Applications of a lower bound for linear forms in two logarithms to exponential Diophantine equations

by

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1. Introduction. In 1956, Sierpiński [Si] showed that the equation $3^x + 4^y = 5^z$ has only the positive integral solution $(x, y, z) = (2, 2, 2)$. Jeśmanowicz [J] conjectured that if $a, b, c$ are Pythagorean triples, i.e., positive integers satisfying $a^2 + b^2 = c^2$, then the equation $a^x + b^y = c^z$ has only the positive integral solution $(x, y, z) = (2, 2, 2)$. This conjecture has been proved to be true in many special cases (cf. Guo-Le [GL], Le [Le] and Takakuwa [Ta]). It is, however, still unsolved.

As an analog to this conjecture, we propose the following (cf. Terai [Te1]):

**Conjecture.** If $a, b, c, p, q, r$ are fixed positive integers satisfying $a^p + b^q = c^r$ with $p, q, r \geq 2$ and $(a, b) = 1$, then the Diophantine equation

$$a^x + b^y = c^z \quad (1)$$

has only the positive integral solution $(x, y, z) = (p, q, r)$ except for three cases (taking $a < b$), where (1) has only the following solutions, respectively:

$$(a, b, c) = (2, 3, 5), \quad (x, y, z) = (1, 1, 1), (4, 2, 2);$$
$$(a, b, c) = (2, 7, 3), \quad (x, y, z) = (1, 1, 2), (5, 2, 4);$$
$$(a, b, c) = (1, 2, 3), \quad (x, y, z) = (m, 1, 1), (n, 3, 2)$$

with $m, n$ arbitrary (cf. Nagell [N4], Cao [Cao]).

In our previous papers [Te2]–[Te4], we considered the conjecture above when $p = 2$, $q = 2$ and $r$ is an odd prime. In [Te2] and [Te3], we reduced (1) to certain quartic equations, which have no non-trivial solutions by the method of infinite descent. In [Te4], we reduced (1) to Thue equations, and used the known estimates of linear forms in logarithms due to Mignotte and Waldschmidt [MW] and Bugeaud and Győry [BG].

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In this paper, we apply a lower bound for linear forms in two logarithms due to Mignotte [M] which is a corollary to a theorem of Laurent–Mignotte–Nesterenko [LMN] to the Diophantine equation

\[ a^x + b^y = c^z, \]

where \( n \) is a given “small” positive integer (Main Theorem). The Main Theorem shows that if the upper bound \( n \) of the solution \( y \) of (1) is attained (and small), then the solution \( x \) of (1) satisfies

\[ x \leq n + p - q \]

under a certain condition on \( a, b \) when \( a, b, c, p, q, r \) are as in the Main Theorem. By an elementary or algebraic method, we can attain the upper bound \( n \). Indeed, in our theorems, the upper bound \( n \) is derived by using congruences modulo 3, 8 etc. and results concerning the Diophantine equations of the form \( x^2 + D = y^r \).

The Main Theorem has a number of applications. An easy consequence is that if \( A, B, C \) are fixed positive integers satisfying \( A - B = C > 1 \), \( (A, B) = 1 \) and \( B \geq 1697C \), then the Diophantine equation

\[ A^x - B^y = C \]

has only the positive integral solution \((x, y) = (1, 1)\) (Theorem 3 in Section 4). In Section 3, using the Main Theorem, we show that the conjecture above holds under some conditions on \( a, b, c \) (Theorems 1, 2 in Section 3). In particular, there are infinitely many \( a, b, c \) such that it holds when \((p, q, r) = (2, 2, 3)\). In Section 4, we illustrate in detail how the upper bound \( n \) is determined and the Main Theorem is applied to equation (1) for various degrees \( p, q, r \geq 1 \). In some of the theorems of that section, we verify that the condition “\( a \geq \kappa b^{\theta / p} \)” in the Main Theorem can easily be eliminated.

2. Main Theorem. We use the following result of Mignotte [M] to prove the Main Theorem, which plays an important role in the proofs.

Let \( \alpha \) be an algebraic number of degree \( d \) with minimal polynomial

\[ a_0 x^d + a_1 x^{d-1} + \ldots + a_d = a_0 \prod_{i=1}^{d} (x - \alpha_i), \]

where the \( a_i \)'s are relatively prime integers with \( a_0 > 0 \) and the \( \alpha_i \)'s are conjugates of \( \alpha \). Then

\[ h(\alpha) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \log \max(1, |\alpha_i|) \right) \]

is called the **absolute logarithmic height** of \( \alpha \). In particular, if \( \alpha \in \mathbb{Q} \), say \( \alpha = p/q \) as a fraction in lowest terms, then \( h(\alpha) = \log \max(|p|, |q|) \).
Let $\alpha_1, \alpha_2$ be two non-zero algebraic numbers, and let $\log \alpha_1$ and $\log \alpha_2$ be any determinations of their logarithms. We consider the linear form

$$L = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where $b_1$ and $b_2$ are positive integers. Without loss of generality, we suppose that $|\alpha_1|$ and $|\alpha_2|$ are $\geq 1$. Put

$$D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] / [\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}].$$

**Lemma 1** (Mignotte [M]). Let $a_1, a_2, h$ be real positive numbers, and $\varrho$ a real number $> 1$. Put $\lambda = \log \varrho$ and suppose that

$$h \geq \max \left\{ \frac{D \log 2}{2}, C\lambda, D \left( \log \left( \frac{b_1}{a_2} + \frac{b_2}{a_1} \right) + \log \lambda + f(K_0) + 0.189 \right) \right\}$$

with $C \geq 2$,

$$a_i \geq \max \{ 2, \varrho | \log \alpha_i | - \log |\alpha_i| + 2Dh(\alpha_i) \} \quad (i = 1, 2),$$

where

$$f(x) = \log \left( \frac{1 + \sqrt{x - 1}}{x - 1} \right) \frac{\sqrt{x}}{x - 1} + \frac{\log x}{6x(x - 1)} + \frac{3}{2} + \log \frac{3}{4} + \frac{\log \frac{x}{x - 1}}{x - 1}.$$

Suppose also that

$$\frac{1}{a_1} + \frac{1}{a_2} \leq \frac{2}{\lambda}$$

and that there exists an integer $K_0$ such that

$$8(1 + C)a_1a_2 \frac{4(a_1 + a_2)}{3\lambda} + 8\sqrt{2(1 + C)a_1a_2} \frac{4\lambda}{3\lambda} > K_0 - 1 \geq 33.$$

