ACTA ARITHMETICA
XC.1 (1999)

Applications of a lower bound for linear forms in two logarithms to exponential Diophantine equations

by

NOBUHIRO TERAI (Ashikaga)

1. Introduction. In 1956, Sierpiński [Si] showed that the equation $3^x + 4^y = 5^z$ has only the positive integral solution $(x, y, z) = (2, 2, 2)$. Jeśmanowicz [J] conjectured that if a, b, c are Pythagorean triples, i.e., positive integers satisfying $a^2 + b^2 = c^2$, then the equation $a^x + b^y = c^z$ has only the positive integral solution $(x, y, z) = (2, 2, 2)$. This conjecture has been proved to be true in many special cases (cf. Guo-Le [GL], Le [Le] and Takakuwa [Ta]). It is, however, still unsolved.

As an analog to this conjecture, we propose the following (cf. Terai [Te1]):

CONJECTURE. *If a, b, c, p, q, r are fixed positive integers satisfying $a^p + b^q = c^r$ with $p, q, r \geq 2$ and $(a, b) = 1$, then the Diophantine equation*

$$a^x + b^y = c^z \tag{1}$$

has only the positive integral solution $(x, y, z) = (p, q, r)$ except for three cases (*taking $a < b$*), *where* (1) *has only the following solutions*, *respectively*

$$(a, b, c) = (2, 3, 5), \quad (x, y, z) = (1, 1, 1),\ (4, 2, 2);$$
$$(a, b, c) = (2, 7, 3), \quad (x, y, z) = (1, 1, 2),\ (5, 2, 4);$$
$$(a, b, c) = (1, 2, 3), \quad (x, y, z) = (m, 1, 1),\ (n, 3, 2)$$

with m, n arbitrary (cf. Nagell [N4], Cao [Cao]).

In our previous papers [Te2]–[Te4], we considered the conjecture above when $p = 2$, $q = 2$ and r is an odd prime. In [Te2] and [Te3], we reduced (1) to certain quartic equations, which have no non-trivial solutions by the method of infinite descent. In [Te4], we reduced (1) to Thue equations, and used the known estimates of linear forms in logarithms due to Mignotte and Waldschmidt [MW] and Bugeaud and Győry [BG].

1991 *Mathematics Subject Classification*: Primary 11D61.

In this paper, we apply a lower bound for linear forms in two logarithms due to Mignotte [M] which is a corollary to a theorem of Laurent–Mignotte–Nesterenko [LMN] to the Diophantine equation

$$a^x + b^n = c^z,$$

where n is a given "small" positive integer (Main Theorem). The Main Theorem shows that if the upper bound n of the solution y of (1) is attained (and small), then the solution x of (1) satisfies

$$x \le n + p - q$$

under a certain condition on a, b when a, b, c, p, q, r are as in the Main Theorem. By an elementary or algebraic method, we can attain the upper bound n. Indeed, in our theorems, the upper bound n is derived by using congruences modulo 3, 8 etc. and results concerning the Diophantine equations of the form $x^2 + D^u = y^v$.

The Main Theorem has a number of applications. An easy consequence is that if A, B, C are fixed positive integers satisfying $A - B = C > 1$, $(A, B) = 1$ and $B \ge 1697C$, then the Diophantine equation

$$A^x - B^y = C$$

has only the positive integral solution $(x, y) = (1, 1)$ (Theorem 3 in Section 4). In Section 3, using the Main Theorem, we show that the conjecture above holds under some conditions on a, b, c (Theorems 1, 2 in Section 3). In particular, there are infinitely many a, b, c such that it holds when $(p, q, r) = (2, 2, 3)$. In Section 4, we illustrate in detail how the upper bound n is determined and the Main Theorem is applied to equation (1) for various degrees $p, q, r \ge 1$. In some of the theorems of that section, we verify that the condition "$a \ge \kappa b^{q/p}$" in the Main Theorem can easily be eliminated.

2. Main Theorem. We use the following result of Mignotte [M] to prove the Main Theorem, which plays an important role in the proofs.

Let α be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \ldots + a_d = a_0 \prod_{i=1}^{d} (x - \alpha_i),$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the α_i's are conjugates of α. Then

$$h(\alpha) = \frac{1}{d}\Big(\log a_0 + \sum_{i=1}^{d} \log \max(1, |\alpha_i|)\Big)$$

is called the *absolute logarithmic height* of α. In particular, if $\alpha \in \mathbb{Q}$, say $\alpha = p/q$ as a fraction in lowest terms, then $h(\alpha) = \log \max(|p|, |q|)$.

Let α_1, α_2 be two non-zero algebraic numbers, and let $\log\alpha_1$ and $\log\alpha_2$ be any determinations of their logarithms. We consider the linear form

$$\Lambda = b_2\log\alpha_2 - b_1\log\alpha_1,$$

where b_1 and b_2 are positive integers. Without loss of generality, we suppose that $|\alpha_1|$ and $|\alpha_2|$ are ≥ 1. Put

$$D = [\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}]/[\mathbb{R}(\alpha_1,\alpha_2):\mathbb{R}].$$

Lemma 1 (Mignotte [M]). *Let a_1, a_2, h be real positive numbers, and ϱ a real number >1. Put $\lambda=\log\varrho$ and suppose that*

$$h \ge \max\left\{\frac{D\log 2}{2}, C\lambda, D\left(\log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log\lambda+f(K_0)+0.189\right)\right\} \quad \text{with } C\ge 2,$$

$$a_i \ge \max\{2, \varrho|\log\alpha_i| - \log|\alpha_i| + 2Dh(\alpha_i)\} \qquad (i=1,2),$$

where

$$f(x) = \log\frac{(1+\sqrt{x-1})\sqrt{x}}{x-1} + \frac{\log x}{6x(x-1)} + \frac{3}{2} + \log\frac{3}{4} + \frac{\log\frac{x}{x-1}}{x-1}.$$

Suppose also that

$$\frac{1}{a_1}+\frac{1}{a_2} \le \frac{2}{\lambda}$$

and that there exists an integer K_0 such that

$$\frac{8(1+C)a_1a_2}{9\lambda^2} + \frac{4(a_1+a_2)}{3\lambda} + \frac{8\sqrt{2(1+C)a_1a_2}}{3\lambda} > K_0 - 1 \ge 33.$$

If α_1 and α_2 are multiplicatively independent, we have the lower bound

$$\begin{aligned}\log|\Lambda| \ge {} & -\frac{\lambda a_1a_2}{9}\left(\frac{4h}{\lambda^2}+\frac{4}{\lambda}+\frac{1}{h}\right)^2 - \frac{2\lambda}{3}(a_1+a_2)\left(\frac{4h}{\lambda^2}+\frac{4}{\lambda}+\frac{1}{h}\right)\\ & - \frac{16\sqrt{2a_1a_2}}{3}\left(1+\frac{h}{\lambda}\right)^{3/2} - 2(\lambda+h) - \log\left(a_1a_2\left(1+\frac{h}{\lambda}\right)^2\right)\\ & + \frac{\lambda}{2} + \log\lambda - 0.88.\end{aligned}$$

