974), 1067–1086.

[Ch] B. R. Choe, *An elementary proof of* $\sum_{n=1}^{\infty}1/n^2=\pi^2/6$, ibid. 94 (1987), 662–663.

[CK] D. Cvijović and J. Klinowski, *New rapidly convergent series representations for* $\zeta(2n+1)$, Proc. Amer. Math. Soc. 125 (1997), 1263–1271.

[Ew1] J. A. Ewell, *A new series representation for* $\zeta(3)$, Amer. Math. Monthly 97 (1990), 219–220.

[Ew2] —, *On values of the Riemann zeta function at integral arguments*, Canad. Math. Bull. 34 (1991), 60–66.

[Ew3] —, *On the zeta function values* $\zeta(2k+1)$, $k=1,2,\ldots$, Rocky Mountain J. Math. 23 (1995), 1003–1012.

[Iv] A. Ivić, *The Riemann Zeta-Function*, Wiley, New York, 1985.

[Ka1] M. Katsurada, *Power series with the Riemann zeta-function in the coefficients*, Proc. Japan Acad. Ser. A 72 (1996), 61–63.

[Ka2] M. Katsurada, *On Mellin–Barnes type of integrals and sums associated with the Riemann zeta-function*, Publ. Inst. Math. (Beograd) (N.S.) 62 (76) (1997), 13–25.

[Ka3] —, *Power series and asymptotic series associated with the Lerch zeta-function*, Proc. Japan Acad. Ser. A 74 (1998), 167–170.

[Ra] V. Ramaswami, *Notes on Riemann's* ζ*-function*, J. London Math. Soc. 9 (1934), 165–169.

[Sr1] H. M. Srivastava, *A unified presentation of certain classes of series of the Riemann zeta function*, Riv. Mat. Univ. Parma (4) 14 (1988), 1–23.

[Sr2] —, *Certain families of rapidly convergent series representations for* $\zeta(2n+1)$, Math. Sci. Research Hot-Line 1 (6) (1997), 1–6.

[Sr3] —, *Some rapidly converging series for* $\zeta(2n+1)$, Proc. Amer. Math. Soc. 127 (1999), 385–396.

[Ti] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd ed., revised by D. R. Heath-Brown, Oxford Univ. Press, 1986.

[Wa] L. C. Washington, *Introduction to Cyclotomic Fields*, Springer, New York, 1982.

[Wi] J. R. Wilton, *A proof of Burnside's formula for* $\log\Gamma(x+1)$ *and certain allied properties of Riemann's* ζ*-function*, Messenger Math. 52 (1922/1923), 90–93.

[YW] Z.-N. Yue and K. S. Williams, *Some series representations of* $\zeta(2n+1)$, Rocky Mountain J. Math. 23 (1993), 1581–1591.

Department of Mathematics and Computer Science
Kagoshima University
1-21-35, Korimoto
Kagoshima 890-0065, Japan

Present address:
Mathematics, Hiyoshi Campus
Keio University
4-1-1, Hiyoshi, Kouhoku-ku
Yokohama 223-8521, Japan
E-mail: masanori@math.hc.keio.ac.jp

Received on 5.10.1998
and in revised form on 12.3.1999 (3476)