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On the average number of direct factors of finite abelian groups (II)

by

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1. Introduction. Let $a : n \to a(n)$ be as usual the arithmetic function which counts the number of finite abelian groups of given order n and define $t = a * a$ and $\omega = a * a * a$. We shall be concerned with obtaining estimates for the sums $T(x) = \sum_{n \le x} t(n)$ and $W(x) = \sum_{n \le x} \omega(n)$.

The asymptotic behaviour of $T(x)$ was first studied by Cohen [1], who derived that

$$T(x) = c_1 x(\log x + 2\gamma - 1) + c_2 x + \Delta_0(x) \tag{1.1}$$

with $\Delta_0(x) \ll \sqrt{x}\log x$. Krätzel [6] improved this result to

$$\Delta_0(x) = c_3\sqrt{x}((\log x)/2 + 2\gamma - 1) + c_4\sqrt{x} + \Delta_1(x) \tag{1.2}$$

with $\Delta_1(x) \ll x^{5/12}\log^4 x$. The exponent 5/12 was improved to 83/201, 45/109, 9/22, 3/8, 7/19, 4/11 by Menzer [8], Menzer and Seibold [10], Menzer [9], Yu [14], Liu [7], Zhai and Cao [15], respectively. It should be mentioned that recently J. Wu [13] has obtained a better exponent 47/131.

H. Menzer [9] studied the asymptotic behaviour of $W(x)$. He proved that

$$W(x) = xP_1^{(2)}(\log x) + \sqrt{x}P_2^{(2)}(\log x) + O(x^{76/153}\log^4 x), \tag{1.3}$$

where $P_j^{(2)}$ $(j = 1, 2)$ denotes a polynomial of degree 2.

The aim of this short note is to further improve Menzer's result by a different approach giving

Theorem 1. *We have the asymptotic formula*

$$W(x) = xP_1^{(2)}(\log x) + \sqrt{x}P_2^{(2)}(\log x) + O(x^{53/116+\varepsilon}). \tag{1.4}$$

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Following H. Menzer [9], we only need to study the asymptotic behaviour of the divisor function

$$d(1,1,1,2,2,2;n) = \sum_{n_1 n_2 n_3 m_1^2 m_2^2 m_3^2 = n} 1.$$

Let $\Delta(1,1,1,2,2,2;x)$ denote the error term of the summation function

$$D(1,1,1,2,2,2;x) = \sum_{n\le x} d(1,1,1,2,2,2;n).$$

Then Theorem 1 follows from

THEOREM 2. *We have*

$$\Delta(1,1,1,2,2,2;x) = O(x^{53/116+\varepsilon}). \tag{1.5}$$

The key of the proof is contained in Lemma 8 of Section 2, which connects the problem with the well-known Piltz divisor problem. So the corresponding exponential sums are bilinear forms which can be estimated by the well-known double large sieve inequality due to Bombieri and Iwaniec (Proposition 1 of Fouvry and Iwaniec [2]; see Lemma 1 below). A detailed proof of Theorem 2 is given in Section 3.

Notations. $e(t) = \exp(2\pi it)$. $n \sim N$ means $C_1 N < n < C_2 N$ for some absolute constants C_1 and C_2. ε is a sufficiently small number which may be different at each occurrence. $\Delta_3(t)$ always denotes the error term of the Piltz divisor problem. We use notation $\mathrm{SC}(\Sigma)$ to denote the summation conditions of the sum Σ if these conditions are complicated. For example, instead of

$$F(x) = \sum_{a\le n\le x} f(n)$$

we write

$$F(x) = \sum f(n), \quad \mathrm{SC}(\Sigma): a\le n\le x.$$

2. Some preliminary lemmas. We need the following lemmas.

LEMMA 1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite sets of real numbers*, $\mathcal{X}\subset[-X,X]$, $\mathcal{Y}\subset[-Y,Y]$. *Then for any complex functions $u(x)$ and $v(y)$ we have*

$$\Big|\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} u(x)v(y)e(xy)\Big|^2 \le 20(1+XY) \sum_{\substack{x,x'\in\mathcal{X}\\ |x-x'|\le Y^{-1}}} |u(x)u(x')| \sum_{\substack{y,y'\in\mathcal{Y}\\ |y-y'|\le X^{-1}}} |v(y)v(y')|.$$

Proof. This is Proposition 1 of Fouvry and Iwaniec [2].

