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On the number of coprime integer pairs within a circle

by

Wenguang Zhai (Jinan) and Xiaodong Cao (Beijing)

1. Introduction. Let $P(x)$ denote the number of integer pairs within the circle $a^2+b^2\le x$, and $E(x)$ denote the difference $P(x)-\pi x$. Then the well-known *circle problem* is to estimate the upper bound of $E(x)$ and the best result at present is

$$E(x)=O(x^{23/73+\varepsilon}). \tag{1.1}$$

See Huxley [6].

Let $V(x)$ denote the number of coprime integer pairs within the circle $a^2+b^2\le x$. It is an exercise to deduce that

$$V(x)=\frac{6}{\pi}x+O(x^{1/2}\exp(-c\log^{3/5}x(\log\log x)^{-2/5})), \tag{1.2}$$

where c is some absolute constant. The problem of reducing the exponent 1/2 is open. One way to make progress is to assume the Riemann Hypothesis (RH). W. G. Nowak [11] proved that RH implies

$$V(x)=\frac{6}{\pi}x+O(x^{15/38+\varepsilon}). \tag{1.3}$$

D. Hensley [5] also got a result of this type, but with a larger exponent.

The aim of this paper is to further improve this result. We have the following

Theorem. *If RH is true, then*

$$V(x)=\frac{6}{\pi}x+O(x^{11/30+\varepsilon}). \tag{1.4}$$

Notations. $e(x)=\exp(2\pi ix)$. $m\sim M$ means $c_1M\le m\le c_2M$ for absolute constants c_1 and c_2. $E(x)$ always denotes the error term in the circle problem. ε denotes an arbitrary small positive number and may be different at each occurrence.

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2. Some preliminary lemmas and results. The following lemmas are needed.

LEMMA 1. *Suppose* $0 < c_1\lambda_1 \le |f'(n)| \le c_2\lambda_1$ *and* $|f''(n)| \sim \lambda_1 N^{-1}$ *for* $N \le n \le cN$. *Then*

$$\sum_{N<n\le cN} e(f(n)) \ll \lambda_1^{-1} + \lambda_1^{1/2} N^{1/2}.$$

Proof. If $c_2\lambda_1 \le 1/2$, this estimate is contained in the Kuz'min–Landau inequality; otherwise, the estimate follows from the well-known van der Corput's estimate for the second order derivative.

LEMMA 2. *Suppose* $a(n) = O(1), 0 < L \le M < N \le cL, L \gg 1, T \ge 2$. *Then*

$$\sum_{M<n\le N} a(n) = \frac{1}{2\pi i}\int_{-T}^{T} \sum_{L<l\le cL} \frac{a(l)}{l^{it}} \cdot \frac{N^{it}-M^{it}}{t}\,dt + O\bigg(\min\bigg(1, \frac{L}{T\|M\|}\bigg) + \min\bigg(1, \frac{L}{T\|N\|}\bigg) + O\bigg(\frac{L\log(1+L)}{T}\bigg)\bigg).$$

Proof. This is the well-known Perron formula.

LEMMA 3. *Suppose* $f(n) \ll P$ *and* $f'(n) \gg \Delta$ *for* $n \sim N$. *Then*

$$\sum_{n\sim N} \min\bigg(D, \frac{1}{\|f(n)\|}\bigg) \ll (P+1)(D+\Delta^{-1})\log(2+\Delta^{-1}).$$

Proof. This is contained in Lemma 2.8 of Krätzel [9].

LEMMA 4. *Suppose* $a(n)$ *are any complex numbers and* $1 \le Q \le N$. *Then*

$$\Big|\sum_{N<n\le cN} a(n)\Big|^2 \ll \frac{N}{Q}\sum_{0\le q\le Q}\bigg(1-\frac{q}{Q}\bigg)\Re \sum_{N<n\le cN-q} a(n)\overline{a(n+q)}.$$

Proof. This is Weyl's inequality.

LEMMA 5. *Let* $M \le N < N_1 \le M_1$ *and* $a(n)$ *be any complex numbers. Then*

$$\Big|\sum_{N<n\le N_1} a(n)\Big| \le \int_{-\infty}^{\infty} K(\theta)\Big|\sum_{M<n\le M_1} a(m)e(m\theta)\Big|\,d\theta$$

with $K(\theta) = \min(M_1 - M + 1, (\pi|\theta|)^{-1}, (\pi|\theta|)^{-2})$ *and*

$$\int_{-\infty}^{\infty} K(\theta)\, d\theta \le 3\log(2 + M_1 - M).$$

P r o o f. This is Lemma 2.2 of Bombieri and Iwaniec [2].

LEMMA 6. *Let* $\alpha\beta \ne 0, \Delta > 0, M \ge 1, N \ge 1$. *Let* $\mathcal{A}(M,N,\Delta)$ *be the number of quadruples* $(m, \widetilde{m}, n, \widetilde{n})$ *such that*

$$|(\widetilde{m}/m)^{\alpha} - (\widetilde{n}/n)^{\beta}| < \Delta$$

with $M \le m, \widetilde{m} \le 2M$ *and* $N \le n, \widetilde{n} \le 2N$. *Then*

$$\mathcal{A}(M,N,\Delta) \ll MN\log 2MN + \Delta M^2N^2.$$

P r o o f. This is Lemma 1 of [3].