Main Theorem. *Let a, b, c, p, q, r be fixed positive integers satisfying $a^p+b^q=c^r$ with $(a,b)=1$, $a>b>1$, $c\ge 3$ and $p\ge q$. Let n be a given positive integer with $q\le n\le 1722$. If $a\ge\kappa b^{q/p}$ and the Diophantine equation*

$$a^x + b^n = c^z \tag{2}$$

has positive integral solutions x, z with $(x,n)\ne(p,q)$, then

$$x < n+p-q,$$

where

$$\kappa = \left\{ \exp\left(\frac{\delta}{n+1696} \right) - 1 \right\}^{-1/p}$$

and $\delta = 1$ *or* 2 *according as* $rx - pz$ *is odd or even.*

REMARK. We note that the Main Theorem can also be applied to the case of $p = 1, q = 1$ or $r = 1$. The table of values of κ for some p, n, δ is as follows. (These values will be used in the theorems.)

Table

κ	p	n	δ
6.41783...	4	1	1
9.46524...	3	1	2
29.12044...	2	1	2
29.14618...	2	4	2
41.18859...	2	1	1
848.00009...	1	1	2
1696.50004...	1	1	1

Proof (of the Main Theorem). Suppose that $x \geq n + p - q$. From $a^p + b^q = c^r$ and $a^x + b^n = c^z$, we now consider the following linear forms in two logarithms:

$$\Lambda_1 = r \log c - p \log a \quad (> 0), \qquad \Lambda_2 = z \log c - x \log a \quad (> 0).$$

Using the inequality $\log(1 + t) < t$ for $t > 0$, we have

$$0 < \Lambda_2 = \log\left(\frac{c^z}{a^x}\right) = \log\left(1 + \frac{b^n}{a^x}\right) < \frac{b^n}{a^x}.$$

Hence

$$\log \Lambda_2 < n \log b - x \log a. \tag{3}$$

On the other hand, we use Lemma 1 to obtain a lower bound for Λ_2. We keep the notations of Lemma 1. Put $\varrho = 4.9$ and $\lambda = \log \varrho$. We take

$$a_1 = (\varrho - 1) \log a + 2 \log a = (\varrho + 1) \log a > \lambda,$$
$$a_2 = (\varrho - 1) \log c + 2 \log c = (\varrho + 1) \log c > \lambda.$$

Then it is clear that $1/a_1 + 1/a_2 \leq 2/\lambda$. In Lemma 1, we choose $C = 4.5$. Then we take $K_0 = 177$ and $f(K_0) = 1.2879$. Since

$$\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) = \log\left(\frac{x}{\log c} + \frac{z}{\log a}\right) - \log(\varrho + 1),$$

we can take

$$h = \max\left\{ \log\left(\frac{x}{\log c} + \frac{z}{\log a}\right) + 0.17, 9 \right\}.$$

Hence Lemma 1 shows that

$$\begin{aligned}(4)\qquad \log \Lambda_2 \geq\ &-13.09h^2 \log a \log c - 11.73h(\log a + \log c) - 2h \\ &- 28.35h^{3/2}(\log a \log c)^{1/2} - \log(h^2 \log a \log c) - 5.75,\end{aligned}$$

where $h = \max\{\log B + 0.17, 9\}$ and $B = x/\log c + z/\log a$.

If a, b, c are primes ≤ 7, Nagell [N4] completely determined the solutions of the equation $a^x + b^y = c^z$ using the theory of quadratic fields and cubic fields. In view of his result, if a, b, c are positive integers satisfying $a^p + b^q = c^r$ with $(a, b) = 1$, $a > b > 1$, $c \geq 3$, $p \geq q$ and $a, b, c \leq 9$, then the solution x of $a^x + b^n = c^z$ satisfies $x \leq n + p - q$, where n is a fixed positive integer. (The cases where a, b, c are composite can be treated similarly.) Hence we may suppose that

$$(5)\qquad a \geq 10,\ c \geq 3 \quad \text{or} \quad a \geq 3,\ c \geq 10.$$

Now we distinguish two cases: (i) $B \leq e^{8.83}$ $(= 6836.2868\ldots)$ and (ii) $B > e^{8.83}$.

Case (i): $B \leq e^{8.83}$. Then we show that making Λ_1 small yields a contradiction. (In case (ii), we do not use Λ_1.) Since $h = 9$, (4) implies

$$\begin{aligned}\log \Lambda_2 \geq\ &-1060.29 \log a \log c - 105.53(\log a + \log c) \\ &- 765.39(\log a \log c)^{1/2} - \log(81 \log a \log c) - 12.26,\end{aligned}$$

so

$$\begin{aligned}\frac{\log \Lambda_2}{\log a \log c} \geq\ &-1060.29 - 105.53\left(\frac{1}{\log a} + \frac{1}{\log c}\right) - 765.39(\log a \log c)^{-1/2} \\ &- \frac{\log 81 + 12.26}{\log a \log c} - \frac{\log(\log a \log c)}{\log a \log c} \\ \geq\ &-1696 \quad \text{(from (5))}.\end{aligned}$$

From (3), we have

$$(6)\qquad x < n \cdot \frac{\log b}{\log a} - \frac{\log \Lambda_2}{\log a} < n - \frac{\log \Lambda_2}{\log a},$$

since $a > b$.

We want to obtain a lower bound for x. We now show $rx - pz > 0$. By our assumptions, we have

$$\begin{aligned}(a^p + b^q)^x &= \sum_{j=0}^{x} \binom{x}{j} (a^p)^{x-j} (b^q)^j = \sum_{j=0}^{x} \binom{x}{j} a^{px-pj} b^{qj} \\ &= \sum_{j=0}^{x} \binom{x}{j} a^{px-(n+p-q)j} a^{(n-q)j} b^{qj}\end{aligned}$$

$$\geq \sum_{j=0}^{x} \binom{x}{j} a^{px-(n+p-q)j} b^{nj} \qquad \text{(since } a > b \text{ and } n \geq q\text{)}$$

$$\geq \sum_{j=0}^{p} \binom{p}{j} a^{px-xj} b^{nj} = (a^x + b^n)^p \qquad \text{(since } x \geq n+p-q \geq p\text{)}$$

with ">" in the first inequality except when $n = q$ and with ">" in the second inequality except when $x = n + p - q$. In conclusion, we obtain $(a^p + b^q)^x > (a^x + b^n)^p$ when $(x, n) \neq (p, q)$. This implies that $c^{rx} = (a^p + b^q)^x > (a^x + b^n)^p = c^{pz}$ when $(x, n) \neq (p, q)$. Thus we have $rx - pz > 0$. In particular, $rx - pz \geq \delta$, where $\delta = 1$ or 2 according as $rx - pz$ is odd or even.

