vanish. Hence $C_0 = I_{n_0}$. Since the columns of D_0 are orthogonal to the columns of T, it follows that n_0 rows of T vanish and the proof is complete.

We have already remarked that, in the case of quadratic forms Q associated with p-blocks of positive defect of finite groups, no row of the matrix T can vanish. Hence Q cannot represent a form Q_0 of determinant 1. In particular, Q cannot represent the number 1.

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On Epstein's zeta function

by

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Dedicated to Professor L. J. Mordell on the occasion of his seventy-fifth birthday

§ 1. Introduction and statement of results. The purpose of this paper is to give detailed proofs of two theorems on the Epstein zeta function which were announced without proof by S. Chowla and A. Selberg about fifteen years ago [1]. The two results which they announced are our Theorem 1 and a slightly weaker form of our Theorem 3. Our Theorem 2 was not stated explicitly by Chowla and Selberg in their paper, but they did indicate that they were in possession of a result of the same nature as our Theorem 2, that is, one giving a good approximation to the Epstein zeta function in the critical strip, particularly on or near the real line.

Throughout this paper a, b, and c will denote real numbers with $a > 0$ and $d = b^2 - 4ac < 0$, so that $am^2 + bmn + cn^2$ is a positive definite quadratic form. The Epstein zeta function associated with this form is given by

$$(1) \qquad Z(s) = \tfrac{1}{2} \sum{}' (am^2 + bmn + cn^2)^{-s} \qquad (\mathrm{Re}\, s > 1),$$

where the stroke on the sign of summation indicates that the summation is to be extended over all pairs (m, n) of integers other than the pair $(0, 0)$. It will be convenient to define a positive number k by putting

$$k^2 = \frac{|d|}{4a^2} = \frac{4ac - b^2}{4a^2} = \frac{c}{a} - \left(\frac{b}{2a}\right)^2.$$

As usual ζ will denote the Riemann zeta function. We shall also require the Bessel function defined for arbitrary ν and $|\arg z| < \pi/2$ by

$$(2) \quad K_\nu(z) = \tfrac{1}{2}\int_0^\infty e^{-z(u+u^{-1})/2} u^{\nu-1}\, du = \tfrac{1}{2}\int_{-\infty}^\infty e^{-z\cosh t} e^{\nu t}\, dt = \int_0^\infty e^{-z\cosh t} \cosh \nu t\, dt.$$

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Finally for arbitrary ν and positive integral n we write

$$\sigma_\nu(n) = \sum_{d|n} d^\nu = \sum_{d|n} (n/d)^\nu.$$

We shall prove the following theorems.

THEOREM 1. *When* $\mathrm{Re}\, s > 1$ *we have*

$$(3)\qquad a^s Z(s) = \zeta(2s) + k^{1-2s}\zeta(2s-1)\frac{\Gamma(s-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(s)} + \frac{\pi^s}{\Gamma(s)} k^{1/2-s} H(s),$$

where

$$(4)\qquad H(s) = 4\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)\cos(n\pi b/a) K_{s-1/2}(2\pi k n).$$

Further $H(s)$ *is an entire function of* s *such that*

$$H(s) = H(1-s).$$

The identity (3) provides the analytic continuation of $Z(s)$. In fact, since the sum of the first two terms on the right-hand side of (3) has a removable singularity at $s = 1/2$, a simple pole at $s = 1$ with residue $\pi/(2k)$, and no other finite singularities, it follows that $Z(s)$ has a continuation into the entire finite plane except for a simple pole at $s = 1$ with residue $\pi/(2ka) = \pi/|d|^{1/2}$. If we apply the functional equation for the Riemann zeta function to the second term on the right-hand side of (3), the formula (3) may be written

$$(5)\qquad (ak/\pi)^s\Gamma(s)Z(s) = (k/\pi)^s\Gamma(s)\zeta(2s) + (k/\pi)^{1-s}\Gamma(1-s)\zeta(2-2s) + k^{1/2}H(s).$$

Thus, if we put for the moment

$$\varphi(s) = (ak/\pi)^s\Gamma(s)Z(s) = |d|^{s/2}(2\pi)^{-s}\Gamma(s)Z(s),$$

we see that the Epstein zeta function satisfies the functional equation

$$(6)\qquad \varphi(s) = \varphi(1-s).$$

Many proofs of the continuability of $Z(s)$ and of the functional equation (6) are known, and accordingly a variety of other formulas for $H(s)$ are known (cf. [2], [3], [4], [6], [7]). However, the formula (4) given here is particularly good for estimating $H(s)$ on or near the segment $(0, 1)$ of the real line.