If $\alpha_1$ and $\alpha_2$ are multiplicatively independent, we have the lower bound

$$\log |L| \geq -\frac{\lambda a_1a_2}{9} \left( \frac{4h}{3\lambda} + \frac{4}{\lambda} + \frac{1}{h} \right)^2 - 2\lambda \frac{a_1 + a_2}{3} \left( \frac{4h}{3\lambda} + \frac{4}{\lambda} + \frac{1}{h} \right) - \frac{16\sqrt{2a_1a_2}}{3} \left( 1 + \frac{h}{\lambda} \right)^{3/2} - 2(\lambda + h) - \log \left( a_1a_2 \left( 1 + \frac{h}{\lambda} \right)^2 \right) + \frac{\lambda}{2} + \log \lambda - 0.88.$$

**Main Theorem.** Let $a, b, c, p, q, r$ be fixed positive integers satisfying $a^p + b^q = c^r$ with $(a, b) = 1$, $a > b > 1$, $c \geq 3$ and $p \geq q$. Let $n$ be a given positive integer with $q \leq n \leq 1722$. If $a \geq \kappa b^{q/p}$ and the Diophantine equation

$$a^x + b^n = c^z$$

has positive integral solutions $x, z$ with $(x, n) \neq (p, q)$, then

$$x < n + p - q,$$
where

\[ \kappa = \left\{ \exp\left(\frac{\delta}{n + 1696}\right) - 1 \right\}^{-1/p} \]

and \( \delta = 1 \) or 2 according as \( rx - pz \) is odd or even.

**Remark.** We note that the Main Theorem can also be applied to the case of \( p = 1, q = 1 \) or \( r = 1 \). The table of values of \( \kappa \) for some \( p, n, \delta \) is as follows. (These values will be used in the theorems.)

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>( p )</th>
<th>( n )</th>
<th>( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.41783...</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>9.46524...</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>29.12044...</td>
<td>2</td>
<td>1</td>
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<td>1</td>
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</tr>
<tr>
<td>848.0009...</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1696.50004...</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Proof (of the Main Theorem).** Suppose that \( x \geq n + p - q \). From \( a^p + b^q = c^r \) and \( a^x + b^n = c^z \), we now consider the following linear forms in two logarithms:

\[
A_1 = r \log c - p \log a \quad (>0), \quad A_2 = z \log c - x \log a \quad (>0).
\]

Using the inequality \( \log(1 + t) < t \) for \( t > 0 \), we have

\[ 0 < A_2 = \log \left( \frac{c^z}{a^x} \right) = \log \left( 1 + \frac{b^n}{a^x} \right) < \frac{b^n}{a^x}. \]

Hence

\[ (3) \quad \log A_2 < n \log b - x \log a. \]

On the other hand, we use Lemma 1 to obtain a lower bound for \( A_2 \). We keep the notations of Lemma 1. Put \( \varrho = 4.9 \) and \( \lambda = \log \varrho \). We take

\[
a_1 = (\varrho - 1) \log a + 2 \log a = (\varrho + 1) \log a > \lambda, \quad a_2 = (\varrho - 1) \log c + 2 \log c = (\varrho + 1) \log c > \lambda.
\]

Then it is clear that \( 1/a_1 + 1/a_2 \leq 2/\lambda \). In Lemma 1, we choose \( C = 4.5 \). Then we take \( K_0 = 177 \) and \( f(K_0) = 1.2879 \). Since

\[ \log \left( \frac{b_1}{a_2} + \frac{b_2}{a_1} \right) = \log \left( \frac{x}{\log c} + \frac{z}{\log a} \right) - \log(\varrho + 1), \]

we can take

\[ h = \max \left\{ \log \left( \frac{x}{\log c} + \frac{z}{\log a} \right) + 0.17, 9 \right\}. \]
Hence Lemma 1 shows that

\[(4) \quad \log A_2 \geq -13.09h^2 \log a \log c - 11.73h(\log a + \log c) - 2h - 28.35h^{3/2}(\log a \log c)^{1/2} - \log(h^2 \log a \log c) - 5.75,\]

where \(h = \max\{\log B + 0.17, 9\}\) and \(B = x/\log c + z/\log a\).

If \(a, b, c\) are primes \(\leq 7\), Nagell [N4] completely determined the solutions of the equation \(a^x + b^y = c^z\) using the theory of quadratic fields and cubic fields. In view of his result, if \(a, b, c\) are positive integers satisfying \(a^p + b^q = c^r\) with \((a, b) = 1, a > b > 1, c \geq 3, p \geq q\) and \(a, b, c \leq 9\), then the solution \(x\) of \(a^x + b^y = c^z\) satisfies \(x \leq n + p - q\), where \(n\) is a fixed positive integer. (The cases where \(a, b, c\) are composite can be treated similarly.) Hence we may suppose that

\[(5) \quad a \geq 10, \quad c \geq 3 \quad \text{or} \quad a \geq 3, \quad c \geq 10.\]

Now we distinguish two cases: (i) \(B \leq e^{8.83} (= 6836.2868\ldots)\) and (ii) \(B > e^{8.83}\).

Case (i): \(B \leq e^{8.83}\). Then we show that making \(A_1\) small yields a contradiction. (In case (ii), we do not use \(A_1\).) Since \(h = 9\), (4) implies

\[
\log A_2 \geq -1060.29 \log a \log c - 105.53(\log a + \log c) - 765.39(\log a \log c)^{1/2} - \log(81 \log a \log c) - 12.26,
\]

so

\[
\frac{\log A_2}{\log a \log c} \geq -1060.29 - 105.53 \left(\frac{1}{\log a} + \frac{1}{\log c}\right) - 765.39(\log a \log c)^{-1/2} - \frac{\log 81 + 12.26}{\log a \log c} - \log(\log a \log c).
\]

\[
\geq -1696 \quad \text{(from (5)).}
\]

From (3), we have

\[(6) \quad x < n + \frac{\log b}{\log a} - \frac{\log A_2}{\log a} < n + \frac{\log A_2}{\log a},\]

since \(a > b\).