Eliminating a from the defining equations for Λ_1, Λ_2 yields

$$x\Lambda_1 - p\Lambda_2 = (rx - pz)\log c,$$

so

$$x = \frac{rx - pz}{\Lambda_1} \cdot \log c + \frac{p\Lambda_2}{\Lambda_1} > \frac{\delta}{\Lambda_1} \cdot \log c,$$

since $rx - pz \geq \delta$ and $\Lambda_1, \Lambda_2 > 0$.

Therefore we obtain

$$n - \frac{\log \Lambda_2}{\log a} > \frac{\delta}{\Lambda_1} \cdot \log c,$$

and thus

$$\Lambda_1 = \log\left(1 + \frac{b^q}{a^p}\right) > \frac{\delta \log c}{n - \frac{\log \Lambda_2}{\log a}} = \frac{\delta}{\frac{n}{\log c} - \frac{\log \Lambda_2}{\log a \log c}} > \frac{\delta}{n + 1696},$$

since $c \geq 3$. Hence

$$\frac{b^q}{a^p} > \exp\left(\frac{\delta}{n + 1696}\right) - 1,$$

which implies

$$a < \left\{\exp\left(\frac{\delta}{n + 1696}\right) - 1\right\}^{-1/p} b^{q/p} =: \kappa b^{q/p}.$$

Therefore if $a \geq \kappa b^{q/p}$, then (2) has no positive integral solutions x, z with $x \geq n + p - q$ and $(x, n) \neq (p, q)$.

Case (ii): $B > e^{8.83}$. Then $h = \log B + 0.17$. Since $\Lambda_2 = z \log c - x \log a$, we have

$$B = \frac{2x}{\log c} + \frac{\Lambda_2}{\log a \log c}.$$

From (6), we have

$$\frac{2x}{\log c} < \frac{2n}{\log c} - \frac{2\log \Lambda_2}{\log a \log c}.$$

Note that $\Lambda_2 < 1$. In fact, $\Lambda_2 < b^n/a^x \le (b/a)^n < 1$, since $x \ge n+p-q \ge n$ from $p \ge q$ and $a > b$.

Hence

$$\begin{aligned}
B &< \frac{2n}{\log c} + \frac{\Lambda_2}{\log a \log c} - \frac{2\log \Lambda_2}{\log a \log c} \\
&< 2n + \frac{1}{\log a \log c} - \frac{2\log \Lambda_2}{\log a \log c} \\
&< 2n + 1 + 26.18h^2 + 23.46h\left(\frac{1}{\log a} + \frac{1}{\log c}\right) + \frac{4h + 4\log h}{\log a \log c} \\
&\quad + 56.7h^{3/2}(\log a \log c)^{-1/2} + \frac{2\log(\log a \log c) + 11.5}{\log a \log c} \qquad \text{(from (4))} \\
&\le 26.18(\log B + 0.17)^2 + 33.12(\log B + 0.17) \\
&\quad + 35.65(\log B + 0.17)^{3/2} + 1.59\log(\log B + 0.17) + 3451.34
\end{aligned}$$

(from (5) and $n \le 1722$). Therefore $B \le 6836$, which contradicts $B > e^{8.83}$. This completes the proof of the Main Theorem. ∎

3. Applications of the Main Theorem to the Conjecture. Applying the Main Theorem to the Conjecture with $p = 2, q = 2$ and r odd ≥ 3, we prove the following:

Theorem 1. *Let a, b, c be fixed positive integers satisfying $a^2 + b^2 = c^r$ with $(a,b) = 1$ and r odd ≥ 3. Suppose that*

$$a \equiv 3 \pmod 8, \qquad 2 \parallel b, \qquad \left(\frac{b}{l}\right) = -1, \qquad a \ge 30b,$$

where $l > 1$ is a divisor of a and $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol. Then equation (1) *has only the positive integral solution $(x,y,z) = (2,2,r)$.*

We first need two lemmas. (We prove Lemmas 2 and 3 under slightly weaker conditions than those of Theorem 1.)

Lemma 2. *Let a, b, c be fixed positive integers satisfying $a^2 + b^2 = c^r$ with $(a,b) = 1$ and r odd ≥ 3. Suppose that*

$$a \equiv 3 \pmod 4, \qquad 2 \mid b, \qquad \left(\frac{b}{l}\right) = -1.$$

If equation (1) *has positive integral solutions (x,y,z), then x and y are even.*

Proof. Since $a^2 + b^2 = c^r$ and r is odd, we have $1 = \left(\frac{b}{l}\right)^2 = \left(\frac{c}{l}\right)^r$, so $\left(\frac{c}{l}\right) = 1$. Thus since $\left(\frac{b}{l}\right) = -1$, y must be even from (1).

Note that $c \equiv 1 \pmod 4$ from $a^2 + b^2 = c^r$. Since $a \equiv 3 \pmod 4$ and $b^2 \equiv 0 \pmod 4$, we have $3^x \equiv 1 \pmod 4$. Thus x is even. ∎

LEMMA 3. *Let a, b, c be fixed positive integers satisfying $a^2 + b^2 = c^r$ with $(a,b) = 1$ and r odd ≥ 3. Suppose that*

$$a \equiv 3 \pmod 8, \qquad 2 \parallel b, \qquad \left(\frac{b}{l}\right) = -1.$$

If equation (1) *has positive integral solutions (x,y,z), then either*

(i) *x is even, $y = 2$, z is odd, or*
(ii) *x is even, $y = 4$, z is even.*

Proof. Lemma 2 implies that x and y are even. Note that $c \equiv 5 \pmod 8$. In fact, $c \equiv c^r = a^2 + b^2 \equiv 1 + 4 \equiv 5 \pmod 8$, since $2 \parallel b$.

CASE (i): z is odd. Then it follows from (1) that $1 + b^y \equiv 5 \pmod 8$. Since $2 \parallel b$, we have $y = 2$.

CASE (ii): z is even. Then from (1), we have

$$a^X = u^2 - v^2, \qquad b^Y = 2uv, \qquad c^Z = u^2 + v^2,$$

where $x = 2X, y = 2Y, z = 2Z$ and u, v are integers such that $(u,v) = 1$ and $u \not\equiv v \pmod 2$.

Since $2 \parallel b$, we have $Y > 1$. If $Y > 2$, then $uv \equiv 0 \pmod 4$ and so

$$a^X \equiv \pm 1 \pmod 8, \qquad c^Z \equiv 1 \pmod 8.$$

In view of $a \equiv 3 \pmod 8$ and $c \equiv 5 \pmod 8$, we see that X and Z are even. Then equation (1) leads to

$$(a^{x/4})^4 + (b^{y/2})^2 = (c^{z/4})^4,$$

which has no non-trivial solutions by the method of infinite descent (cf. Ribenboim [Ri], p. 38). Hence $Y = 2$ and so $y = 4$. ■

We are now ready to apply the Main Theorem and prove Theorem 1.