THEOREM 2. *If* $\frac{1}{2} \leqslant \sigma = \mathrm{Re}\, s \leqslant 1$, *we have*

$$(7)\qquad H(s) = 4\cos(\pi b/a)K_{s-1/2}(2\pi k) + \theta\frac{2\Gamma(\sigma)}{k^{1/2}|\Gamma(s)|}\sum_{n=2}^{\infty}\sigma_{-1}(n)e^{-2\pi k n},$$

where $|\theta| \leqslant 1$, *or more crudely*

$$(8)\qquad |H(s)| < \frac{\Gamma(\sigma)}{|\Gamma(s)|}\left\{\frac{2e^{-2\pi k}}{k^{1/2}} + \frac{3e^{-4\pi k}}{k^{1/2}(1-e^{-2\pi k})^2}\right\}.$$

Since $H(1-s) = H(s)$, the estimates in Theorem 2 imply estimates for $0 \leqslant \sigma \leqslant \frac{1}{2}$ as well.

THEOREM 3. *If* $k \geqslant \sqrt{3}/2$, *then*

$$(9)\qquad a^{1/2}Z(\tfrac{1}{2}) = \gamma + \log k - \log 4\pi + 2\theta k^{-1/2}e^{-2\pi k},$$

where γ *is Euler's constant and* $-1 < \theta < 1$. *Thus*

$$(10)\qquad Z(\tfrac{1}{2}) > 0 \quad \text{if} \quad k \geqslant 7.0556$$

(*that is, if* $k^2 \geqslant 49.79$ *or if* $4k^2 = |d|/a^2 \geqslant 199.2$), *but*

$$(11)\qquad Z(\tfrac{1}{2}) < 0 \quad \text{if} \quad \sqrt{3}/2 \leqslant k \leqslant 7.0554$$

(*that is, if* $3/4 \leqslant k^2 \leqslant 49.77$ *or if* $3 \leqslant 4k^2 = |d|/a^2 \leqslant 199.1$).

The inequality $k \geqslant \sqrt{3}/2$ is not particularly crucial. Some positive lower bound on k must be assumed in order to avoid degenerate cases. The inequality $k \geqslant \sqrt{3}/2$ is certainly fulfilled if a is the minimum of the quadratic form $am^2 + bmn + cn^2$ for integral values of m and n not both zero, in particular, if the form is reduced in the sense that $|b| \leqslant a \leqslant c$.

Since $Z(s)$ approaches $-\infty$ when s approaches 1 from below, it follows from Theorem 3 that $Z(s)$ vanishes in $(\frac{1}{2}, 1)$ if $k \geqslant 7.0556$. On the other hand, it is probable that $Z(s)$ is negative throughout $(0, 1)$ if $k \leqslant 7.0554$, but this would require a more detailed numerical analysis than that needed to prove Theorem 3. If $Z(s)$ is actually negative throughout $(0, 1)$ for all k between $\sqrt{3}/2$ and 7.0554 inclusive, it would follow by superposition that the Dedekind zeta function of any imaginary quadratic field with discriminant between -3 and -199 inclusive is negative throughout $(0, 1)$ and so has no zeros there. This last assertion has in fact been proved by Rosser [8], [9] in another way.

§ 2. Necessary lemmas.

LEMMA 1. *If* $\mathrm{Re}\,\nu > -\frac{1}{2}$ *and* $|\arg z| < \pi/2$, *we have the following alternative expressions for the Bessel function defined by* (2):

$$(12)\qquad K_\nu(z) = \frac{\Gamma(\nu+\frac{1}{2})(2z)^\nu}{\Gamma(\frac{1}{2})}\int_0^\infty \frac{\cos u\, du}{(u^2+z^2)^{\nu+1/2}},$$

$$(13)\qquad K_\nu(z) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zu}(u^2-1)^{\nu-1/2}du.$$

For proofs of these formulas see [5], pp. 50-52 or [10], pp. 185-187.