We want to obtain a lower bound for \(x\). We now show \(rx - pz > 0\). By our assumptions, we have

\[
(a^p + b^q)^x = \sum_{j=0}^{x} \binom{x}{j} (a^p)^x-j(b^q)^j = \sum_{j=0}^{x} \binom{x}{j} a^{px-pj} b^{qj}
\]

\[
= \sum_{j=0}^{x} \binom{x}{j} a^{px-(n+p-q)j} a^{(n-q)j} b^{qj}
\]
\[
\geq \sum_{j=0}^{x} \binom{x}{j} a^{px-(n+p-q)j} b^{nj} \quad (\text{since } a > b \text{ and } n \geq q)
\]
\[
\geq \sum_{j=0}^{p} \binom{p}{j} a^{px-xj} b^{nj} = (a^x + b^n)^p \quad (\text{since } x \geq n + p - q \geq p)
\]
with “>” in the first inequality except when \(n = q\) and with “>” in the second inequality except when \(x = n + p - q\). In conclusion, we obtain
\[
x \Lambda_1 - p \Lambda_2 = (rx - pz) \log c,
\]
so
\[
x = \frac{rx - pz \log c + p \Lambda_2}{\Lambda_1} > \frac{\delta}{\Lambda_1} \log c,
\]
since \(rx - pz \geq \delta\) and \(\Lambda_1, \Lambda_2 > 0\).

Therefore we obtain
\[
n - \log \frac{A_2}{\log a} > \frac{\delta}{\Lambda_1} \log c,
\]
and thus
\[
A_1 = \log \left(1 + \frac{b^q}{a^p}\right) > \frac{\delta \log c}{n - \log A_2 / \log a} = \frac{n}{\log c} - \frac{\log A_2}{\log a \log c} > \frac{\delta}{n + 1696},
\]
since \(c \geq 3\). Hence
\[
\frac{b^q}{a^p} > \exp \left(\frac{\delta}{n + 1696}\right) - 1,
\]
which implies
\[
a < \left\{ \exp \left(\frac{\delta}{n + 1696}\right) - 1 \right\}^{-1/p} b^{q/p} =: \kappa b^{q/p}.
\]
Therefore if \(a \geq \kappa b^{q/p}\), then (2) has no positive integral solutions \(x, z\) with \(x \geq n + p - q\) and \((x, n) \neq (p, q)\).

Case (ii): \(B > e^{8.83}\). Then \(h = \log B + 0.17\). Since \(A_2 = z \log c - x \log a\), we have
\[
B = \frac{2x}{\log c} + \frac{A_2}{\log a \log c}.
\]
From (6), we have
\[
\frac{2x}{\log c} < \frac{2n}{\log c} - \frac{2 \log A_2}{\log a \log c}.
\]
Note that $A_2 < 1$. In fact, $A_2 < b^n/a^x \leq (b/a)^n < 1$, since $x \geq n + p - q \geq n$ from $p \geq q$ and $a > b$.

Hence

$$B < \frac{2n}{\log c} + \frac{A_2}{\log a \log c} - \frac{2 \log A_2}{\log a \log c}$$

$$< 2n + 1 + 26.18h^2 + 23.46h \left( \frac{1}{\log a} + \frac{1}{\log c} \right) + \frac{4b + 4 \log b}{\log a \log c}$$

$$+ 56.7h^{3/2} (\log a \log c)^{-1/2} + \frac{2 \log (\log a \log c) + 11.5}{\log a \log c}$$

(from (4))

$$\leq 26.18(\log B + 0.17)^2 + 33.12(\log B + 0.17)$$

$$+ 35.65(\log B + 0.17)^{3/2} + 1.59 \log(\log B + 0.17) + 3451.34$$

(from (5) and $n \leq 1722$). Therefore $B \leq 6836$, which contradicts $B > e^{8.83}$. This completes the proof of the Main Theorem. 

3. Applications of the Main Theorem to the Conjecture. Applying the Main Theorem to the Conjecture with $p = 2, q = 2$ and $r$ odd $\geq 3$, we prove the following:

**Theorem 1.** Let $a, b, c$ be fixed positive integers satisfying $a^2 + b^2 = c^r$ with $(a, b) = 1$ and $r$ odd $\geq 3$. Suppose that

$$a \equiv 3 \pmod{8}, \quad 2 \mid b, \quad \left( \frac{b}{l} \right) = -1, \quad a \geq 30b,$$

where $l > 1$ is a divisor of $a$ and $\left( \frac{\cdot}{l} \right)$ denotes the Jacobi symbol. Then equation (1) has only the positive integral solution $(x, y, z) = (2, 2, r)$.

We first need two lemmas. (We prove Lemmas 2 and 3 under slightly weaker conditions than those of Theorem 1.)

**Lemma 2.** Let $a, b, c$ be fixed positive integers satisfying $a^2 + b^2 = c^r$ with $(a, b) = 1$ and $r$ odd $\geq 3$. Suppose that

$$a \equiv 3 \pmod{4}, \quad 2 \mid b, \quad \left( \frac{b}{l} \right) = -1.$$

If equation (1) has positive integral solutions $(x, y, z)$, then $x$ and $y$ are even.

**Proof.** Since $a^2 + b^2 = c^r$ and $r$ is odd, we have $1 = \left( \frac{b}{l} \right)^2 = \left( \frac{z}{l} \right)^r$, so $\left( \frac{z}{l} \right) = 1$. Thus since $\left( \frac{b}{l} \right) = -1$, $y$ must be even from (1).

Note that $c \equiv 1 \pmod{4}$ from $a^2 + b^2 = c^r$. Since $a \equiv 3 \pmod{4}$ and $b^2 \equiv 0 \pmod{4}$, we have $3^r \equiv 1 \pmod{4}$. Thus $x$ is even. 

Lemma 3. Let $a, b, c$ be fixed positive integers satisfying $a^2 + b^2 = c^r$ with $(a, b) = 1$ and $r$ odd $\geq 3$. Suppose that

$$a \equiv 3 \pmod{8}, \quad 2 \parallel b, \quad \left(\frac{b}{7}\right) = -1.$$ 

If equation (1) has positive integral solutions $(x, y, z)$, then either

(i) $x$ is even, $y = 2$, $z$ is odd, or

(ii) $x$ is even, $y = 4$, $z$ is even.

Proof. Lemma 2 implies that $x$ and $y$ are even. Note that $c \equiv 5 \pmod{8}.$ In fact, $c \equiv c^r = a^2 + b^2 \equiv 1 + 4 \equiv 5 \pmod{8}$, since $2 \parallel b$.

Case (i): $z$ is odd. Then it follows from (1) that $1 + b^y \equiv 5 \pmod{8}$.