Proof of Theorem 1. It follows from Lemma 3 that x is even and $y = 2, 4$. In the Main Theorem, let $p = 2$, $q = 2$, $n = 2, 4$ and $\delta = 2$. Then by the Main Theorem, if (1) has positive integral solutions with $(x,n) \neq (2,2)$, then

$$x < n + p - q \leq 4 + 2 - 2 = 4$$

under the condition $a \geq 30b$ (cf. Table). Since x is even, we have $x = 2$. If $y = 2$, then $c^z = a^x + b^y = a^2 + b^2 = c^r$. Thus $z = r$. If $y = 4$, then $c^z = a^2 + b^4 = (c^r - b^2) + b^4$ and so $b^2(b^2 - 1) = c^r(c^{z-r} - 1)$. Since $(b,c) = 1$, we have $c^r \mid (b^2 - 1)$. Hence

$$c^r \leq b^2 - 1 < a^2 + b^2 = c^r,$$

which is impossible. ■

Now, consider the case $r = 3$ in Theorem 1. The general integral solutions of $a^2 + b^2 = c^3$ are as follows:

LEMMA 4 [Te2]. *The integral solutions of the equation* $a^2 + b^2 = c^3$ *with* $(a,b) = 1$ *are given by*

$$a = \pm u(u^2 - 3v^2), \quad b = \pm v(v^2 - 3u^2), \quad c = u^2 + v^2,$$

where u, v *are integers such that* $(u,v) = 1$ *and* $u \not\equiv v \pmod 2$.

Let a, b, c be as in Lemma 4 with $v = 2$. Then we can eliminate the conditions $\left(\frac{b}{l}\right) = -1$ and $a \geq 30b$ in Theorem 1. Indeed, we show the following:

COROLLARY. *Let* $a = u(u^2 - 12)$, $b = 2(3u^2 - 4)$, $c = u^2 + 4$ *with* $u \equiv -1 \pmod 8$ (> 0). *Then equation* (1) *has only the positive integral solution* $(x, y, z) = (2, 2, 3)$.

REMARK. By the Corollary, we see that when $(p, q, r) = (2, 2, 3)$, there are infinitely many a, b, c such that the Conjecture holds.

Proof (of Corollary). It follows from $u \equiv -1 \pmod 8$ that $a \equiv 3 \pmod 8$, and $2 \parallel b$.

We also see that $\left(\frac{b}{a}\right) = -1$. In fact,

$$\begin{aligned}\left(\frac{b}{a}\right) &= \left(\frac{2(3u^2 - 4)}{a}\right) = -\left(\frac{3u^2 - 4}{a}\right) \\ &= -\left(\frac{3u^2 - 4}{u}\right)\left(\frac{3u^2 - 4}{u^2 - 12}\right) = -\left(\frac{-4}{u}\right)\left(\frac{32}{u^2 - 12}\right) \\ &= (-1) \cdot (-1) \cdot (-1) = -1.\end{aligned}$$

The inequality $a \geq 30b$ implies that $u \geq 183$. Hence if $u \equiv -1 \pmod 8$ and $u \geq 183$, then the conditions of Theorem 1 are all satisfied. Thus our assertion follows.

It remains to consider the case $u < 183$. We show that if $r = 3$, then case (ii) in Lemma 3 does not occur except for the case $u = 7$. (Note that if $u > 7$, then $a > b$.) On the contrary, suppose that case (ii) occurs. We keep the notation of Lemma 3. We may suppose that X and Z are odd, since the equations $A^4 + B^4 = C^2$, $A^2 + B^4 = C^4$ have no non-trivial solutions (cf. Ribenboim [Ri], pp. 37, 38). The equation $a^{2X} + b^4 = c^{2Z}$ implies that

$$b^4 = (c^Z + a^X)(c^Z - a^X) \geq c^Z + a^X > c^Z.$$

On the other hand, from $a^2 + b^2 = c^3$, we have $b^2 < c^3$ and so $b^4 < c^6$. Hence $Z < 6$. Since Z is odd > 1, $Z = 3, 5$.

CASE 1: $Z = 3$. Then $a^{2X} + b^4 = c^6 = (a^2 + b^2)^2 = a^4 + 2a^2b^2 + b^4$. Thus $a^{2X} = a^4 + 2a^2b^2$, which is impossible, since $(a, b) = 1$.

CASE 2: $Z = 5$. If $X \leq 3$, then

$$c^{10} = a^{2X} + b^4 \leq a^6 + b^4 < (a^2 + b^2)^3 = c^9,$$

which is impossible. If $X \ge 5$, then from $a > b$ (except for $u = 7$), we have

$$a^{10} \le a^{2X} < a^{2X} + b^4 = c^{10} < c^{12} = (a^2+b^2)^4 < (2a^2)^4 < a^9,$$

which is impossible. Hence when $r = 3$, case (ii) in Lemma 3 does not occur except for the case $u = 7$.

Therefore Lemma 3 shows that x is even, $y = 2$ and z is odd except for the case $u = 7$.

We need the following claim, which is simple and useful:

CLAIM 1. *Let a, b, c be positive integers satisfying $a^2 + b^2 = c^3$ with $(a,b) = 1$. Suppose that there is a prime l such that $ab(a \pm 1) \equiv 0 \pmod{l}$ and $e \equiv 0 \pmod{3}$, where e is the order of c modulo l. Then*

(C_1) *If $ab \equiv 0 \pmod{l}$ and $a^x + b^y = c^z$, then $z \equiv 0 \pmod{3}$.*

(C_2) *If $a \pm 1 \equiv 0 \pmod{l}$ and $a^x + b^2 = c^z$ with x even, then $z \equiv 0 \pmod{3}$.*

Proof. (C_1) See Lemma 3 in [Te2].

(C_2) If $a \pm 1 \equiv 0 \pmod{l}$, then $1 + b^2 \equiv c^3 \equiv c^z \pmod{l}$. Hence from $e \equiv 0 \pmod{3}$, we obtain $z \equiv 0 \pmod{3}$. ∎

For all a, b, c such that $u \equiv -1 \pmod{8}$ (> 0) and $u < 183$, we verified that $e \equiv 0 \pmod{3}$ by computer.

By Claim 1, the fact that $e \equiv 0 \pmod{3}$ implies that $z \equiv 0 \pmod{3}$. Note that x is even and $y = 2$ ($y = 2$ or 4 if $u = 7$). Hence using Lemma 4, we can determine x, z in a finite number of steps.

CASE (1): $u = 7$. Then $(7\cdot 37)^X = \pm U(U^2 - 3V^2)$, $2\cdot 11\cdot 13$ or $(2\cdot 11\cdot 13)^2 = \pm V(V^2 - 3U^2)$, $53^Z = U^2 + V^2$, where $x = 2X, z = 3Z$. Thus we obtain $U = \pm 7$, $V = \pm 2$ and so $X = 1$, $Z = 1$, $x = 2$, $z = 3$, $y = 2$.