LEMMA 7. *Suppose* $f(x)$ *and* $g(x)$ *are algebraic functions in* $[a,b]$ *and*

$$|f''(x)| \sim \frac{1}{R}, \qquad |f'''(x)| \ll \frac{1}{RU},$$
$$|g(x)| \ll G, \qquad |g'(x)| \ll GU_1^{-1}, \qquad U, U_1 \ge 1.$$

Then

$$\sum_{a<n\le b} g(n)e(f(n)) = \sum_{\alpha<u\le\beta} b_u \frac{g(n_u)}{\sqrt{|f''(n_u)|}} e(f(n_u) - un_u + 1/8)$$
$$+ O(G\log(\beta - \alpha + 2) + G(b - a + R)(U^{-1} + U_1^{-1}))$$
$$+ O(G\min(\sqrt{R}, 1/\langle\alpha\rangle) + G\min(\sqrt{R}, 1/\langle\beta\rangle)),$$

where $[\alpha,\beta]$ *is the image of* $[a,b]$ *under the mapping* $y = f'(x)$, n_u *is the solution of the equation* $f'(x) = u$,

$$b_u = \begin{cases} 1 & \text{for } \alpha < u < \beta, \\ \frac{1}{2} & \text{for } u = \alpha \in \mathbb{Z} \text{ or } u = \beta \in \mathbb{Z}, \end{cases}$$

and the function $\langle t\rangle$ *is defined as follows*:

$$\langle t\rangle = \begin{cases} \|t\| & \text{if } t \text{ is not an integer}, \\ \beta - \alpha & \text{otherwise}, \end{cases}$$

where $\|t\| = \min_{n\in\mathbb{Z}}\{|t - n|\}$.

P r o o f. This is Theorem 2.2 of Min [10].

LEMMA 8. *Suppose* A_i, B_j, a_i *and* b_j *are all positive numbers. If* Q_1 *and* Q_2 *are real with* $0 < Q_1 \le Q_2$, *then there exists some* q *such that* $Q_1 \le q \le Q_2$ *and*

$$\sum_{i=1}^{m} A_i q^{a_i} + \sum_{j=1}^{n} B_j q^{-b_j} \le 2^{m+n}\Big(\sum_{i=1}^{m}\sum_{j=1}^{n}(A_i^{b_j}B_j^{a_i})^{1/(a_i+b_j)} + \sum_{i=1}^{m} A_i Q_1^{a_i} + \sum_{j=1}^{n} B_j Q_2^{-b_j}\Big).$$

Proof. This is Lemma 3 of Srinivasan [12].

LEMMA 9. *Suppose x is a large positive real number, M, N, U are positive integers such that $1 \le N \le M^{1/4}, x^{\varepsilon} \le M \le x^{4/15}$ and $M^{1/6} \le U \le M^{5/6}$. Then*

$$S = \sum_{n\sim N} a_n \sum_{u\sim U} b_u \sum_{v\sim MU^{-1}} c_v e\Big(\frac{\sqrt{nx}}{uv}\Big) \ll (x^{1/12}M^{7/12}N^{11/12} + M^{11/12}N^{1/2})\log x, \tag{2.1}$$

where a_n, b_u, c_v are coefficients satisfying $|a_n| \le 1, |b_u| \le 1, |c_v| \le 1$.

Proof. We use Heath-Brown's method [4].

Without loss of generality, suppose $M^{1/2} \le U \le M^{5/6}$; otherwise we change the order of U and $V = MU^{-1}$. Suppose $1 \le Q \le VN$ is a parameter to be determined. For each q $(1 \le q \le VN)$ define

$$w_q = \Big\{(v,n)\,\Big|\, v\sim V,\ n\sim N,\ \frac{2\sqrt{N}(q-1)}{VQ} < \frac{\sqrt{n}}{v} < \frac{2\sqrt{N}q}{VQ}\Big\}.$$

Then

$$S = \sum_{u\sim U} b_u \sum_{q=1}^{Q}\Big(\sum_{(v,n)\in w_q} a_n c_v e\Big(\frac{\sqrt{nx}}{uv}\Big)\Big).$$

By Cauchy's inequality

$$\begin{aligned}|S|^2 &\ll UQ \sum_{u\sim U}\sum_{q=1}^{Q}\sum_{(v_1,n_1),(v_2,n_2)\in w_q} a_{n_1}\overline{a}_{n_2} \\ &\qquad \times c_{v_1}\overline{c}_{v_2} e\Big(\frac{\sqrt{x}}{u}\Big(\frac{\sqrt{n_1}}{v_1} - \frac{\sqrt{n_2}}{v_2}\Big)\Big) \\ &\ll UQ \sum_{(*)}\Big|\sum_{u\sim U} e\Big(\frac{\sqrt{x}}{u}\Big(\frac{\sqrt{n_1}}{v_1} - \frac{\sqrt{n_2}}{v_2}\Big)\Big)\Big|,\end{aligned} \tag{2.2}$$

where $(*)$ denotes the condition

$$\Big|\frac{\sqrt{n_1}}{v_1} - \frac{\sqrt{n_2}}{v_2}\Big| \le \frac{2\sqrt{N}}{VQ}, \qquad v_1, v_2 \sim V, n_1, n_2 \sim N. \tag{2.3}$$

Let $\lambda = \sqrt{n_1}/v_1 - \sqrt{n_2}/v_2$. Then by Lemma 1, the inner sum in (2.2) can be estimated by

$$\min\left(U, \frac{U^2}{\sqrt{x}|\lambda|}\right) + x^{1/4}|\lambda|^{1/2}U^{-1/2}. \tag{2.4}$$

By Lemma 6, the contribution of U (namely $|\lambda| \le Ux^{-1/2}$) to $|S|^2$ is

$$U^2Q(VN\log 2VN + Ux^{-1/2}N^{3/2}V^3) \ll U^2QVN\log 2VN. \tag{2.5}$$

The contribution of $U^2x^{-1/2}|\lambda|^{-1}$ ($|\lambda| > Ux^{-1/2}$) is

$$U^2QVN(\log 2VN)^2$$

by a similar argument. The contribution of $x^{1/4}|\lambda|^{1/2}U^{-1/2}$ is (note $|\lambda| \ll \sqrt{N}/(VQ), Q \ll VN$)

$$x^{1/4}U^{1/2}N^{9/4}V^{3/2}Q^{-1/2}.$$

Now, taking $Q = 1 + [x^{1/6}U^{-1}N^{5/6}V^{1/3}]$, we get

$$|S|\log^{-1} 2VN \ll UV^{1/2}N^{1/2} + x^{1/12}U^{1/2}V^{2/3}N^{11/12}, \tag{2.6}$$

whence the lemma follows since $M^{1/2} \le U \le M^{5/6}$.