Case (ii): $z$ is even. Then from (1), we have $a^X = u^2 - v^2, \quad b^Y = 2uv, \quad c^Z = u^2 + v^2,$

where $x = 2X, y = 2Y, z = 2Z$ and $u, v$ are integers such that $(u, v) = 1$ and $u \neq v \pmod{2}$.

Since $2 \parallel b$, we have $Y > 1$. If $Y > 2$, then $uv \equiv 0 \pmod{4}$ and so

$$a^X \equiv \pm 1 \pmod{8}, \quad c^Z \equiv 1 \pmod{8}.$$ 

In view of $a \equiv 3 \pmod{8}$ and $c \equiv 5 \pmod{8}$, we see that $X$ and $Z$ are even. Then equation (1) leads to

$$(a^{x/4})^4 + (b^{y/2})^2 = (c^{z/4})^4,$$

which has no non-trivial solutions by the method of infinite descent (cf. Ribenboim [Ri], p. 38). Hence $Y = 2$ and so $y = 4$. ■

We are now ready to apply the Main Theorem and prove Theorem 1.

Proof of Theorem 1. It follows from Lemma 3 that $x$ is even and $y = 2, 4$.

In the Main Theorem, let $p = 2, q = 2, n = 2, 4$ and $\delta = 2$. Then by the Main Theorem, if (1) has positive integral solutions with $(x, n) \neq (2, 2)$, then

$$x < n + p - q \leq 4 + 2 - 2 = 4$$

under the condition $a \geq 30b$ (cf. Table). Since $x$ is even, we have $x = 2$. If $y = 2$, then $c^z = a^2 + b^2 = a^2 + b^2 = c^r$. Thus $z = r$. If $y = 4$, then $c^z = a^2 + b^4 = (c^r - b^2) + b^4$ and so $b^2(b^2 - 1) = c^r(c^{z-r} - 1)$. Since $(b,c) = 1$, we have $c^r | (b^2 - 1)$. Hence

$$c^r \leq b^2 - 1 < a^2 + b^2 = c^r,$$

which is impossible. ■

Now, consider the case $r = 3$ in Theorem 1. The general integral solutions of $a^2 + b^2 = c^3$ are as follows:
**Lemma 4** [Te2]. The integral solutions of the equation $a^2 + b^2 = c^3$ with $(a, b) = 1$ are given by
\[ a = \pm u(u^2 - 3v^2), \quad b = \pm v(v^2 - 3u^2), \quad c = u^2 + v^2, \]
where $u, v$ are integers such that $(u, v) = 1$ and $u \not\equiv v \pmod{2}$.

Let $a, b, c$ be as in Lemma 4 with $v = 2$. Then we can eliminate the conditions $(\frac{b}{a}) = -1$ and $a \geq 30b$ in Theorem 1. Indeed, we show the following:

**Corollary.** Let $a = u(u^2 - 12), \ b = 2(3u^2 - 4), \ c = u^2 + 4$ with $u \equiv -1 \pmod{8}$ ($> 0$). Then equation (1) has only the positive integral solution $(x, y, z) = (2, 2, 3)$.

**Remark.** By the Corollary, we see that when $(p, q, r) = (2, 2, 3)$, there are infinitely many $a, b, c$ such that the Conjecture holds.

**Proof** (of Corollary). It follows from $u \equiv -1 \pmod{8}$ that $a \equiv 3 \pmod{8}$, and $2 \not\parallel b$.

We also see that $(\frac{b}{a}) = -1$. In fact,
\[
\left( \frac{b}{a} \right) = \left( \frac{2(3u^2 - 4)}{a} \right) = -\left( \frac{3u^2 - 4}{a} \right) = -\left( \frac{3u^2 - 4}{u} \right) \left( \frac{3u^2 - 4}{u^2 - 12} \right) = -\left( \frac{-4}{u} \right) \left( \frac{32}{u^2 - 12} \right) = (-1) \cdot (-1) \cdot (-1) = -1.
\]

The inequality $a \geq 30b$ implies that $u \geq 183$. Hence if $u \equiv -1 \pmod{8}$ and $u \geq 183$, then the conditions of Theorem 1 are all satisfied. Thus our assertion follows.

It remains to consider the case $u < 183$. We show that if $r = 3$, then case (ii) in Lemma 3 does not occur except for the case $u = 7$. (Note that if $u > 7$, then $a > b$.) On the contrary, suppose that case (ii) occurs. We keep the notation of Lemma 3. We may suppose that $X$ and $Z$ are odd, since the equations $A^4 + B^4 = C^2, \ A^2 + B^4 = C^4$ have no non-trivial solutions (cf. Ribenboim [Ri], pp. 37, 38). The equation $a^{2X} + b^4 = c^{2Z}$ implies that
\[ b^4 = (c^2 + a^X)(c^2 - a^X) \geq c^2 + a^X > c^2. \]

On the other hand, from $a^2 + b^2 = c^3$, we have $b^2 < c^3$ and so $b^4 < c^6$. Hence $Z < 6$. Since $Z$ is odd $> 1$, $Z = 3, 5$.

**Case 1:** $Z = 3$. Then $a^{2X} + b^4 = c^6 = (a^2 + b^2)^2 = a^4 + 2a^2b^2 + b^4$. Thus $a^{2X} = a^2 + 2a^2b^2$, which is impossible, since $(a, b) = 1$.

**Case 2:** $Z = 5$. If $X \leq 3$, then $c^{10} = a^{2X} + b^4 \leq a^6 + b^4 < (a^2 + b^2)^3 = c^9$, etc.
which is impossible. If $X \geq 5$, then from $a > b$ (except for $u = 7$), we have

$$a^{10} \leq a^{2X} < a^{2X} + b^4 = c^{10} < c^{12} = (a^2 + b^2)^4 < (2a^2)^4 < a^9,$$

which is impossible. Hence when $r = 3$, case (ii) in Lemma 3 does not occur except for the case $u = 7$.

Therefore Lemma 3 shows that $x$ is even, $y = 2$ and $z$ is odd except for the case $u = 7$.

We need the following claim, which is simple and useful:

**Claim 1.** Let $a, b, c$ be positive integers satisfying $a^2 + b^2 = c^3$ with $(a, b) = 1$. Suppose that there is a prime $l$ such that $ab(a \pm 1) \equiv 0 \pmod{l}$ and $e \equiv 0 \pmod{3}$, where $e$ is the order of $c$ modulo $l$. Then

1. If $ab \equiv 0 \pmod{l}$ and $a^x + b^y = c^z$, then $z \equiv 0 \pmod{3}$.
2. If $a \pm 1 \equiv 0 \pmod{l}$ and $a^x + b^2 = c^z$ with $x$ even, then $z \equiv 0 \pmod{3}$.