LEMMA 2. *If* $\operatorname{Re}\nu \geqslant 0$ *and* z *is positive, then*

$$|K_{-\nu}(z)| = |K_\nu(z)| \leqslant e^{-z}\left(\frac{\pi}{2z}\right)^{1/2} \sum_{j=0}^{r} \binom{r}{j} \frac{\Gamma(\operatorname{Re}\nu+\frac{1}{2}+j)}{|\Gamma(\nu+\frac{1}{2})|}\cdot\frac{1}{(2z)^j},$$

where r *is any non-negative integer not less than* $\operatorname{Re}\nu-\frac{1}{2}$. *In particular, if* $0 \leqslant \operatorname{Re}\nu \leqslant \frac{1}{2}$, *then*

$$|K_{-\nu}(z)| = |K_\nu(z)| \leqslant e^{-z}\left(\frac{\pi}{2z}\right)^{1/2} \frac{\Gamma(\operatorname{Re}\nu+\frac{1}{2})}{|\Gamma(\nu+\frac{1}{2})|}.$$

To prove this we make the substitution $u = t/z+1$ in (13) and obtain

$$(14)\qquad K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} \frac{e^{-z}}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty e^{-t} t^{\nu-1/2}\left(1+\frac{t}{2z}\right)^{\nu-1/2} dt.$$

Hence

$$|K_\nu(z)| \leqslant \left(\frac{\pi}{2z}\right)^{1/2} \frac{e^{-z}}{|\Gamma(\nu+\frac{1}{2})|} \int_0^\infty e^{-t} t^{\operatorname{Re}\nu-1/2}\left(1+\frac{t}{2z}\right)^{r} dt.$$

Expanding the rth power by the binomial theorem, we obtain the result stated, since $K_{-\nu}(z) = K_\nu(z)$ by (2).

LEMMA 3. *If* $0 \leqslant \nu \leqslant \frac{1}{2}$ *and* $z > 0$, *then*

$$0 < \left(\frac{2z}{\pi}\right)^{1/2} e^z K_\nu(z) \leqslant 1,$$

$$1-\frac{1-4\nu^2}{8z} \leqslant \left(\frac{2z}{\pi}\right)^{1/2} e^z K_\nu(z) \leqslant 1-\frac{1-4\nu^2}{8z}+\frac{(1-4\nu^2)(9-4\nu^2)}{2!\,(8z)^2},$$

and so on.

To prove this, let h be a fixed positive integer. Then, if $t > 0$, we have by Taylor's theorem

$$\left(1+\frac{t}{2z}\right)^{\nu-1/2} = \sum_{j=0}^{h-1}\binom{\nu-\frac{1}{2}}{j}\left(\frac{t}{2z}\right)^j + \theta_t\binom{\nu-\frac{1}{2}}{h}\left(\frac{t}{2z}\right)^h,$$

where $0 < \theta_t < 1$. Substituting this in (14), we get

$$\left(\frac{2z}{\pi}\right)^{1/2} e^z K_\nu(z) = 1+\sum_{j=1}^{h-1}\frac{(4\nu^2-1)(4\nu^2-3^2)\ldots(4\nu^2-\{2j-1\}^2)}{j!\,(8z)^j} + $$
$$+\,\theta\,\frac{(4\nu^2-1)(4\nu^2-3^2)\ldots(4\nu^2-\{2h-1\}^2)}{h!\,(8z)^h},$$

where $0 < \theta < 1$. Since $0 \leqslant \nu \leqslant \frac{1}{2}$, the last term has the sign of $(-1)^h$ and so we get the results stated by taking $h = 1, 2, \ldots$

§ 3. Proofs of the theorems. We begin with the first part of Theorem 1. Suppose $\operatorname{Re}s > 1$. From (1) we have

$$(15)\qquad a^s Z(s) = \frac{1}{2}\sum{}' \{(m+yn)^2+k^2n^2\}^{-s}$$
$$= \frac{1}{2}\sum_{m\neq 0} m^{-2s} + \frac{1}{2}\sum_{n\neq 0}\sum_{m=-\infty}^{\infty}\{(m+yn)^2+k^2n^2\}^{-s}$$
$$= \zeta(2s) + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\{(m+yn)^2+k^2n^2\}^{-s},$$

where $y = b/(2a)$. Now for fixed k and s the right-hand side of (15) is a function of y with period 1 and a bounded derivative. Thus

$$(16)\qquad a^s Z(s) = \tfrac{1}{2}A_0 + \sum_{r=1}^{\infty} A_r\cos(2\pi r y),$$

where

$$A_r = 2\int_0^1 \Big(\zeta(2s)+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\{(m+xn)^2+k^2n^2\}^{-s}\Big)\cos(2\pi r x)\,dx.$$