LEMMA 10. *Suppose x, M, N satisfy the conditions of Lemma 9. Then*

$$\begin{aligned} T(M,N) = \sum_{n\sim N} a(n) \sum_{m\sim M} \mu(m) e\left(\frac{\sqrt{nx}}{m}\right) \\ \ll (x^{1/12}M^{7/12}N^{11/12} + M^{11/12}N^{1/2})x^{\varepsilon}, \end{aligned} \tag{2.7}$$

where $|a(n)| \le 1$.

Proof. By Heath-Brown's identity ($k = 4$), $T(M,N)$ can be written as $O(\log^8 x)$ sums of the form

$$T = \sum_{n\sim N} a(n) \sum_{m_1\sim M_1} \cdots \sum_{m_8\sim M_8} \mu(m_1)\ldots\mu(m_4) e\left(\frac{\sqrt{nx}}{m_1\ldots m_8}\right), \tag{2.8}$$

where $M \ll M_1\ldots M_8 \ll M, M_1, M_2, M_3, M_4 \le (2M)^{1/4}$. Some m_i may only take value 1.

To prove the lemma we consider three cases.

CASE 1: There is some M_i such that $M_i > M^{5/6}$. It must follow that $i \ge 5; i = 8$ for example. We use the exponent pair $(1/6, 4/6)$ to estimate the sum on m_8 and estimate other variables trivially, to get ($F = \sqrt{Nx}M^{-1}$)

$$\begin{aligned} T &\ll NMM_8^{-1}\left(\frac{M_8}{F} + \left(\frac{F}{M_8}\right)^{1/6} M_8^{2/3}\right) \\ &\ll x^{-1/2}M^2N^{1/2} + x^{1/12}M^{5/12}N^{13/12} \\ &\ll x^{1/12}M^{5/12}N^{13/12} \ll x^{1/12}M^{7/12}N^{11/12}. \end{aligned} \tag{2.9}$$

CASE 2: There is some M_i satisfying $M^{1/6} \le M_i \le M^{5/6}$. Let $u = m_i$, $v = \prod_{j\ne i} m_j$, $U = M_i$. Then by Lemma 9 we get

$$x^{-\varepsilon}T \ll x^{1/12}M^{7/12}N^{11/12} + M^{11/12}N^{1/2}, \tag{2.10}$$

where x^{ε} comes from the divisor argument.

CASE 3: All M_i satisfy $M_i \le M^{1/6}$. Without loss of generality, suppose $M_1 \ge M_2 \ge \ldots \ge M_8$. Let l be the first positive integer j such that

$$M_1 \ldots M_j > M^{1/6}; \tag{2.11}$$

then $l \ge 2$. Thus we have

$$M_1 \ldots M_l \le (M_1 \ldots M_{l-1})M_l \le M^{2/6}.$$

Now take $u = m_1 \ldots m_j, v = m_{j+1} \ldots m_6, U = M_1 \ldots M_j$. By Lemma 9, (2.10) still holds.

Lemma 10 follows from the three cases.

Now we prove the following proposition. It plays an important role in this paper. The idea of the proof has been used by several authors; see Jia [8], Baker and Harman [1], for example.

PROPOSITION 1. *Suppose M, N are large positive numbers, $A > 0$, α, β are rational numbers (not non-negative numbers). Suppose $m \sim M$, $n \sim N$, $F = AM^{\alpha}N^{\beta} \gg N, |a(m)| \le 1, |b(n)| \le 1$. Then*

$$\begin{aligned} S &= \sum_{M<m\le 2M} \sum_{N<n\le 2N} a(m)b(n)e(Am^{\alpha}n^{\beta}) \\ &\ll (MN^{1/2} + F^{4/20}M^{13/20}N^{15/20} + F^{4/23}M^{15/23}N^{18/23} \\ &\quad + F^{1/6}M^{2/3}N^{7/9} + F^{1/5}M^{3/5}N^{4/5} + F^{1/10}M^{4/5}N^{7/10}) \log^4 F. \end{aligned}$$

Proof. Without loss of generality, we suppose $\beta > 0$; for $\beta < 0$, the proof is the same. By Cauchy's inequality and Lemma 4 we get

$$\begin{aligned} |S|^2 &\ll M \sum_{M<m\le 2M} \Big| \sum_{N<n\le 2N} b(n)e(Am^{\alpha}n^{\beta}) \Big|^2 \\ &\ll \frac{M^2N^2}{Q} + \frac{MN}{Q}|\Sigma_1| \end{aligned} \tag{2.12}$$

with

$$\Sigma_1 = \sum_{q=1}^{Q} \left(1 - \frac{q}{Q}\right) \sum_{N<n\le 2N-q} \overline{b(n)}b(n+q) \sum_{M<m\le 2M} e(Am^{\alpha}g(n,q)),$$

where Q is a parameter satisfying $\log N \le Q \le N\log^{-1} N$, $g(n,q) = (n+q)^{\beta} - n^{\beta}$.