**Proof.** (C1) See Lemma 3 in [Te2].

(C2) If $a \pm 1 \equiv 0 \pmod{l}$, then $1 + b^2 \equiv e \equiv c^3 \pmod{l}$. Hence from $e \equiv 0 \pmod{3}$, we obtain $z \equiv 0 \pmod{3}$. ■

For all $a, b, c$ such that $u \equiv -1 \pmod{8}$ ($> 0$) and $u < 183$, we verified that $e \equiv 0 \pmod{3}$ by computer.

By Claim 1, the fact that $e \equiv 0 \pmod{3}$ implies that $z \equiv 0 \pmod{3}$. Note that $x$ is even and $y = 2$ ($y = 2$ or $4$ if $u = 7$). Hence using Lemma 4, we can determine $x, z$ in a finite number of steps.

**Case (1):** $u = 7$. Then $(7\cdot37)^X = \pm U(U^2 - 3V^2)$, $2 \cdot 11 \cdot 13$ or $(2 \cdot 11 \cdot 13)^2 = \pm V(V^2 - 3U^2)53Z = U^2 + V^2$, where $x = 2X, z = 3Z$. Thus we obtain $U = \pm 7, V = \pm 2$ and so $X = 1, Z = 1, x = 2, z = 3, y = 2$.

**Case (2):** $u = 15$. Then $(3^2 \cdot 5 \cdot 71)^X = \pm U(U^2 - 3V^2), 2 \cdot 11 \cdot 61 = \pm V(V^2 - 3U^2), 229^2 = U^2 + V^2$, where $x = 2X, z = 3Z$. Thus we obtain $U = \pm 15, V = \pm 2$ and so $X = 1, Z = 1, x = 2, z = 3$.

The other cases can be treated similarly. ■

In the same way as in the proof of Theorem 1, we obtain the following (cf. Theorem in [Le]):

**Theorem 2.** Let $a, b, c$ be fixed positive integers satisfying $a^2 + b^2 = c^2$ with $(a, b) = 1$. Suppose that

$$a \equiv 3 \pmod{8}, \quad b \equiv 4 \pmod{8}, \quad \left( \frac{b}{a} \right) = -1, \quad a \geq 30b.$$  

Then equation (1) has only the positive integral solution $(x, y, z) = (2, 2, 2)$.

**Proof.** Let $(x, y, z)$ be a solution of (1) with $(x, y, z) \neq (2, 2, 2)$. Then Lemma 2 in [GL] shows that $2 \mid x, y = 1$ and $2 \mid z$.  

In the Main Theorem, let \((p, q, r) = (2, 2, 2), n = 1\) and \(\delta = 2\). Note that \(n = 1 < 2 = q\), but \(rx - p^2 = 2x - 2z > 0\) when \(y = n = 1\). In fact, otherwise, \((a^x + b)^2 = c^{2z} \geq c^{2x} = (a^2 + b^2)^x\), which is impossible, since \(x \geq 2\). Then by the Main Theorem, if \((1)\) has positive integral solutions, then

\[ x \leq n + p - q = 1 + 2 - 2 = 1 \]

under the condition \(a \geq 30\) (cf. Table). Thus \(x = 1\), which is impossible, since \(x\) is even.

4. Other applications of the Main Theorem. In the proof of the theorems in this section, we need the following lemmas. Cohn [Co3] discussed in detail the Diophantine equation \(x^2 + C = y^n\). He collected together some of the known results, and obtained many new ones for values of \(C \leq 100\).

**Lemma 5** (Nagell [N3]). Let \(n\) be odd \(\geq 3\). Then the Diophantine equation

\[ x^2 + 4 = y^n \]

has only the positive integral solutions \((x, y, n) = (2, 2, 3), (11, 5, 3)\).

**Lemma 6** (Nagell [N2], Cohn [Co2]). Let \(m\) be a non-negative integer. Then the Diophantine equation

\[ x^2 + 2^{2m+1} = y^n \]

has only the positive integral solutions \((x, y, m, n) = (5, 3, 0, 3), (7, 3, 2, 4)\) with \((y, 2) = 1\) and \(n \geq 3\).

**Lemma 7** (Nagell [N3]). Let \(n\) be an odd integer \(\geq 3\) and \(A\) a square-free odd integer \(\geq 3\). Let \(h(-2A)\) be the class number of the imaginary quadratic field \(\mathbb{Q}(\sqrt{-2A})\). If \(h(-2A) \not\equiv 0 \pmod{n}\), then the Diophantine equation

\[ Ax^2 + 2 = y^n \]

has no integral solutions \(x, y, n\).

**Lemma 8** (Rabinowitz [Ra]). Let \(m\) be a positive integer. Then the Diophantine equation

\[ x^3 + 3^m = y^2 \]

has only the positive integral solutions \((x, y, m) = (1, 2, 1), (40, 253, 2)\) with \((y, 3) = 1\).

**Lemma 9** (Brown [B1], [B2]). Let \(m\) be a non-negative integer and \(p\) an odd prime. Then the Diophantine equation

\[ x^2 + 3^{2m+1} = y^p \]

has only the positive integral solution \((x, y, m, p) = (10, 7, 2, 3)\) with \((y, 3) = 1\).
LEMMA 10 (Nagell [N1]). Let \( n \) be an integer \( \geq 2 \). Then the Diophantine equation
\[
x^2 + 5 = y^n
\]
has only the positive integral solution \((x, y, n) = (2, 3, 2)\).

Using the Main Theorem with \((p, q, r) = (1, 1, 1)\), \( n = 1 \) and \( \delta = 1 \), we immediately obtain the following (cf. Table):

**THEOREM 3.** Let \( A, B, C \) be fixed positive integers satisfying \( A - B = C > 1 \) with \( (A, B) = 1 \). If \( B \geq 1697C \), then the Diophantine equation
\[
A^x - B^y = C
\]
(7)
has only the positive integral solution \((x, y, n) = (1, 1)\).

In the case where \( A - B^2 = 2 \), the condition "\( a \geq \kappa b^{\alpha / p} \)" in the Main Theorem can easily be eliminated. In some other theorems of this section, we also adopt the following way of eliminating it, which is of use and interest:

**THEOREM 4.** Let \( A, B \) be fixed positive integers satisfying \( A - B^2 = 2 \) with \( B \geq 3 \) and \( (A, B) = 1 \). Then the Diophantine equation
\[
A^x - B^y = 2
\]
(8)
has only the positive integral solution \((x, y) = (1, 2)\).