CASE (2): $u = 15$. Then $(3^2\cdot 5\cdot 71)^X = \pm U(U^2 - 3V^2)$, $2\cdot 11\cdot 61 = \pm V(V^2 - 3U^2)$, $229^Z = U^2 + V^2$, where $x = 2X$, $z = 3Z$. Thus we obtain $U = \pm 15$, $V = \pm 2$ and so $X = 1$, $Z = 1$, $x = 2$, $z = 3$.

The other cases can be treated similarly. ∎

In the same way as in the proof of Theorem 1, we obtain the following (cf. Theorem in [Le]):

THEOREM 2. *Let a, b, c be fixed positive integers satisfying $a^2 + b^2 = c^2$ with $(a,b) = 1$. Suppose that*

$$a \equiv 3 \pmod{8}, \qquad b \equiv 4 \pmod{8}, \qquad \left(\frac{b}{a}\right) = -1, \qquad a \ge 30b.$$

Then equation (1) *has only the positive integral solution $(x,y,z) = (2,2,2)$.*

Proof. Let (x,y,z) be a solution of (1) with $(x,y,z) \ne (2,2,2)$. Then Lemma 2 in [GL] shows that $2 \mid x, y = 1$ and $2 \nmid z$.

In the Main Theorem, let $(p,q,r)=(2,2,2), n=1$ and $\delta=2$. Note that $n=1<2=q$, but $rx-pz=2x-2z>0$ when $y=n=1$. In fact, otherwise, $(a^x+b)^2=c^{2z}\ge c^{2x}=(a^2+b^2)^x$, which is impossible, since $x\ge 2$. Then by the Main Theorem, if (1) has positive integral solutions, then

$$x\le n+p-q=1+2-2=1$$

under the condition $a\ge 30b$ (cf. Table). Thus $x=1$, which is impossible, since x is even. ∎

4. Other applications of the Main Theorem. In the proof of the theorems in this section, we need the following lemmas. Cohn [Co3] discussed in detail the Diophantine equation $x^2+C=y^n$. He collected together some of the known results, and obtained many new ones for values of $C\le 100$.

LEMMA 5 (Nagell [N3]). *Let n be odd ≥ 3. Then the Diophantine equation*

$$x^2+4=y^n$$

has only the positive integral solutions $(x,y,n)=(2,2,3),(11,5,3)$.

LEMMA 6 (Nagell [N2], Cohn [Co2]). *Let m be a non-negative integer. Then the Diophantine equation*

$$x^2+2^{2m+1}=y^n$$

has only the positive integral solutions $(x,y,m,n)=(5,3,0,3),(7,3,2,4)$ with $(y,2)=1$ and $n\ge 3$.

LEMMA 7 (Nagell [N3]). *Let n be an odd integer ≥ 3 and A a square-free odd integer ≥ 3. Let $h(-2A)$ be the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-2A})$. If $h(-2A)\not\equiv 0 \pmod n$, then the Diophantine equation*

$$Ax^2+2=y^n$$

has no integral solutions x,y,n.

LEMMA 8 (Rabinowitz [Ra]). *Let m be a positive integer. Then the Diophantine equation*

$$x^3+3^m=y^2$$

has only the positive integral solutions $(x,y,m)=(1,2,1),(40,253,2)$ with $(y,3)=1$.

LEMMA 9 (Brown [B1], [B2]). *Let m be a non-negative integer and p an odd prime. Then the Diophantine equation*

$$x^2+3^{2m+1}=y^p$$

has only the positive integral solution $(x,y,m,p)=(10,7,2,3)$ with $(y,3)=1$.

Lemma 10 (Nagell [N1]). *Let n be an integer ≥ 2. Then the Diophantine equation*

$$x^2 + 5 = y^n$$

has only the positive integral solution $(x, y, n) = (2, 3, 2)$.

Using the Main Theorem with $(p, q, r) = (1, 1, 1)$, $n = 1$ and $\delta = 1$, we immediately obtain the following (cf. Table):

Theorem 3. *Let A, B, C be fixed positive integers satisfying $A - B = C > 1$ with $(A, B) = 1$. If $B \geq 1697C$, then the Diophantine equation*

$$A^x - B^y = C \tag{7}$$

has only the positive integral solution $(x, y) = (1, 1)$.

In the case where $A - B^2 = 2$, the condition "$a \geq \kappa b^{q/p}$" in the Main Theorem can easily be eliminated. In some other theorems of this section, we also adopt the following way of eliminating it, which is of use and interest:

Theorem 4. *Let A, B be fixed positive integers satisfying $A - B^2 = 2$ with $B \geq 3$ and $(A, B) = 1$. Then the Diophantine equation*

$$A^x - B^y = 2 \tag{8}$$

has only the positive integral solution $(x, y) = (1, 2)$.

Proof. In the Main Theorem, let $(p, q, r) = (2, 1, 1)$, $n = 1$ and $\delta = 1$. Then by the Main Theorem, (8) has only the positive integral solution $(x, y) = (1, 2)$ under the condition $B \geq 41.19 \cdot \sqrt{2} = 58.251\ldots$ (cf. Table).

The condition $B \geq 59$ can easily be eliminated.

Let y be even. Then x is odd. Hence by Lemma 6 (with $m = 0$), we obtain $x = 1$ and so $y = 2$.

Let y be odd. If $\left(\frac{B}{A}\right) = -1$, then (8) has no solutions. Since $A - B^2 = 2$, it follows that if $B \equiv 5$ or $7 \pmod 8$, then $\left(\frac{B}{A}\right) = -1$. Thus we may suppose that $B \equiv 1$ or $3 \pmod 8$. From $A - B^2 = 2$ and (8), we have

$$A(A^{x-1} - 1) = B^2(B^{y-2} - 1).$$

In particular,

$$2^{x-1} \equiv 1 \pmod B \quad \text{and} \quad B^{y-2} \equiv 1 \pmod A.$$

For all B such that $B < 59$ and $B \equiv 1$ or $3 \pmod 8$, the order of 2 modulo B is even. Hence x is odd. We also see that for all B above except $B = 3, 9, 25, 33$, the order of B modulo A is even, which implies that y is even. In view of Lemma 6 (with $m = 0$), B is never a square. Consequently, $B = 3$ or 33.

Since y is odd, (8) can be written as

$$B(B^{(y-1)/2})^2 + 2 = A^x \quad (\text{with } x \text{ odd}).$$

Since $h(-6) = 2$ and $h(-66) = 8$, this equation has no solutions from Lemma 7. ∎

Remark. The example above shows that the estimate of linear forms of Lemma 1 is fairly sharp. Indeed, if $B \geq 59$ and $B \equiv 1$ or $3 \pmod 8$, then there are some exceptions in using Lemma 7, namely $B = 67, 91, 123$: $h(-134) = 14$, $e(67) = 249$, $d(67) = 66$; $h(-182) = 12$, $e(91) = 25$, $d(91) = 12$; $h(-246) = 12$, $e(123) = 7565$, $d(123) = 20$, where $e(B), d(B)$ denote the order of B modulo A and the order of 2 modulo B, respectively (cf. Theorems 6, 7).