We write

$$(2.13)\qquad \Sigma_1 = \sum_{1\le q\le Q}\ \sum_{2N-Q<n\le 2N-q}\ \sum_{m} + \sum_{q\le B}\ \sum_{N<n\le 2N-Q}\ \sum_{m} + \sum_{B\le q\le Q}\ \sum_{N<n\le 2N-Q}\ \sum_{m} = \Sigma_2+\Sigma_3+\Sigma_4,$$

where $B=\max(\log N, MN/(c(\alpha,\beta))F)$ and

$$c(\alpha,\beta)=2|\alpha\beta|\max(2^{\alpha-1},1)\max(2^{\beta-1},1).$$

By Lemma 1 we have

$$(2.14)\qquad \Sigma_2 \ll \left(\frac{MNQ}{F}+\frac{F^{1/2}Q^{5/2}}{N^{1/2}}\right)\log N,$$

$$(2.15)\qquad \Sigma_3 \ll \left(\frac{MN^2}{F}+F^{1/2}N^{1/2}\right)\log^2 N.$$

So we only need to bound Σ_4. Notice that Σ_4 can be written as the sum of $O(\log Q)$ exponential sums of the form

$$(2.16)\quad \Sigma_5 = \sum_{Q_1<q\le 2Q_1} c(q) \sum_{N<n\le 2N-Q} \overline{b(n)}b(n+q) \sum_{M<m\le 2M} e(Am^\alpha g(n,q)),$$

where $B\le Q_1\le Q/2$, $c(q)=1-q/Q$.

By Lemma 7 we have

$$(2.17)\qquad \sum_{M<m\le 2M} e(Am^\alpha g(n,q)) = c_3 \sum_{U_1<u\le U_2} \frac{(Ag)^{1/(2(1-\alpha))}}{u^{(1-\alpha/2)/(1-\alpha)}} e(c_4A^{1/(1-\alpha)}g^{1/(1-\alpha)}u^{-\alpha/(1-\alpha)}) + O\left(\log F+\frac{MN}{Fq}+\min\left(\frac{MN^{1/2}}{F^{1/2}q^{1/2}}, \max\left(\frac{1}{\langle U_2\rangle},\frac{1}{\langle U_1\rangle}\right)\right)\right),$$

where $U_1=c_5AM^{\alpha-1}g$, $U_2=c_6AM^{\alpha-1}g$, $g=g(n,q)$. By Lemma 3, the contribution of the error term to Σ_5 is

$$NQ_1\log F + MN^2F^{-1}\log Q_1 + F^{1/2}Q_1^{3/2}N^{-1/2}.$$

It can be easily seen that $g(n,q)<\beta qn^{\beta-1}$ for $0<\beta<1$, $g(n,q)>\beta qn^{\beta-1}$ for $\beta>1$. If $\beta qAM^{\alpha-1}n^{\beta-1}-AM^{\alpha-1}g>\log^{-1}N$, then by the trivial estimate we have $(i=5,6)$

$$(2.18)\qquad \sum_{q\sim q_1}\sum_{n\sim N}\left|\sum_{c_iAM^{\alpha-1}g<u\le c_i\beta qAM^{\alpha-1}n^{\beta-1}} \frac{(Ag)^{1/(2(1-\alpha))}}{u^{(1-\alpha/2)/(1-\alpha)}} \times e(c_4A^{1/(1-\alpha)}g^{1/(1-\alpha)}u^{-\alpha/(1-\alpha)})\right| \ll F^{1/2}Q_1^{5/2}N^{-1/2}\log^2 N.$$

Now suppose $\beta q A M^{\alpha-1} n^{\beta-1} - A M^{\alpha-1} g \le \log^{-1} N$. Notice that the fact that

$$[c_i \beta q A M^{\alpha-1} n^{\beta-1}] - [c_i A M^{\alpha-1} g] = 1$$

always implies

$$\|c_i A M^{\alpha-1} g\| \le c_i \beta q A M^{\alpha-1} n^{\beta-1} - c_i A M^{\alpha-1} g = \delta_i.$$

We have

$$\begin{aligned}(2.19)\qquad & \sum_{c_i A M^{\alpha-1} g < u \le c_i \beta q A M^{\alpha-1} n^{\beta-1}} \frac{(Ag)^{1/(2(1-\alpha))}}{u^{(1-\alpha/2)/(1-\alpha)}} \\ &\ll \frac{MN^{1/2}}{F^{1/2} Q_1^{1/2}} \sum_{c_i A M^{\alpha-1} g < u \le c_i \beta q A M^{\alpha-1} n^{\beta-1}} 1 \\ &\ll \frac{MN^{1/2}}{F^{1/2} Q_1^{1/2}} \min\left(1, \frac{\delta_i}{\|c_i A M^{\alpha-1} g\|}\right) \\ &\ll \frac{MN^{1/2}}{F^{1/2} Q_1^{1/2}} \min\left(1, \frac{FQ_1^2}{MN^2} \cdot \frac{1}{\|c_i A M^{\alpha-1} g\|}\right),\end{aligned}$$

thus by Lemma 3,

$$\begin{aligned}(2.20)\qquad & \sum_{q\sim q_1} \sum_{n\sim N} \Bigg| \sum_{c_i A M^{\alpha-1} g < u \le c_i \beta q A M^{\alpha-1} n^{\beta-1}} \frac{(Ag)^{1/(2(1-\alpha))}}{u^{(1-\alpha/2)/(1-\alpha)}} \\ &\qquad \times e\big(c_4 A^{1/(1-\alpha)} g^{1/(1-\alpha)} u^{-\alpha/(1-\alpha)}\big)\Bigg| \\ &\ll F^{1/2} Q_1^{3/2} N^{-1/2} \log^2 N.\end{aligned}$$

Now it suffices to bound

$$\begin{aligned}(2.21)\qquad \Sigma_6 = & \sum_{Q_1 < q \le 2Q_1} c(q) \sum_{N < n \le 2N - Q} \overline{b(n)} b(n+q) \\ & \times \sum_u \frac{(Ag)^{1/(2(1-\alpha))}}{u^{(1-\alpha/2)/(1-\alpha)}} e\big(c_4 A^{1/(1-\alpha)} g^{1/(1-\alpha)} u^{-\alpha/(1-\alpha)}\big),\end{aligned}$$

where

$$c_5 \beta q A M^{\alpha-1} n^{\beta-1} < u \le c_6 \beta q A M^{\alpha-1} n^{\beta-1}.$$