**Proof.** In the Main Theorem, let \((p, q, r) = (2, 1, 1)\), \( n = 1 \) and \( \delta = 1 \). Then by the Main Theorem, (8) has only the positive integral solution \((x, y) = (1, 2)\) under the condition \( B \geq 41.19 \cdot \sqrt{2} = 58.251 \ldots \) (cf. Table).

The condition \( B \geq 59 \) can easily be eliminated.

Let \( y \) be even. Then \( x \) is odd. Hence by Lemma 6 (with \( m = 0 \)), we obtain \( x = 1 \) and so \( y = 2 \).

Let \( y \) be odd. If \( \left( \frac{B}{A} \right) = -1 \), then (8) has no solutions. Since \( A - B^2 = 2 \), it follows that if \( B \equiv 5 \) or \( 7 \) (mod 8), then \( \left( \frac{B}{A} \right) = -1 \). Thus we may suppose that \( B \equiv 1 \) or \( 3 \) (mod 8). From \( A - B^2 = 2 \) and (8), we have
\[
A(A^{x-1} - 1) = B^2(B^{y-2} - 1).
\]
In particular,
\[
2^{x-1} \equiv 1 \pmod{B} \quad \text{and} \quad B^{y-2} \equiv 1 \pmod{A}.
\]

For all \( B \) such that \( B < 59 \) and \( B \equiv 1 \) or \( 3 \) (mod 8), the order of 2 modulo \( B \) is even. Hence \( x \) is odd. We also see that for all \( B \) above except \( B = 3, 9, 25, 33 \), the order of \( B \) modulo \( A \) is even, which implies that \( y \) is even. In view of Lemma 6 (with \( m = 0 \)), \( B \) is never a square. Consequently, \( B = 3 \) or \( 33 \).

Since \( y \) is odd, (8) can be written as
\[
B(B^{(y-1)/2})^2 + 2 = A^x \quad \text{(with \( x \) odd)}.
\]
Since \( h(-6) = 2 \) and \( h(-66) = 8 \), this equation has no solutions from Lemma 7.

**Remark.** The example above shows that the estimate of linear forms of Lemma 1 is fairly sharp. Indeed, if \( B \geq 59 \) and \( B \equiv 1, 3 \pmod{8} \), then there are some exceptions in using Lemma 7, namely \( B = 67, 91, 123 \):

- \( h(-134) = 14 \), \( e(67) = 249 \), \( d(67) = 66 \); \( h(-182) = 12 \), \( e(91) = 25 \), \( d(91) = 12 \); \( h(-246) = 12 \), \( e(123) = 7565 \), \( d(123) = 20 \), where \( e(B), d(B) \) denote the order of \( B \) modulo \( A \) and the order of 2 modulo \( B \), respectively (cf. Theorems 6, 7).

We now make some comments on equation (7), where \( A > 1, B > 1, C \geq 1 \) are any integers. Pillai [P1] showed that (7) has only finitely many positive integral solutions \((x, y)\). Pillai [P2] also showed that if \( C \) is sufficiently great with respect to \( A \) and \( B \), then (7) has at most one solution. LeVeque [Lv] and Cassels [Ca] independently established that for \( C = 1 \), there is at most one solution with \( y \) even and at most one with \( y \) odd, except for five specific choices of \((A, B, C)\).

Moreover, we make a remark on the equation \( a^x + b^y = c^z \), where \( a, b, c \) are any positive integers \( > 1 \) with \((a, b) = 1\). Using the theory of imaginary quadratic fields, Scott [Sc] proved that if \( c \) is prime, then this equation has at most two solutions \((x, y, z)\) in positive integers when \( c \neq 2 \), and at most one solution when \( c = 2 \), except for two cases (taking \( a < b \)): \((a, b, c) = (3, 5, 2) \) and \((a, b, c) = (3, 13, 2) \), when there are exactly three solutions \((x, y, z) = (1, 1, 3), (3, 1, 5), (1, 3, 7) \) and exactly two solutions \((x, y, z) = (1, 1, 4), (5, 1, 8) \), respectively (cf. Guy [G], Section D9).

When \( a, b, c \) are fixed positive integers satisfying \( a^p + b^q = c^r \), we apply the Main Theorem to the equation \( a^x + b^y = c^z \) for various degrees \( p, q, r \geq 1 \).

By an argument similar to the one used in Theorem 1, we obtain the following:

**Theorem 5.** Let \( a, c \) be fixed positive integers satisfying \( a + 2 = c \) with \( a \equiv 3 \) or 5 \( \pmod{8} \). If \( a \geq 1697 \), then the Diophantine equation

\[
a^x + 2^y = c^z
\]

has only the positive integral solution \((x, y, z) = (1, 1, 1)\).

**Proof.** Let \( a \equiv 3 \pmod{8} \). Then \( c = a + 2 \equiv 5 \pmod{8} \). From (9), we have \( 3^x + 2^y = 5^z \pmod{8} \). If \( y = 1 \), then we easily see that \( x \) and \( z \) are odd. If \( y = 2 \), then \( x \) is even and \( z \) is odd. Then (9) becomes

\[
(a^x/2)^2 + 4 = c^z,
\]

which has no solutions by Lemma 5.
If \( y \geq 3 \), then \( x \) and \( z \) are even, say \( x = 2X, z = 2Z \). From (9), we have \( 2^y = (c^Z + a^X)(c^Z - a^X) \) and so \( c^Z + a^X = 2^{y-1}, c^Z - a^X = 2 \). Hence
\[
 c^Z - 2^{y-2} = 1,
\]
which has no solutions by the following claim:

**Claim 2.** Let \( c \) be odd \( \geq 3 \) and \( x, y > 1 \). The Diophantine equation
\[
 c^x - 2^y = 1
\]
has only the solution \((x, y, c) = (2, 3, 3)\).

**Proof.** Suppose that \( x \) is even, say \( x = 2X \). Then \((c^Z + 1)(c^Z - 1) = 2^y\) and so \( c^Z + 1 = 2^{y-1}, c^Z - 1 = 2 \). Thus \( 2^{y-1} - 2 = 2 \). Hence \( y = 3, x = 2 \) and \( c = 3 \).