We now make some comments on equation (7), where $A>1$, $B>1$, $C\geq 1$ are any integers. Pillai [P1] showed that (7) has only finitely many positive integral solutions (x, y). Pillai [P2] also showed that if C is sufficiently great with respect to A and B, then (7) has at most one solution. LeVeque [Lv] and Cassels [Ca] independently established that for $C = 1$, there is at most one solution to (7) unless $(A, B) = (3, 2)$, when there are two solutions $(x, y) = (1, 1), (2, 3)$. Scott [Sc] proved that if A is prime, then (7) has at most one solution with y even and at most one with y odd, except for five specific choices of (A, B, C).

Moreover, we make a remark on the equation $a^x + b^y = c^z$, where a, b, c are any positive integers > 1 with $(a, b) = 1$. Using the theory of imaginary quadratic fields, Scott [Sc] proved that if c is prime, then this equation has at most two solutions (x, y, z) in positive integers when $c \neq 2$, and at most one solution when $c = 2$, except for two cases (taking $a < b$): $(a, b, c) = (3, 5, 2)$ and $(a, b, c) = (3, 13, 2)$, when there are exactly three solutions $(x, y, z) = (1, 1, 3), (3, 1, 5), (1, 3, 7)$ and exactly two solutions $(x, y, z) = (1, 1, 4), (5, 1, 8)$, respectively (cf. Guy [G], Section D9).

When a, b, c are fixed positive integers satisfying $a^p + b^q = c^r$, we apply the Main Theorem to the equation $a^x + b^y = c^z$ for various degrees $p, q, r \geq 1$.

By an argument similar to the one used in Theorem 1, we obtain the following:

Theorem 5. *Let a, c be fixed positive integers satisfying $a + 2 = c$ with $a \equiv 3$ or $5 \pmod 8$. If $a \geq 1697$, then the Diophantine equation*

$$a^x + 2^y = c^z \tag{9}$$

has only the positive integral solution $(x, y, z) = (1, 1, 1)$.

Proof. Let $a \equiv 3 \pmod 8$. Then $c = a + 2 \equiv 5 \pmod 8$. From (9), we have $3^x + 2^y \equiv 5^z \pmod 8$. If $y = 1$, then we easily see that x and z are odd. If $y = 2$, then x is even and z is odd. Then (9) becomes

$$(a^{x/2})^2 + 4 = c^z,$$

which has no solutions by Lemma 5.

If $y \geq 3$, then x and z are even, say $x = 2X, z = 2Z$. From (9), we have $2^y = (c^Z + a^X)(c^Z - a^X)$ and so $c^Z + a^X = 2^{y-1}, c^Z - a^X = 2$. Hence

$$c^Z - 2^{y-2} = 1,$$

which has no solutions by the following claim:

CLAIM 2. *Let c be odd ≥ 3 and $x, y > 1$. The Diophantine equation*

$$c^x - 2^y = 1$$

has only the solution $(x, y, c) = (2, 3, 3)$.

Proof. Suppose that x is even, say $x = 2X$. Then $(c^X + 1)(c^X - 1) = 2^y$ and so $c^X + 1 = 2^{y-1}, c^X - 1 = 2$. Thus $2^{y-1} - 2 = 2$. Hence $y = 3$, $x = 2$ and $c = 3$.

Suppose that x is odd. Then $(c-1)\left(\frac{c^x - 1}{c-1}\right) = 2^y$. Since $(c^x - 1)/(c-1)$ is odd, we have $c - 1 = 2^y$ and $(c^x - 1)/(c-1) = 1$, which is impossible, since $x > 1$. ∎

Let $a \equiv 5 \pmod 8$. Then $c = a + 2 \equiv 7 \pmod 8$. From (9), we have $5^x + 2^y \equiv 7^z \pmod 8$. If $y = 1$, then we see that x and z are odd. If $y = 2$, then x is odd and z is even, say $z = 2Z$. Then $(c^Z + 2)(c^Z - 2) = a^x$ and so $c^Z + 2 = a_1^x$, $c^Z - 2 = a_2^x$ with $a = a_1 a_2$. Thus $a_1^x - a_2^x = 4$, which is impossible. If $y \geq 3$, then x and z are even. As above, (9) has no solutions.

Hence if $a \equiv 3$ or $5 \pmod 8$, then x, z are odd and $y = 1$. In the Main Theorem, let $(p, q, r) = (1, 1, 1)$, $n = 1$ and $\delta = 2$. Then by the Main Theorem, if (9) has positive integral solutions, then

$$x \leq n + p - q = 1 + 1 - 1 = 1$$

under the condition $a \geq 848.1 \cdot 2 = 1696.2$ (cf. Table). Thus $x = 1$ and so $z = 1$. ∎

THEOREM 6. *Let a, c be fixed positive integers satisfying $a^3 + 2 = c$ with $a \equiv 3$ or $5 \pmod 8$. Then the Diophantine equation*

$$a^x + 2^y = c^z$$

has only the positive integral solution $(x, y, z) = (3, 1, 1)$.

Proof. In the same way as in the proof of Theorem 5, we see that x and z are odd, and $y = 1$. In the Main Theorem, let $(p, q, r) = (3, 1, 1)$, $n = 1$ and $\delta = 2$. Then by the Main Theorem, if (9) has positive integral solutions with $(x, n) \neq (3, 1)$, then

$$x < n + p - q = 1 + 3 - 1 = 3$$

under the condition $a \geq 9.47 \cdot 2^{1/3} = 11.931\ldots$ (cf. Table). Hence from $a^3 + 2 = c$, (9) has only the solution $x = 3$, $y = 1$, $z = 1$.

The condition $a \geq 12$ can easily be eliminated. If $a < 12$, then the pairs of (a, c) are only $(3, 29), (5, 127)$ and $(11, 1333)$. Since x is odd and $y = 1$, (9) can be written as

$$a(x^{(x-1)/2})^2 + 2 = c^z \quad \text{(with } z \text{ odd).}$$

Since $h(-6) = h(-10) = h(-22) = 2$, we obtain $x = 3$, $z = 1$ for the pairs of (a, c) above from Lemma 7. ∎

THEOREM 7. *Let a, c be fixed positive integers satisfying $a^4 + 8 = c$ with $a \equiv 3, 5$ or $7 \pmod 8$. Then the Diophantine equation*

$$a^x + 2^y = c^z$$

has only the positive integral solution $(x, y, z) = (4, 3, 1)$.

Proof. Since a is odd and $c = a^4 + 8$, we have $c \equiv 1 \pmod 8$.