By Lemmas 2 and 3 we get (choose $T = F^{10}$)

$$(2.22)\qquad \Sigma_6 \ll |\Sigma_7| + F^{1/2} Q_1^{3/2} N^{-1/2} \log N$$

with

$$(2.23)\quad \Sigma_7 = \sum_{Q_1<q\le 2Q_1} c(q) \sum_{N<n\le 2N-Q} \overline{b(n)}b(n+q) \sum_u \frac{(Ag)^{1/(2(1-\alpha))}}{u^{(1-\alpha/2)/(1-\alpha)}} \times \frac{q^{it}A^{it}M^{(\alpha-1)it}n^{(\beta-1)it}}{u^{it}} e(c_4A^{1/(1-\alpha)}g^{1/(1-\alpha)}u^{-\alpha/(1-\alpha)}),$$

where t is a real number independent of the variables and

$$c_7FQ_1(MN)^{-1} < u \le c_8FQ_1(MN)^{-1}.$$

It is easy to see that

$$\begin{aligned}(2.24)\quad \Sigma_7 &= \sum_u \frac{A^{1/(2(1-\alpha))+it}M^{(\alpha-1)it}}{u^{(1-\alpha/2)/(1-\alpha)+it}} \sum_{N<n\le 2N-Q} \overline{b(n)}n^{(\beta-1)it} \\ &\quad \times \sum_{Q_1<q\le 2Q_1} j(q)b(n+q)g^{1/(2(1-\alpha))} \\ &\quad \times e(c_4A^{1/(1-\alpha)}g^{1/(1-\alpha)}u^{-\alpha/(1-\alpha)}) \\ &\ll \sum_u \frac{(Ag)^{1/(2(1-\alpha))}}{u^{(1-\alpha/2)/(1-\alpha)}}(Q_1N^{\beta-1})^{1/(2(1-\alpha))}\Sigma_8(u) \\ &\ll F^{1/2}Q_1^{1/2}N^{-1/2}\Sigma_8(u_0)\end{aligned}$$

for some u_0 with

$$(2.25)\quad \Sigma_8(u) = \sum_{N<n\le 2N-Q} \Big| \sum_{Q_1<q\le 2Q_1} j(q)b(n+q)g_0^{1/(2(1-\alpha))} \times e(c_4A^{1/(1-\alpha)}g^{1/(1-\alpha)}u^{-\alpha/(1-\alpha)})\Big|,$$

where $j(q) = c(q)q^{it}, 1 \ll g_0 = g(Q_1N^{\beta-1})^{-1} \ll 1$.

Suppose $10 \le R \le Q_1 \log^{-1/2} N$ is a parameter to be determined. By Cauchy's inequality and Lemma 4 we get

$$\begin{aligned}(2.26)\quad \Sigma_8(u)^2 &\ll N \sum_n \Big| \sum_q j(q)b(n+q)g_0^{1/(2(1-\alpha))} \\ &\quad \times e(c_4A^{1/(1-\alpha)}g^{1/(1-\alpha)}u^{-\alpha/(1-\alpha)})\Big|^2 \\ &\ll \frac{N^2Q_1^2}{R} + \frac{NQ_1}{R}\sum_{1\le r\le R}|E_r|,\end{aligned}$$

where

$$\begin{aligned}E_r = &\sum_{N<n\le 2N-Q}\ \sum_{Q_1<q\le 2Q_1-r} j(q)b(n+q)g_0^{1/(2(1-\alpha))}(n,q) \\ &\times \overline{j(q+r)b(n+q+r)}g_0^{1/(2(1-\alpha))}(n,q+r) \\ &\times e(c_4A^{1/(1-\alpha)}u^{-\alpha/(1-\alpha)}(g^{1/(2(1-\alpha))}(n,q) - g^{1/(2(1-\alpha))}(n,q+r))).\end{aligned}$$

So it reduces to bound E_r for fixed r. Making the change of variable $n+q=l$, we have

$$\begin{aligned}E_r = &\sum_{Q_1<q\le 2Q_1-r} j(q)\overline{j(q+r)} \sum_{N+q<l\le 2N+q-Q} b(l)\overline{b(l+r)} g_0^{1/(2(1-\alpha))}(l-q,q)\\ &\times g_0^{1/(2(1-\alpha))}(l-q,q+r)\\ &\times e(c_4A^{1/(1-\alpha)}u^{-\alpha/(1-\alpha)}(g^{1/(2(1-\alpha))}(l,q)-g^{1/(2(1-\alpha))}(l,q+r)))\\ = &\sum_{N+Q_1<n\le 2N+2Q_1-r-Q} b(n)\overline{b(n+r)}\\ &\times \sum_{\max(Q_1,n-2N+Q)<q\le\min(2Q_1-r,n-N)} j(q)\overline{j(q+r)}\\ &\times g_0^{1/(2(1-\alpha))}(n-q,q)g_0^{1/(2(1-\alpha))}(n-q,q+r)\\ &\times e(c_4A^{1/(1-\alpha)}u^{-\alpha/(1-\alpha)}k(n,q,r)),\end{aligned}$$

where

$$k(n,q,r) = g^{1/(2(1-\alpha))}(n-q,q) - g^{1/(2(1-\alpha))}(n-q,q+r).$$

By Lemma 5 we get

$$\begin{aligned}E_r\log^{-1}N \ll &\sum_{n\sim N}\Big|\sum_{q\sim Q_1} j(q)\overline{j(q+r)}g_0^{1/(2(1-\alpha))}(n-q,q)\\ &\times g_0^{1/(2(1-\alpha))}(n-q,q+r)e(c_4A^{1/(1-\alpha)}u^{-\alpha/(1-\alpha)}k(n,q,r)+\theta_0 q)\Big|,\end{aligned}$$

where θ_0 is a real number independent of n and q.