Suppose that \( x \) is odd. Then \((c - 1)(c^z - 1) = 2^y\). Since \((c^x - 1)/(c - 1)\) is odd, we have \( c - 1 = 2^y \) and \((c^x - 1)/(c - 1) = 1\), which is impossible, since \( x > 1 \).

Let \( a \equiv 5 \pmod{8} \). Then \( c = a + 2 \equiv 7 \pmod{8} \). From (9), we have \( 5^z + 2^y \equiv 7^z \pmod{8} \). If \( y = 1 \), then we see that \( x \) and \( z \) are odd. If \( y = 2 \), then \( x \) is odd and \( z \) is even, say \( z = 2Z \). Then \((c^Z + 2)(c^Z - 2) = a^z\) and so \( c^Z + 2 = a_1^z, c^Z - 2 = a_2^z \) with \( a = a_1a_2 \). Thus \( a_1^z - a_2^z = 4 \), which is impossible. If \( y \geq 3 \), then \( x \) and \( z \) are even. As above, (9) has no solutions.

Hence if \( a \equiv 3 \) or \( 5 \pmod{8} \), then \( x, z \) are odd and \( y = 1 \). In the Main Theorem, let \((p, q, r) = (1, 1, 1), n = 1 \) and \( \delta = 2 \). Then by the Main Theorem, if (9) has positive integral solutions, then
\[
 x \leq n + p - q = 1 + 1 - 1 = 1
\]
under the condition \( a \geq 848.1 \cdot 2 = 1696.2 \) (cf. Table). Thus \( x = 1 \) and so \( z = 1 \).

**Theorem 6.** Let \( a, c \) be fixed positive integers satisfying \( a^3 + 2 = c \) with \( a \equiv 3 \) or \( 5 \pmod{8} \). Then the Diophantine equation
\[
 a^x + 2^y = c^z
\]
has only the positive integral solution \((x, y, z) = (3, 1, 1)\).

**Proof.** In the same way as in the proof of Theorem 5, we see that \( x \) and \( z \) are odd, and \( y = 1 \). In the Main Theorem, let \((p, q, r) = (3, 1, 1), n = 1 \) and \( \delta = 2 \). Then by the Main Theorem, if (9) has positive integral solutions with \((x, n) \neq (3, 1)\), then
\[
 x < n + p - q = 1 + 3 - 1 = 3
\]
under the condition \( a \geq 9.47 \cdot 2^{1/3} = 11.931 \ldots \) (cf. Table). Hence from \( a^3 + 2 = c \), (9) has only the solution \( x = 3, y = 1, z = 1 \).
The condition \( a \geq 12 \) can easily be eliminated. If \( a < 12 \), then the pairs of \((a,c)\) are only \((3,29),(5,127)\) and \((11,1333)\). Since \( x \) is odd and \( y = 1 \), \((9)\) can be written as

\[
a(x^{(x-1)/2})^2 + 2 = c^z\quad \text{(with \( z \) odd).}
\]

Since \( h(-6) = h(-10) = h(-22) = 2 \), we obtain \( x = 3, z = 1 \) for the pairs of \((a,c)\) above from Lemma 7.

**Theorem 7.** Let \( a, c \) be fixed positive integers satisfying \( a^4 + 8 = c \) with \( a \equiv 3, 5 \) or \( 7 \pmod{8} \). Then the Diophantine equation

\[
a^x + 2^y = c^z
\]

has only the positive integral solution \((x,y,z) = (4,3,1)\).

**Proof.** Since \( a \) is odd and \( c = a^4 + 8 \), we have \( c \equiv 1 \pmod{8} \).

Let \( y = 2 \). Then \( a^x \equiv 5 \pmod{8} \), which is clearly impossible if \( a \equiv 3 \) or \( 7 \pmod{8} \). If \( a \equiv 5 \pmod{8} \), then \( \left( \frac{z}{a} \right) = \left( \frac{2}{a} \right) = -1 \) and so \( z \) is even from \((9)\). This is impossible from \( a^x + 4 = c^z \).

Let \( y \geq 3 \). Then \( a^x \equiv 1 \pmod{8} \), which implies that \( x \) is even, since \( a \equiv 3, 5 \) or \( 7 \pmod{8} \). As in the proof of Theorem 5, it follows from Claim 2 that \( z \) is odd. We show that \( y \) is odd. If \( a \not\equiv 0 \pmod{3} \), then \( c \equiv 0 \pmod{3} \). Thus \((9)\) implies that \( 1 + (-1)^y \equiv 0 \pmod{3} \) and so \( y \) is odd. If \( a \equiv 0 \pmod{3} \), then \((-1)^y \equiv (-1)^z \pmod{3} \) and so \( y \) is odd, since \( z \) is odd. Hence as \( x \) is even, \( y \) is odd and \( z \) is odd, Lemma 6 implies that \( z = 1 \). Then by \((9)\), we have \( a^x + 2^y = a^4 + 8 \). The case \( x = 2 \) does not occur. In fact, if \( x = 2 \), then we have

\[
(2a^2 - 1)^2 + 31 = 2y^2.
\]

The equation above has no solutions by Browkin and Schinzel [BS], which states that the Diophantine equation \( x^2 + 31 = 2^n \) has only the positive integral solutions \((x,n) = (1,5), (15,8)\). Thus we have \( x = 4, y = 3 \) and so \( z = 1 \).

Let \( y = 1 \). Then \( a^x \equiv -1 \pmod{8} \), which implies that \( x \) is odd and \( a \equiv -1 \pmod{8} \). In the Main Theorem, let \((p,q,r) = (4,3,1), n = 1 \) and \( \delta = 1 \). We may suppose that \( x > 4 \), since \( a^x + 2 = c^z = (a^4 + 8)^z \). Note that \( n = 1 < 3 = q \), but \( rx - pz = x - 4z > 0 \) when \( y = n = 1 \). In fact, otherwise, \((a^x + 2)^4 = c^{4z} \geq c^z = (a^4 + 8)^z \), which is impossible, since \( x > 4 \). Then by the Main Theorem, if \((9)\) has positive integral solutions, then

\[
x \leq n + p - q = 1 + 4 - 3 = 2
\]

under the condition \( a \geq 6.42 \cdot 2^{3/4} = 10.797 \ldots \) (cf. Table). This is impossible, since \( x > 4 \).