Let $y = 2$. Then $a^x \equiv 5 \pmod 8$, which is clearly impossible if $a \equiv 3$ or $7 \pmod 8$. If $a \equiv 5 \pmod 8$, then $\left(\frac{c}{a}\right) = \left(\frac{2}{a}\right) = -1$ and so z is even from (9). This is impossible from $a^x + 4 = c^z$.

Let $y \geq 3$. Then $a^x \equiv 1 \pmod 8$, which implies that x is even, since $a \equiv 3$, 5 or $7 \pmod 8$. As in the proof of Theorem 5, it follows from Claim 2 that z is odd. We show that y is odd. If $a \not\equiv 0 \pmod 3$, then $c \equiv 0 \pmod 3$. Thus (9) implies that $1 + (-1)^y \equiv 0 \pmod 3$ and so y is odd. If $a \equiv 0 \pmod 3$, then $(-1)^y \equiv (-1)^z \pmod 3$ and so y is odd, since z is odd. Hence as x is even, y is odd and z is odd, Lemma 6 implies that $z = 1$. Then by (9), we have $a^x + 2^y = a^4 + 8$. The case $x = 2$ does not occur. In fact, if $x = 2$, then we have

$$(2a^2 - 1)^2 + 31 = 2^{y+2}.$$

The equation above has no solutions by Browkin and Schinzel [BS], which states that the Diophantine equation $x^2 + 31 = 2^n$ has only the positive integral solutions $(x, n) = (1, 5)$, $(15, 8)$. Thus we have $x = 4$, $y = 3$ and so $z = 1$.

Let $y = 1$. Then $a^x \equiv -1 \pmod 8$, which implies that x is odd and $a \equiv -1 \pmod 8$. In the Main Theorem, let $(p, q, r) = (4, 3, 1)$, $n = 1$ and $\delta = 1$. We may suppose that $x > 4$, since $a^x + 2 = c^z = (a^4 + 8)^z$. Note that $n = 1 < 3 = q$, but $rx - pz = x - 4z > 0$ when $y = n = 1$. In fact, otherwise, $(a^x + 2)^4 = c^{4z} \geq c^x = (a^4 + 8)^x$, which is impossible, since $x > 4$. Then by the Main Theorem, if (9) has positive integral solutions, then

$$x \leq n + p - q = 1 + 4 - 3 = 2$$

under the condition $a \geq 6.42 \cdot 2^{3/4} = 10.797\ldots$ (cf. Table). This is impossible, since $x > 4$.

The condition $a \geq 11$ can easily be eliminated. Since $a < 11$ and $a \equiv -1 \pmod 8$, it remains to consider the case $a = 7$. When $a = 7$, taking equation

(9) modulo 5 implies that $x \equiv 1 \pmod 4$ and z is odd. Since x is odd and $y = 1$, (9) can be written as

$$a(x^{(x-1)/2})^2 + 2 = c^z \quad \text{(with } z \text{ odd)}.$$

Since $h(-14) = 4$, (9) has no solutions with $y = 1$ from Lemma 7. ∎

THEOREM 8. *Let a, c be fixed positive integers satisfying $a + 3 = c^2$ with $c \equiv -1 \pmod 9$. If $a \geq 2545$, then the Diophantine equation*

$$a^x + 3^y = c^z \tag{10}$$

has only the positive integral solution $(x, y, z) = (1, 1, 2)$.

Proof. Since $a \equiv 1 \pmod 3$ and $c \equiv -1 \pmod 3$, we have $1 \equiv (-1)^z \pmod 3$ and so z is even.

Let $y \geq 2$. Since $a \equiv -2 \pmod 9$ and $c \equiv -1 \pmod 9$, we have $(-2)^x \equiv 1 \pmod 9$ and so $x \equiv 0 \pmod 3$. In fact, the order of -2 modulo 9 is 3. Thus (10) becomes

$$(a^{x/3})^3 + 3^y = (c^{z/2})^2,$$

which has no solutions by Lemma 8.

Therefore we have $y = 1$. In the Main Theorem, let $(p, q, r) = (1, 1, 2)$, $n = 1$ and $\delta = 2$. Then by the Main Theorem, if (10) has positive integral solutions, then

$$x \leq n + p - q = 1 + 1 - 1 = 1$$

under the condition $a \geq 848.1 \cdot 3 = 2544.3$ (cf. Table). Thus $x = 1$ and so $z = 2$. ∎

REMARK. Let a, c be fixed positive integers satisfying $a^2 + 3 = c$ with $a \equiv -1 \pmod 3$. Then we can solve (10) without using the Main Theorem. In fact, taking (10) modulo 3 and 8 implies that x is even, y is odd and z is odd. Hence in view of Lemma 9, if a, c are as above, then (10) has only the positive integral solution $(x, y, z) = (2, 1, 1)$.

In connection with Theorems 7 and 8, we conclude this section by showing the following:

THEOREM 9. *Let a, c be fixed positive integers satisfying $a^2 + 5 = c$ with $a \equiv -1 \pmod{25}$ and c odd. Then the Diophantine equation*

$$a^x + 5^y = c^z \tag{11}$$

has only the positive integral solution $(x, y, z) = (2, 1, 1)$.

Proof. Since $a \equiv -1 \pmod 5$ and $c \equiv 1 \pmod 5$, we have $(-1)^x \equiv 1 \pmod 5$ and so x is even.

Let $y \geq 2$. Since $a \equiv -1 \pmod{25}$ and $c \equiv 6 \pmod{25}$, we have $1 \equiv 6^z \pmod{25}$ and so $z \equiv 0 \pmod 5$. In fact, the order of 6 modulo 25 is 5.

We next show that y is odd. If $a \not\equiv 0 \pmod 3$, then $c \equiv 0 \pmod 3$. Thus (11) implies that $1 + (-1)^y \equiv 0 \pmod 3$ and so y is odd. If $a \equiv 0 \pmod 3$, then $(-1)^y \equiv (-1)^z \pmod 3$ and so $y \equiv z \pmod 2$. The case where $y \equiv z \equiv 0 \pmod 2$ does not occur. In fact, if $y \equiv z \equiv 0 \pmod 2$, then

$$a^X = 2uv, \quad 5^Y = u^2 - v^2, \quad c^Z = u^2 + v^2,$$

where $x = 2X$, $y = 2Y$, $z = 2Z$ and u, v are integers such that $(u, v) = 1$ and $u \not\equiv v \pmod 2$. Then we have $u+v = 5^Y$ and $u-v = 1$. Thus $5^{2Y}+1 = 2c^Z$, which is impossible, since $c \equiv 1 \pmod 5$. Hence $y \equiv z \equiv 1 \pmod 2$.