Suppose $10\le T\le Q_1\log^{-1/2}N$ is a parameter to be determined. By Cauchy's inequality and Lemma 4 we get

$$|E_r|^2\log^{-2}N \ll \frac{N^2Q_1^2}{T} + \frac{NQ_1}{T}|D_t(r)|, \tag{2.27}$$

with

$$\begin{aligned}D_t(r) = &\sum_{n\sim N}\sum_{Q_1<q\le 2Q_1-t} j(q)\overline{j(q+r)}g_0^{1/(2(1-\alpha))}(n-q,q)\\ &\times g_0^{1/(2(1-\alpha))}(n-q,q+r)j(q+t+r)\overline{j(q+t)}\\ &\times g_0^{1/(2(1-\alpha))}(n-q-t,q+t)g_0^{1/(2(1-\alpha))}(n-q-t,q+t+r)\\ &\times e(c_4A^{1/(1-\alpha)}u^{-\alpha/(1-\alpha)}(k(n,q,r)-k(n,q+t,r)))\\ \ll &\sum_{q\sim Q_1}\Big|\sum_{n\sim N}\phi(n)e(f(n))\Big|,\end{aligned}$$

where

$$\phi(n) = g_0^{1/(2(1-\alpha))}(n-q,q)g_0^{1/(2(1-\alpha))}(n-q,q+r)$$
$$\times g_0^{1/(2(1-\alpha))}(n-q-t,q+t)g_0^{1/(2(1-\alpha))}(n-q-t,q+t+r)$$

and

$$f(n) = c_4 A^{1/(1-\alpha)} u^{-\alpha/(1-\alpha)}(k(n,q,r) - k(n,q+t,r)).$$

It is an easy exercise to verify that $\phi(n)$ is monotonic and

$$|f'(n)| \sim \frac{Frt}{Q_1 N^2}, \qquad |f''(n)| \sim \frac{Frt}{Q_1 N^3} \qquad (N < n \leq 2N);$$

thus by Lemma 1 we get

$$D_t(r) \ll Q_1\left(\frac{Q_1 N^2}{Frt} + \frac{(Frt)^{1/2}}{(Q_1 N)^{1/2}}\right) = \frac{Q_1^2 N^2}{Frt} + \frac{(FrtQ_1)^{1/2}}{N^{1/2}}. \tag{2.28}$$

Inserting (2.28) into (2.27) we get

$$|E_r|^2 \log^{-2} N \ll \frac{Q_1^2 N^2}{T} + \frac{Q_1^3 N^3}{FrT} + Q_1^{3/2}(NFTr)^{1/2}. \tag{2.29}$$

Notice that (2.29) is also true for $0 < T < 10$; then by Lemma 8 choosing a best $T \in (0, Q_1 \log^{-1/2} N)$ we get

$$|E_r|^2 \log^{-2} N \ll N^{1/2}Q_1^{5/6}F^{1/6}r^{1/6} + N^{2/3}Q_1 + NQ_1^{1/2} + \frac{N^{3/2}Q_1}{F^{1/2}r^{1/2}}. \tag{2.30}$$

Inserting (2.30) into (2.26) we have

$$\Sigma_8(u)^2 \log^{-2} N \ll \frac{Q_1^2 N^2}{R} + N^{3/2}Q_1^{11/6}F^{1/6}R^{1/6} + \frac{N^{5/2}Q_1^2}{F^{1/2}R^{1/2}} + N^{5/3}Q_1^2 + N^2 Q_1^{3/2}. \tag{2.31}$$

This is also true for $0 < R \leq 10$. Choosing a best $R \in (0, Q_1 \log^{-1/2} N)$ via Lemma 8 we get

$$\Sigma_8(u) \log^{-2} N \ll N^{11/14}Q_1^{13/14}F^{1/14} + N^{14/16}Q_1^{15/16} + N^{5/4}Q_1^{3/4}F^{-1/4} + N^{5/6}Q_1 + NQ_1^{3/4}. \tag{2.32}$$

Inserting (2.32) into (2.24) we have

$$\Sigma_7 \ll N^{4/14}Q_1^{20/14}F^{8/14} + N^{6/16}Q_1^{23/16}F^{8/16} + N^{3/4}Q_1^{5/4}F^{1/4} + N^{1/3}Q_1^{3/2}F^{1/2} + N^{1/2}Q_1^{5/4}F^{1/2}. \tag{2.33}$$

Combining (2.17), (2.18), (2.20), (2.22) and (2.33) we get

$$\Sigma_5 \log^{-4} F \ll N^{4/14}Q^{20/14}F^{8/14} + N^{6/16}Q^{23/16}F^{8/16} + N^{3/4}Q^{5/4}F^{1/4} + N^{1/3}Q^{3/2}F^{1/2} + N^{1/2}Q^{5/4}F^{1/2} + NQ + MN^2F^{-1} + F^{1/2}Q^{5/2}N^{-1/2}. \tag{2.34}$$

Inserting (2.14), (2.15) and (2.34) into (2.12) we get

$$
\begin{aligned}
(2.35)\quad |S|^2\log^{-6}F \ll{}& \frac{M^2N^2}{Q} + MN^{18/14}Q^{6/14}F^{8/14}\\
&+ MN^{22/16}Q^{7/16}F^{8/16} + MN^{7/4}Q^{1/4}F^{1/4}\\
&+ MN^{4/3}Q^{1/2}F^{1/2} + MN^{3/2}Q^{1/4}F^{1/2} + MN^2\\
&+ M^2N^3Q^{-1}F^{-1} + MN^{1/2}Q^{3/2}F^{1/2}.
\end{aligned}
$$