LEMMA 2. *Suppose that* $0 < a < b \le 2a$ *and* R *is an open convex set in* $\mathbb{C}$ *containing the real segment* $[a,b]$. *Suppose further that* $f(z)$ *is analytic on* R, $f(x)$ *is real for real* x *in* $R, |f''(z)| \le M$ *for* $z \in R$, *and there is a constant* $k > 0$ *such that* $f''(x) \le -kM$ *for all real* x *in* R. *Let* $\alpha = f'(b)$, $\beta = f'(a)$ *and define* x_v *for each integer* v *in the range* $\alpha < v < \beta$ *by* $f'(x_v) = v$. *Then*

$$\sum_{a<n\le b} e(f(n)) = e(-1/8) \sum_{\alpha<v\le\beta} |f''(x_v)|^{-1/2} e(f(x_v) - vx_v) + O(M^{-1/2} + \log(2 + M(b-a))).$$

Proof. This is Lemma 6 of Heath-Brown [3].

LEMMA 3. *Let* $\alpha\beta \ne 0$, $\Delta > 0$, $M \ge 1$ *and* $N \ge 1$. *Let* $\mathcal{A}(M,N;\Delta)$ *be the number of quadruples* (m_1, m_2, n_1, n_2) *such that*

$$\left| \left(\frac{m_1}{m_2}\right)^{\alpha} - \left(\frac{n_1}{n_2}\right)^{\beta} \right| < \Delta$$

with $M \le m_1, m_2 \le 2M$ *and* $N \le n_1, n_2 \le 2N$. *Then*

$$\mathcal{A}(M,N;\Delta) \ll MN \log 2MN + \Delta M^2 N^2.$$

Proof. This is Lemma 1 of Fouvry and Iwaniec [2].

LEMMA 4. *Let* $0 < L \le N < M \le cL$ *and* a_l *be complex numbers such that* $|a_l| \le 1$. *Then*

$$\sum_{N<n\le M} a_n = \frac{1}{2\pi} \int_{-cL}^{cL} \Big(\sum_{L<l\le cL} a_l l^{-it} \Big) (M^{it} - N^{it}) t^{-1}\, dt + O(\log(2+L)).$$

Proof. This is essentially Lemma 6 of Fouvry and Iwaniec [2].

LEMMA 5. *For* $1 \ll Y \ll x^2$ *we have*

$$\Delta_3(x) = \frac{x^{1/3}}{\sqrt{3}\,\pi} \sum_{1\le n\le Y} \frac{d_3(n)}{n^{2/3}} \cos(6(nx)^{1/3}) + O(x^{2/3+\varepsilon} Y^{-1/3}).$$

Proof. See Chapter 3 of Ivić [4] or p. 80 of Min [11].

LEMMA 6. *Let* $M > 0$, $N > 0$, $u_m > 0$, $v_n > 0$, $A_m > 0$, $B_n > 0$ $(1 \le m \le M,\ 1 \le n \le N)$, *and let* Q_1 *and* Q_2 *be given non-negative numbers*, $Q_1 \le Q_2$. *Then there is a* q *such that* $Q_1 \le q \le Q_2$ *and*

$$\sum_{m=1}^{M} A_m q^{u_m} + \sum_{n=1}^{N} B_n q^{-v_n} \ll \sum_{m=1}^{M} \sum_{n=1}^{N} (A_m^{v_n} B_n^{u_m})^{1/(u_m+v_n)} + \sum_{m=1}^{M} A_m Q_1^{u_m} + \sum_{n=1}^{N} B_n Q_2^{-v_n}.$$

Proof. See Srinivasan [12].

LEMMA 7. *Suppose $M \geq 2$, $N \geq 2$, $z \geq 2$ are positive numbers, α and β are real non-integer constants and $F = zM^\alpha N^\beta$, $a(m) \ll d_3(m)$, $b(n) \ll d_3(n)$. Then*

$$\sum_{m\sim M} a(m) \sum_{n\sim N} b(n)e(zm^\alpha n^\beta) \ll (FMN)^{1/2}\left(1+\frac{M}{F}\right)^{1/2}\left(1+\frac{N}{F}\right)^{1/2}\log^8 FMN.$$

P r o o f. This can be easily derived from Theorem 1 of Fouvry and Iwaniec [2] if we notice $\sum_{n\leq x} d_3^2(n) \ll x\log^8 x$.

LEMMA 8. *We have*

$$\Delta(1,1,1,2,2,2;x) = \sum_{m\leq x^{1/3}} d_3(m)\Delta_3\left(\frac{x}{m^2}\right) + \sum_{m\leq x^{1/3}} d_3(m)\Delta_3\left(\sqrt{\frac{x}{m}}\right) + O(x^{1/3+\varepsilon}). \tag{2.1}$$

P r o o f. This lemma plays an important role in our proof and the same idea has been used in Zhai and Cao [15].