The condition \( a \geq 11 \) can easily be eliminated. Since \( a < 11 \) and \( a \equiv -1 \pmod{8} \), it remains to consider the case \( a = 7 \). When \( a = 7 \), taking equation
modulo 5 implies that \( x \equiv 1 \pmod{4} \) and \( z \) is odd. Since \( x \) is odd and \( y = 1 \), (9) can be written as

\[
a(x^{(x-1)/2})^2 + 2 = c^z \quad \text{(with } z \text{ odd)}.
\]

Since \( h(-14) = 4 \), (9) has no solutions with \( y = 1 \) from Lemma 7. \( \blacksquare \)

**Theorem 8.** Let \( a, c \) be fixed positive integers satisfying \( a + 3 = c^2 \) with \( c \equiv -1 \pmod{9} \). If \( a \geq 2545 \), then the Diophantine equation

\[
a^x + 3^y = c^z
\]

has only the positive integral solution \((x, y, z) = (1, 1, 2)\).

**Proof.** Since \( a \equiv 1 \pmod{3} \) and \( c \equiv -1 \pmod{3} \), we have \( 1 \equiv (-1)^x \pmod{3} \) and so \( z \) is even.

Let \( y \geq 2 \). Since \( a \equiv -2 \pmod{9} \) and \( c \equiv -1 \pmod{9} \), we have \((-2)^x \equiv 1 \pmod{9} \) and so \( x \equiv 0 \pmod{3} \). In fact, the order of \(-2\) modulo 9 is 3. Thus (10) becomes

\[
(a^{x/3})^3 + 3^y = (c^{z/2})^2,
\]

which has no solutions by Lemma 8.

Therefore we have \( y = 1 \). In the Main Theorem, let \((p, q, r) = (1, 1, 2)\), \( n = 1 \) and \( \delta = 2 \). Then by the Main Theorem, if (10) has positive integral solutions, then

\[
x \leq n + p - q = 1 + 1 - 1 = 1
\]

under the condition \( a \geq 848.1 \cdot 3 = 2544.3 \) (cf. Table). Thus \( x = 1 \) and so \( z = 2 \). \( \blacksquare \)

**Remark.** Let \( a, c \) be fixed positive integers satisfying \( a^2 + 3 = c \) with \( a \equiv -1 \pmod{3} \). Then we can solve (10) without using the Main Theorem. In fact, taking (10) modulo 3 and 8 implies that \( x \) is even, \( y \) is odd and \( z \) is odd. Hence in view of Lemma 9, if \( a, c \) are as above, then (10) has only the positive integral solution \((x, y, z) = (2, 1, 1)\).

In connection with Theorems 7 and 8, we conclude this section by showing the following:

**Theorem 9.** Let \( a, c \) be fixed positive integers satisfying \( a^2 + 5 = c \) with \( a \equiv -1 \pmod{25} \) and \( c \) odd. Then the Diophantine equation

\[
a^x + 5^y = c^z
\]

has only the positive integral solution \((x, y, z) = (2, 1, 1)\).

**Proof.** Since \( a \equiv -1 \pmod{5} \) and \( c \equiv 1 \pmod{5} \), we have \((-1)^x \equiv 1 \pmod{5} \) and so \( x \) is even.

Let \( y \geq 2 \). Since \( a \equiv -1 \pmod{25} \) and \( c \equiv 6 \pmod{25} \), we have \( 1 \equiv 6^z \pmod{25} \) and so \( z \equiv 0 \pmod{5} \). In fact, the order of 6 modulo 25 is 5.
We next show that $y$ is odd. If $a \not\equiv 0 \pmod{3}$, then $c \equiv 0 \pmod{3}$. Thus (11) implies that $1 + (-1)^y \equiv 0 \pmod{3}$ and so $y$ is odd. If $a \equiv 0 \pmod{3}$, then $(-1)^y \equiv (-1)^z \pmod{3}$ and so $y \equiv z \pmod{2}$. The case where $y \equiv z \equiv 0 \pmod{2}$ does not occur. In fact, if $y \equiv z \equiv 0 \pmod{2}$, then

$$a^X = 2uv, \quad 5^Y = u^2 - v^2, \quad c^Z = u^2 + v^2,$$

where $x = 2X$, $y = 2Y$, $z = 2Z$ and $u, v$ are integers such that $(u, v) = 1$ and $u \not\equiv v \pmod{2}$. Then we have $u + v = 5^Y$ and $u - v = 1$. Thus $5^{2Y} + 1 = 2c^Z$, which is impossible, since $c \equiv 1 \pmod{5}$. Hence $y \equiv z \equiv 1 \pmod{2}$.

Now put $x = 2X, y = 2k + 1, z = 5Z$, where $X \geq 1, k \geq 0, Z \geq 1$ are integers. Since $(a, 5) = 1$ and $c$ is odd, (11) leads to

$$a^X = 5^k \sqrt{-5} = (u + v \sqrt{-5})^5,$$

where $u, v$ are integers such that $(u, v) = 1$ and $c^Z = u^2 + 5v^2$. Equating imaginary parts yields

$$5^k = 5v(u^4 - 10u^2v^2 + 5v^4),$$

so $k \geq 1$ and $5^{k-1} = v(u^4 - 10u^2v^2 + 5v^4)$. Hence since $(u, v) = 1$, we see that either

(12) $v = \pm 1$, $u^4 - 10u^2v^2 + 5v^4 = \pm 5^{k-1}$

or

(13) $v = \pm 5^{k-1}$, $u^4 - 10u^2v^2 + 5v^4 = \pm 1$.

Since $u \not\equiv 0 \pmod{5}$, the relation (12) is impossible. (The case $k = 1$ easily yields a contradiction.) The second equation in (13) can be written as

$$(u^2 - 5v^2)^2 - 20v^4 = \pm 1.$$ 

Note that the $-$ sign must be rejected since $(u^2 - 5v^2)^2 \equiv -1 \pmod{4}$ is impossible. The equation above has no non-trivial solutions from Cohn’s result in [Co1], which states that the Diophantine equation $x^2 - 20y^4 = 1$ has only the positive integral solution $(x, y) = (161, 6)$.

Therefore we have $y = 1$. Then by Lemma 10, we can solve (11) without using the Main Theorem. Since $x$ is even, Lemma 10 implies that $z = 1$ and so $x = 2$.  

Remark. So far as the author knows, at present, it seems that the families of exponential Diophantine equations below cannot be solved completely (cf. Cohn [Co3] and Rabinowitz [Ra]):

$$x^2 + 5^{2m+1} = y^p,$$

$$x^3 \pm 5^m = y^2,$$

where $m$ is a non-negative integer and $p$ is an odd prime.
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