Since $Q < N \ll F$, we have

$$
\begin{aligned}
MN^2 + MN^{7/4}Q^{1/4}F^{1/4} &\ll MN^{3/2}Q^{1/4}F^{1/2},\\
M^2N^3(FQ)^{-1} &\ll M^2N^2Q^{-1}.
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
(2.36)\quad |S|^2\log^{-6}F \ll{}& \frac{M^2N^2}{Q} + MN^{18/14}Q^{6/14}F^{8/14}\\
&+ MN^{22/16}Q^{7/16}F^{8/16} + MN^{4/3}Q^{1/2}F^{1/2}\\
&+ MN^{3/2}Q^{1/4}F^{1/2} + MN^{1/2}Q^{3/2}F^{1/2}.
\end{aligned}
$$

Note that (2.36) is trivial for $0 < Q \le \log N$. Now the proposition follows from choosing a best $Q \in (0, N\log^{-1}N)$ via Lemma 8.

3. An expression of the error term. In the rest of this paper, we always use $E_P(x)$ to denote the difference $V(x) - \frac{6}{\pi}x$. The aim of this section is to give an expression of $E_P(x)$ subject to RH.

Let y be a parameter, $x^{\varepsilon} \le y \le x^{1/2-\varepsilon}$,

$$
(3.1)\qquad f_1(s) = \sum_{n\le y}\mu(n)n^{-s}, \qquad f_2(s) = \zeta^{-1}(s) - f_1(s).
$$

Let $r(n)$ be the number of representations of n as a sum of two squares. Then

$$
\begin{aligned}
(3.2)\qquad V(x) &= \sum_{\substack{a^2+b^2\le x\\ (a,b)=1}} 1 = \sum_{a^2+b^2\le x}\ \sum_{m|(a,b)}\mu(m)\\
&= \sum_{m^2(a^2+b^2)\le x}\mu(m) = \sum_{m^2k\le x}\mu(m)r(k)\\
&= \sum_{m\le y} + \sum_{m>y} = \Sigma_1 + \Sigma_2, \quad \text{say.}
\end{aligned}
$$

Notice that for $\sigma > 1$,

$$
(3.3)\qquad \sum_{n=1}^{\infty} r(n)n^{-s} = 4\zeta(s)L(s,\chi),
$$

where χ is the non-principal character mod 4, we have

$$(3.4)\quad \Sigma_1 = \sum_{m\le y}\mu(m)P\bigg(\frac{x}{m^2}\bigg)$$
$$= \sum_{m\le y}\mu(m)\bigg(\mathrm{Res}_{s=1}\frac{4\zeta(s)L(s,\chi)}{s}\bigg(\frac{x}{m^2}\bigg)^s + E\bigg(\frac{x}{m^2}\bigg)\bigg)$$
$$= \mathrm{Res}_{s=1}(4f_1(2s)\zeta(s)L(s,\chi)x^s s^{-1}) + \sum_{m\le y}\mu(m)E\bigg(\frac{x}{m^2}\bigg).$$

To treat Σ_2 we begin with

$$f_2(s) = \sum_{m>y}\mu(m)m^{-s}$$

for $\sigma > 1$. Hence

$$(3.5)\qquad 4f_2(2s)\zeta(s)L(s,\chi) = \sum_{n=1}^{\infty} b(n)n^{-s} \qquad (\sigma > 1),$$

where

$$b(n) = \sum_{n=m^2k,\, m>y}\mu(m)r(k).$$

By Perron's formula we have

$$(3.6)\qquad \Sigma_2 = \sum_{n\le x} b(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-ix^2}^{1+\varepsilon+ix^2} g(s)\,ds + O(x^\varepsilon),$$

where

$$g(s) = 4f_2(2s)\zeta(s)L(s,\chi)x^s s^{-1}.$$

By Cauchy's theorem, we have

$$(3.7)\qquad \frac{1}{2\pi i}\int_{1+\varepsilon-ix^2}^{1+\varepsilon+ix^2} g(s)\,ds = I_1 + I_2 - I_3 + \mathrm{Res}_{s=1}g(s),$$

where

$$I_1 = \frac{1}{2\pi i}\int_{0.5+\varepsilon+ix^2}^{1+\varepsilon+ix^2} g(s)\,ds, \quad I_2 = \frac{1}{2\pi i}\int_{0.5+\varepsilon-ix^2}^{0.5+\varepsilon+ix^2} g(s)\,ds,$$
$$I_3 = \frac{1}{2\pi i}\int_{0.5+\varepsilon-ix^2}^{1+\varepsilon-ix^2} g(s)\,ds.$$

Since RH is true, it follows that

$$(3.8)\qquad \zeta(s) \ll |t|^\varepsilon + 1, \qquad \sigma \ge 0.5+\varepsilon,$$

$$f_2(2s) \ll y^{-1/2}(|t|^{\varepsilon}+1), \qquad \sigma \ge 0.5+\varepsilon. \tag{3.9}$$

For $L(s,\chi)$, we have

$$L(s,\chi) \ll (|t|+1)^{(1-\sigma)/2}, \qquad 0.5 \le \sigma \le 1. \tag{3.10}$$

Using (3.8)–(3.10) we get

$$I_1 - I_3 \ll y^{-1/2}(1+x^{\varepsilon}), \tag{3.11}$$

$$\begin{aligned} I_2 &\ll y^{-1/2}x^{1/2+\varepsilon}\biggl(\int_1^{x^2} \frac{|L(1/2+\varepsilon+it,\chi)|}{t}\,dt+1\biggr) \\ &\ll y^{-1/2}x^{1/2+\varepsilon}. \end{aligned} \tag{3.12}$$