We only sketch the proof since it is elementary and direct. We leave the details to the reader.

We begin with

$$\begin{aligned} D(1,1,1,2,2,2;x) &= \sum_{n\leq x} d(1,1,1,2,2,2;n) \\ &= \sum_{n_1n_2n_3m_1^2m_2^2m_3^2\leq x} 1 = \sum_{m^2n\leq x} d_3(n)d_3(m) \\ &= \sum_{n\leq x^{1/3}} d_3(n)D_3\left(\sqrt{\frac{x}{n}}\right) + \sum_{m\leq x^{1/3}} D_3\left(\frac{x}{m^2}\right) - D_3^2(x^{1/3}) \\ &= \Sigma_1 + \Sigma_2 - D_3^2(x^{1/3}), \end{aligned} \tag{2.2}$$

say, where $D_3(x) = \sum_{n\leq x} d_3(n)$. We use the following abelian partial summation formula:

$$\sum_{n\leq u} d_3(n)f(n) = D_3(u)f(u) - \int_1^u D_3(t)f'(t)\,dt \tag{2.3}$$

to Σ_1 and Σ_2. For $D_3(u)$, we use

$$D_3(u) = d_1u\log^2 u + d_2u\log u + d_3u + \Delta_3(u) \tag{2.4}$$

with $\Delta_3(u) \ll u^{1/2}$ and

$$\int_1^T \Delta_3(u)\,du \ll T^{1+\varepsilon} \qquad (T \geq 2). \tag{2.5}$$

Formula (2.4) is in Chapter 13 of Ivić [4]. Formula (2.5) can be easily derived from Lemma 5.

After some calculations, we can get

$$D(1,1,1,2,2,2;x) = \text{main terms} + \sum_{m \leq x^{1/3}} d_3(m)\Delta_3\left(\frac{x}{m^2}\right) + \sum_{n \leq x^{1/3}} d_3(n)\Delta_3\left(\sqrt{\frac{x}{n}}\right) + O(x^{1/3+\varepsilon}), \tag{2.6}$$

whence our lemma follows.

3. Proof of Theorem 2. In order to prove Theorem 2, we only need to estimate the two sums in Lemma 8. We first prove the following

PROPOSITION 1. *We have the estimate*

$$\sum_{m \leq x^{1/3}} d_3(m)\Delta_3\left(\frac{x}{m^2}\right) = O(x^{53/116+\varepsilon}).$$

P r o o f. We only need to show that

$$S(M) = \sum_{m \sim M} d_3(m)\Delta_3\left(\frac{x}{m^2}\right) \ll x^{53/116+\varepsilon} \tag{3.1}$$

for any fixed $1 \ll M \ll x^{1/3}$.

CASE 1: $M \ll x^{5/58}$. In this case (3.1) follows from Kolesnik's well-known estimate $\Delta_3(u) \ll u^{43/96+\varepsilon}$ (see Kolesnik [5]).

CASE 2: $x^{5/58} \ll M \ll x^{1/5}$. Suppose $1 \ll Y \ll x$ is a parameter to be determined. By Lemma 5 we get

$$\begin{aligned} S(M) &\ll x^{1/3}\left|\sum_{m \sim M} \frac{d_3(m)}{m^{2/3}} \sum_{n \leq Y} \frac{d_3(n)}{n^{2/3}} e\left(\frac{3(nx)^{1/3}}{m^{2/3}}\right)\right| + \frac{x^{2/3+\varepsilon}}{(YM)^{1/3}} + x^{\varepsilon} \\ &\ll x^{1/3}|S(M,N)|\log x + \frac{x^{2/3+\varepsilon}}{(YM)^{1/3}} + x^{\varepsilon} \end{aligned} \tag{3.2}$$

for some $1 \ll N \ll Y$, where

$$S(M,N) = \sum_{m \sim M} \frac{d_3(m)}{m^{2/3}} \sum_{n \sim N} \frac{d_3(n)}{n^{2/3}} e\left(\frac{3(nx)^{1/3}}{m^{2/3}}\right). \tag{3.3}$$

We choose $Y = xM^{-2}$, thus

$$\frac{x^{2/3+\varepsilon}}{(YM)^{1/3}} \ll (xM)^{1/3} x^{\varepsilon} \ll x^{2/5+\varepsilon}.$$

Let $a(m) = d_3(m)(M/m)^{2/3}$, $b(n) = d_3(n)(N/n)^{2/3}$, $F = (xN)^{1/3}M^{-2/3}$, and

$$T(M,N) = \sum_{m\sim M} a(m) \sum_{n\sim N} b(n) e\biggl(\frac{3(nx)^{1/3}}{m^{2/3}}\biggr).$$