Now put $x = 2X, y = 2k+1, z = 5Z$, where $X \geq 1, k \geq 0, Z \geq 1$ are integers. Since $(a, 5) = 1$ and c is odd, (11) leads to

$$a^X + 5^k\sqrt{-5} = (u + v\sqrt{-5})^5,$$

where u, v are integers such that $(u, v) = 1$ and $c^Z = u^2 + 5v^2$. Equating imaginary parts yields

$$5^k = 5v(u^4 - 10u^2v^2 + 5v^4),$$

so $k \geq 1$ and $5^{k-1} = v(u^4 - 10u^2v^2 + 5v^4)$. Hence since $(u, v) = 1$, we see that either

$$v = \pm 1, \quad u^4 - 10u^2v^2 + 5v^4 = \pm 5^{k-1} \tag{12}$$

or

$$v = \pm 5^{k-1}, \quad u^4 - 10u^2v^2 + 5v^4 = \pm 1. \tag{13}$$

Since $u \not\equiv 0 \pmod 5$, the relation (12) is impossible. (The case $k = 1$ easily yields a contradiction.) The second equation in (13) can be written as

$$(u^2 - 5v^2)^2 - 20v^4 = \pm 1.$$

Note that the $-$ sign must be rejected since $(u^2 - 5v^2)^2 \equiv -1 \pmod 4$ is impossible. The equation above has no non-trivial solutions from Cohn's result in [Co1], which states that the Diophantine equation $x^2 - 20y^4 = 1$ has only the positive integral solution $(x, y) = (161, 6)$.

Therefore we have $y = 1$. Then by Lemma 10, we can solve (11) without using the Main Theorem. Since x is even, Lemma 10 implies that $z = 1$ and so $x = 2$. ∎

Remark. So far as the author knows, at present, it seems that the families of exponential Diophantine equations below cannot be solved completely (cf. Cohn [Co3] and Rabinowitz [Ra]):

$$x^2 + 5^{2m+1} = y^p,$$
$$x^3 \pm 5^m = y^2,$$

where m is a non-negative integer and p is an odd prime.

Acknowledgments. The author would like to thank the referee for his valuable suggestions.

References

[BS] J. Browkin and A. Schinzel, *On the equation* $2^n - D = y^2$, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8 (1960), 311–318.

[B1] E. Brown, *Diophantine equations of the form* $x^2 + D = y^n$, J. Reine Angew. Math. 274/275 (1975), 385–389.

[B2] —, *Diophantine equations of the form* $ax^2 + Db^2 = y^n$, ibid. 291 (1977), 118–127.

[BG] Y. Bugeaud and K. Győry, *Bounds for the solutions of Thue–Mahler equations and norm form equations*, Acta Arith. 74 (1996), 273–292.

[Cao] Z. F. Cao, *A note on the diophantine equation* $a^x + b^y = c^z$, Acta Arith., to appear.

[Ca] J. W. S. Cassels, *On the equation* $a^x - b^y = 1$, Amer. J. Math. 75 (1953), 159–162.

[Co1] J. H. E. Cohn, *Eight Diophantine equations*, Proc. London Math. Soc. (3) 16 (1966), 153–166.

[Co2] —, *The diophantine equation* $x^2 + 2^k = y^n$, Arch. Math. (Basel) 59 (1992), 341–344.

[Co3] —, *The diophantine equation* $x^2 + C = y^n$, Acta Arith. 65 (1993), 367–381.

[GL] Y.-D. Guo and M.-H. Le, *A note on Jeśmanowicz' conjecture concerning Pythagorean numbers*, Comment. Math. Univ. St. Pauli 44 (1995), 225–228.

[G] R. Guy, *Unsolved Problems in Number Theory*, 2nd ed., Springer, 1994.

[J] L. Jeśmanowicz, *Some remarks on Pythagorean numbers*, Wiadom. Mat. 1 (1955/1956), 196–202 (in Polish).

[LMN] M. Laurent, M. Mignotte et Y. Nesterenko, *Formes linéaires en deux logarithmes et déterminants d'interpolation*, J. Number Theory 55 (1995), 285–321.

[Le] M.-H. Le, *On Jeśmanowicz' conjecture concerning Pythagorean numbers*, Proc. Japan Acad. Ser. A 72 (1996), 97–98.

[Lv] W. J. LeVeque, *On the equation* $a^x - b^y = 1$, Amer. J. Math. 74 (1952), 235–331.

[M] M. Mignotte, *A corollary to a theorem of Laurent–Mignotte–Nesterenko*, Acta Arith. 86 (1998), 101–111.

[MW] M. Mignotte and M. Waldschmidt, *Linear forms in two logarithms and Schneider's method III*, Ann. Fac. Sci. Toulouse Math. 97 (1989), 43–75.

[N1] T. Nagell, *Sur l'impossibilité de quelques équations à deux indéterminées*, Norsk. Mat. Forenings Skrigter 13 (1923), 65–82.

[N2] —, *Verallgemeinerung eines Fermatschen Satzes*, Arch. Math. (Basel) 5 (1954), 153–159.

[N3] —, *Contributions to the theory of a category of diophantine equations of the second degree with two unknowns*, Nova Acta Soc. Sci. Upsal. Ser. IV (2) 16 (1955), 1–38.

[N4] —, *Sur une classe d'équations exponentielles*, Ark. Mat. 3 (1958), 569–582.

[P1] S. S. Pillai, *On the inequality* $0 < a^x - b^y \le n$, J. Indian Math. Soc. (1) 19 (1931), 1–11.

[P2] —, *On* $a^x - b^y = c$, ibid. (2) 2 (1936), 19–122; Corr. ibid., 2, 215.

[Ra] S. Rabinowitz, *On Mordell's equation* $y^2 + k = x^3$ *with* $k = \pm 2^n 3^m$, Doctoral dissertation at the City University of New York, 1971.

[Ri] P. Ribenboim, *13 Lectures on Fermat's Last Theorem*, Springer, 1979.

[Sc] R. Scott, *On the equations* $p^x - b^y = c$ *and* $a^x + b^y = c^z$, J. Number Theory 44 (1993), 153–165.

[Si] W. Sierpiński, *On the equation* $3^x + 4^y = 5^z$, Wiadom. Mat. 1 (1955/1956), 194–195 (in Polish).

[Ta] K. Takakuwa, *A remark on Jeśmanowicz' conjecture*, Proc. Japan Acad. Ser. A 72 (1996), 109–110.

[Te1] N. Terai, *The Diophantine equation* $x^2 + q^m = p^n$, Acta Arith. 63 (1993), 351–358.

[Te2] —, *The Diophantine equation* $a^x + b^y = c^z$, Proc. Japan Acad. Ser. A 70 (1994), 22–26.

[Te3] —, *The Diophantine equation* $a^x + b^y = c^z$ *II*, ibid. 71 (1995), 109–110.

[Te4] —, *The Diophantine equation* $a^x + b^y = c^z$ *III*, ibid. 72 (1996), 20–22.

Division of General Education
Ashikaga Institute of Technology
268-1 Omae, Ashikaga, Tochigi 326
Japan
E-mail: terai@aitsun5.ashitech.ac.jp

Received on 10.12.1996
and in revised form on 26.10.1998 (3094)