Combining (3.6)–(3.12) we get

$$\Sigma_2 = \mathrm{Res}_{s=1}(4f_2(2s)\zeta(s)L(s,\chi)x^s s^{-1}) + O(y^{-1/2}x^{1/2+\varepsilon}). \tag{3.13}$$

From (3.2), (3.4) and (3.13) we get

$$\begin{aligned} V(x) &= \mathrm{Res}_{s=1}(4\zeta^{-1}(2s)\zeta(s)L(s,\chi)x^s s^{-1}) \\ &\quad + \sum_{m\le y}\mu(m)E\biggl(\frac{x}{m^2}\biggr) + O(y^{-1/2}x^{1/2+\varepsilon}) \\ &= \frac{6}{\pi}x + \sum_{m\le y}\mu(m)E\biggl(\frac{x}{m^2}\biggr) + O(y^{-1/2}x^{1/2+\varepsilon}). \end{aligned} \tag{3.14}$$

Now we obtain the main result of this section.

PROPOSITION 2. *If RH is true, then for* $x^{\varepsilon} \le y \le x^{1/2-\varepsilon}$, *we have*

$$E_P(x) = \sum_{m\le y}\mu(m)E\biggl(\frac{x}{m^2}\biggr) + O(y^{-1/2}x^{1/2+\varepsilon}). \tag{3.15}$$

4. Proof of the Theorem. Take $y = x^{4/15}$ in Proposition 2. We only need to estimate the sum

$$\sum_{m\sim M}\mu(m)E\biggl(\frac{x}{m^2}\biggr)$$

for $x^{1/10} \ll M \ll y$. For $M \ll x^{1/10}$, we have trivially

$$\sum_{m\le x^{1/10}}\mu(m)E\biggl(\frac{x}{m^2}\biggr) \ll \sum_{m\le x^{1/10}} x^{1/3}m^{-2/3} \ll x^{11/30}. \tag{4.1}$$

By the well-known Voronoi formula of $E(t)$ (see [7], 13.8) we get

$$x^{-\varepsilon}\sum_{m\sim M}\mu(m)E\biggl(\frac{x}{m^2}\biggr) \ll \biggl|\sum_{m\sim M}\frac{\mu(m)x^{1/4}}{m^{1/2}}\sum_{n\le x^{4/15}}\frac{r(n)}{n^{3/4}}e\biggl(\frac{\sqrt{nx}}{m}\biggr)\biggr| + x^{11/30}. \tag{4.2}$$

It now suffices to show that $(1 \ll N \ll Y)$

$$S(M,N) = \sum_{m\sim M}\frac{\mu(m)x^{1/4}}{m^{1/2}}\sum_{n\sim N}\frac{r(n)}{n^{3/4}}e\biggl(\frac{\sqrt{nx}}{m}\biggr) \ll x^{11/30+\varepsilon}. \tag{4.3}$$

We consider three cases.

CASE 1: $M \le N \le x^{4/15}$. Let

$$T(M,N) = \sum_{m\sim M}\frac{\mu(m)M^{1/2}}{m^{1/2}}\sum_{n\sim N}\frac{r(n)N^{3/4-\varepsilon}}{n^{3/4}}e\biggl(\frac{\sqrt{nx}}{m}\biggr).$$

By Proposition 1 (take $(X,Y)=(N,M)$) we get

$$\begin{aligned} x^{-\varepsilon}T(M,N) \ll{}& NM^{1/2} + x^{2/20}N^{15/20}M^{11/20} \\ &+ x^{2/23}N^{17/23}M^{14/23} + x^{1/12}N^{15/20}M^{11/18} \\ &+ x^{2/20}N^{14/20}M^{12/20} + x^{1/20}N^{17/20}M^{12/20}, \end{aligned} \tag{4.4}$$

whence

$$\begin{aligned} x^{-\varepsilon}S(M,N) &\ll x^{1/4}N^{1/4} + x^{7/20}M^{1/20} + x^{31/92}M^{9/92} \\ &\quad + x^{1/3}M^{1/9} + x^{3/10}N^{1/10}M^{1/10} \\ &\ll x^{109/300} \ll x^{11/30}. \end{aligned} \tag{4.5}$$

CASE 2: $M^{1/4} \le N \le M$. We again use Proposition 1 to bound $S(M,N)$ (now take $(X,Y)=(M,N)$) and get

$$\begin{aligned} x^{-\varepsilon}S(M,N) &\ll x^{7/20}N^{1/10}M^{-1/20} + x^{31/92}N^{11/92}M^{-1/46} \\ &\quad + x^{1/3}N^{1/9} + x^{7/20}N^{3/20}M^{-1/10} + x^{3/10}M^{1/5} \\ &\quad + x^{1/4}N^{-1/4}M^{1/2} \\ &\ll x^{109/300} + x^{1/4}M^{7/16} \ll x^{11/30}. \end{aligned} \tag{4.6}$$

CASE 3: $N < M^{1/4}$. We use Lemma 10 to bound $T(M,N)$ and get

$$x^{-\varepsilon}T(M,N) \ll x^{1/12}N^{11/12}M^{1/12} + N^{1/2}M^{11/12},$$

whence we have

$$\begin{aligned} x^{-\varepsilon}S(M,N) &\ll x^{1/3}N^{1/6}M^{1/12} + x^{1/4}M^{5/12} \\ &\quad + x^{1/3}M^{1/8} + x^{1/4}M^{5/12} \\ &\ll x^{11/30}. \end{aligned} \tag{4.7}$$

This completes the proof of the Theorem.

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Department of Mathematics
Shandong Normal University
Jinan, 250014, Shandong
P.R. China
E-mail: wgzhai@jn-public.sd.cninfo.net

Beijing Institute
of Petrochemical Technology
Daxing, Beijing 102600
P.R. China
E-mail: biptiao@info.ind.cn.net

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