Obviously

$$S(M,N) \ll x^{1/3}(MN)^{-2/3}|T(M,N)|. \tag{3.4}$$

Since $M \ll x^{1/5}$, it is easy to check that $F \gg M$. If $N \ll x^{1/2}/M$, then $F \gg N$. By Lemma 7 we can get

$$T(M,N) \ll (FMN)^{1/2} \log^8 x,$$

which combined with (3.4) gives

$$S(M,N) \ll x^{1/2}M^{-1/2} \log^8 x \ll x^{53/116+\varepsilon}. \tag{3.5}$$

Now suppose $N \gg x^{1/2}/M$. Using the expression $d_3(n) = \sum_{n=uvw} 1$, we find that $T(M,N)$ can be divided into $O(\log^3 x)$ sums of the form

$$T(M,U,V,W) = \sum_{m\sim M} a(m) \sum_{(u,v,w)} e\biggl(\frac{3(uvwx)^{1/3}}{m^{2/3}}\biggr),$$

where

$$\mathrm{SC}\Bigl(\sum_{(u,v,w)}\Bigr) : \begin{array}{l} N \le uvw \le 2N,\ U < u \le 2U,\ V < v \le 2V, \\ W < w \le 2W,\ u \le v \le w,\ UVW \sim N. \end{array}$$

It follows that $W \gg N^{1/3}$. If $W \gg N^{2/3}$, then by the exponent pair $(1/2, 1/2)$ we have

$$\begin{aligned} T(M,U,V,W) \log^{-8} x &\ll MUVF^{1/2} + MN/F \\ &\ll F^{1/2}MN^{1/3} + MN/F, \end{aligned} \tag{3.6}$$

whose contribution to $S(M,N)$ is

$$\begin{aligned} &\ll F^{3/2}MN^{-2/3} \log^8 x + M \log^8 x \ll x^{1/2}N^{-1/6} \log^8 x + M \log^8 x \\ &\ll x^{5/12}M^{1/6} \log^8 x + M \log^8 x \ll x^{9/20} \log^8 x, \end{aligned}$$

where we used the assumptions $N \gg x^{1/2}/M$ and $M \ll x^{1/5}$.

Later we always suppose $N^{1/3} \ll W \ll N^{2/3}$, namely, $N^{1/3} \ll UV \ll N^{2/3}$. Let $a = \max(N/(uv), W, v)$, $b = \min(2W, 2N/(uv))$. Using Lemma 2 to the variable w we have

$$(3.7)\qquad \sum_{a\le w\le b} e\bigg(\frac{3(uvwx)^{1/3}}{m^{2/3}}\bigg) = c_0\sum_r \frac{(xuv)^{1/4}}{m^{2/4}r^{5/4}} e\bigg(\frac{2(xuv)^{1/2}}{mr^{1/2}}\bigg) + O\bigg(\log x + \frac{W}{F^{1/2}}\bigg),$$

where

$$\mathrm{SC}\Big(\sum_r\Big): B = \frac{(xuv)^{1/3}}{(mb)^{2/3}} \le r \le \frac{(xuv)^{1/3}}{(ma)^{2/3}} = A,\ r\sim R = F/W.$$

Using Lemma 4 to the variable r we find that

$$(3.8)\qquad \sum_{a\le w\le b} e\bigg(\frac{3(uvwx)^{1/3}}{m^{2/3}}\bigg) = \frac{c_0}{2\pi}\int_{F/(100W)}^{100F/W}\bigg(\sum_{F/(100W)\le r\le 100F/W} \frac{(xuv)^{1/4}}{m^{2/4}r^{5/4+it}} e\bigg(\frac{2(xuv)^{1/2}}{mr^{1/2}}\bigg)\bigg) \times \frac{A^{it}-B^{it}}{t}\,dt + O(\log x + WF^{-1/2}\log x).$$

Thus we get

$$(3.9)\qquad T(M,U,V,W) \ll \frac{W}{F^{1/2}}\sum_{m\sim M}\sum_{r\sim R} d_3(m)\bigg|\sum_{(u,v)} c_1(u)c_2(v)e\bigg(\frac{2(xuv)^{1/2}}{mr^{1/2}}\bigg)\bigg| + MN^{2/3}\log x + MNF^{-1/2}\log x$$

for some $c_1(u)\ll 1$ and some $c_2(v)\ll 1$, where

$$\mathrm{SC}\Big(\sum_{(u,v)}\Big): U\le u<2U,\ V\le v<2V,\ u\le v.$$

Let T^* denote the exponential sum in the right side of (3.9). By Lemma 1 we get

$$(3.10)\qquad |T^*|^2 \ll F\mathcal{A}\mathcal{B},$$

where

$$\mathcal{A} = \sum_{*} d_3(m)d_3(m_1), \qquad \mathcal{B} = \sum_{**} 1$$

with

$$\mathrm{SC}\Big(\sum_{*}\Big) : |m^{-1}r^{-1/2} - m_1^{-1}r_1^{-1/2}| \ll (xUV)^{-1/2},\ m \sim M,\ r \sim R,$$

$$\mathrm{SC}\Big(\sum_{**}\Big) : |u^{1/2}v^{1/2} - u_1^{1/2}v_1^{1/2}| \ll MR^{1/2}x^{-1/2},\ u \sim U,\ v \sim V.$$

By Lemma 3 we get

$$\mathcal{A}x^{-\varepsilon} \ll MR + (xUV)^{-1/2}M^3R^{5/2} \tag{3.11}$$

and

$$\begin{aligned}\mathcal{B} &\ll UV\log x + MR^{1/2}x^{-1/2}(UV)^{3/2} \\ &\ll UV\log x + (UV)^2/F \ll UV\log x,\end{aligned} \tag{3.12}$$

where we used the fact that $F \gg N^{2/3}$.

Combining (3.9)–(3.12) we get

$$\begin{aligned}T(M,U,V,W)x^{-\varepsilon} &\ll F^{5/4}(UV)^{1/4}M^{3/2}x^{-1/4}W^{-1/4} \\ &\quad + (FMN)^{1/2} + MN^{2/3} + MNF^{-1/2} \\ &\ll (FMN)^{1/2} + x^{-1/4}F^{5/4}M^{3/2}N^{1/12} \\ &\quad + MN^{2/3} + MNF^{-1/2},\end{aligned} \tag{3.13}$$

whose contribution to $S(M,N)$ is

$$\begin{aligned}&\ll \frac{Fx^{\varepsilon}}{N}\bigg((FMN)^{1/2} + x^{-1/4}F^{5/4}M^{3/2}N^{1/12} + MN^{2/3} + \frac{MN}{F^{1/2}}\bigg) \\ &\ll (x^{1/2}M^{-1/2} + x^{1/2}N^{-1/6} + x^{1/3}M^{1/3} + x^{1/6}M^{4/6}N^{1/6})x^{\varepsilon} \\ &\ll (x^{1/2}M^{-1/2} + x^{5/12}M^{1/6} + x^{1/3}M^{1/3})x^{\varepsilon} \ll x^{53/116+\varepsilon},\end{aligned}$$

if we use the assumptions $x^{1/2}M^{-1} \ll N \ll xM^{-2}$ and $M \ll x^{1/5}$.

Combining the above we see that (3.1) holds in Case 2.

Case 3: $x^{1/5} \ll M \ll x^{1/3}$. We begin with (3.2). Using Lemma 7 directly to bound $T(M,N)$ we can get

$$\begin{aligned}S(M)x^{-\varepsilon} &\ll \frac{x^{1/2}}{M^{1/2}} + (xM)^{1/3} + \frac{(xY)^{1/3}}{M^{1/6}} \\ &\quad + x^{1/6}M^{4/6}Y^{1/6} + \frac{x^{2/3}}{(MY)^{1/3}}.\end{aligned} \tag{3.14}$$

Choosing a best $Y \in [1, x^{1/2}]$ via Lemma 6 we get

$$S(M)x^{-\varepsilon} \ll x^{1/2}M^{-1/4} + x^{1/3}M^{1/3} \ll x^{9/20}. \tag{3.15}$$

This completes the proof of Proposition 1.

The second sum in Lemma 8 is handled in

PROPOSITION 2. *We have the estimate*

$$\sum_{m\le x^{1/3}} d_3(m)\Delta_3\Big(\sqrt{\frac{x}{m}}\Big) = O(x^{4/9+\varepsilon}).$$

Proof. It suffices to prove

$$S_1(M) = \sum_{m\sim M} d_3(m)\Delta_3\Big(\sqrt{\frac{x}{m}}\Big) \ll x^{4/9+\varepsilon} \tag{3.16}$$

for $1 \ll M \ll x^{1/3}$.

For $M \ll x^{7/27}$, we use the trivial bound $\Delta_3(u) \ll u^{1/2}$. For $x^{7/27} \ll M \ll x^{1/3}$, the proof of (3.16) is the same as that of Case 3 of Proposition 1. This completes the proof of Proposition 2.

Theorem 2 immediately follows from Lemma 8 and the two propositions.

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