In particular

$$(17)\qquad \tfrac{1}{2}A_0 = \zeta(2s)+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} n^{-1}\int_m^{m+n}(u^2+k^2n^2)^{-s}\,du$$
$$= \zeta(2s)+\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}(u^2+k^2n^2)^{-s}\,du$$
$$= \zeta(2s)+\sum_{n=1}^{\infty}(kn)^{-2s+1}\int_{-\infty}^{\infty}(u^2+1)^{-s}\,du$$
$$= \zeta(2s)+k^{1-2s}\zeta(2s-1)\int_0^1 v^{s-3/2}(1-v)^{-1/2}\,dv$$
$$= \zeta(2s)+k^{1-2s}\zeta(2s-1)\,\frac{\Gamma(s-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(s)}.$$

Also, if $r > 0$, we have (putting $m = qn+l$)

$$A_r = 2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{n}\int_m^{m+n}(u^2+k^2n^2)^{-s}\cos\frac{2\pi r(u-m)}{n}\,du$$

$$= 2\sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=1}^{n}\sum_{q=-\infty}^{\infty}\int_{qn+l}^{(q+1)n+l}(u^2+k^2n^2)^{-s}\cos\frac{2\pi r(u-l)}{n}\,du$$

$$= 2\sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=1}^{n}\int_{-\infty}^{\infty}(u^2+k^2n^2)^{-s}\cos\frac{2\pi r(u-l)}{n}\,du$$

$$= 2\sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=1}^{n}\cos\left(\frac{2\pi rl}{n}\right)\int_{-\infty}^{\infty}(u^2+k^2n^2)^{-s}\cos\left(\frac{2\pi ru}{n}\right)du$$

$$= 2\sum_{n|r}\left(\frac{2\pi r}{n}\right)^{2s-1}\int_{-\infty}^{\infty}\frac{\cos v}{\{v^2+(2\pi kr)^2\}^s}\,dv.$$

Using (12), we get (for r positive)

$$A_r = 2(2\pi r)^{2s-1}\sigma_{1-2s}(r)\frac{2\pi^{1/2}K_{s-1/2}(2\pi kr)}{\Gamma(s)(4\pi kr)^{s-1/2}} \tag{18}$$

$$= 4\frac{\pi^s}{\Gamma(s)}k^{1/2-s}r^{s-1/2}\sigma_{1-2s}(r)K_{s-1/2}(2\pi kr).$$

Inserting (17) and (18) in (16) and recalling that $y = b/(2a)$, we get (3), the first assertion of Theorem 1.

To prove the second part of Theorem 1 we first note that if s is in some bounded subset B of the complex plane and z is positive, then by the first part of Lemma 2

$$|K_{s-1/2}(z)| \leqslant e^{-z}\left(\frac{\pi}{2z}\right)^{1/2}P_B\left(\frac{1}{z}\right),$$

where P_B is a polynomial with positive coefficients depending only on B. Thus the sum in (4) converges uniformly in B. Since each term is an entire function of s, it follows that $H(s)$ is an entire function. Now $K_\nu(z)$ is an even function of ν and

$$n^{-\nu/2}\sigma_\nu(n) = n^{-\nu/2}\sum_{d|n}d^\nu = n^{-\nu/2}\sum_{d|n}(n/d)^\nu = n^{\nu/2}\sigma_{-\nu}(n).$$

Thus each term of the sum in (4) is unchanged if we replace s by $1-s$. Hence $H(s) = H(1-s)$.

To prove Theorem 2 we first remark that if $0 \leqslant \nu \leqslant 1$, then

$$n^{\nu/2}\sigma_{-\nu}(n) = \frac{1}{2}n^{\nu/2}\sum_{d|n}\left\{d^{-\nu}+\left(\frac{n}{d}\right)^{-\nu}\right\} = \frac{1}{2}\sum_{d|n}\left\{\left(\frac{n^{1/2}}{d}\right)^\nu+\left(\frac{d}{n^{1/2}}\right)^\nu\right\}$$

$$\leqslant \frac{1}{2}\sum_{d|n}\left\{\frac{n^{1/2}}{d}+\frac{d}{n^{1/2}}\right\} = n^{1/2}\sigma_{-1}(n).$$

Using this and the second part of Lemma 2, we get

$$|4n^{s-1/2}\sigma_{1-2s}(n)\cos(n\pi b/a)K_{s-1/2}(2\pi kn)| \tag{19}$$

$$\leqslant 4n^{1/2}\sigma_{-1}(n)\frac{e^{-2\pi kn}}{(4kn)^{1/2}}\cdot\frac{\Gamma(\sigma)}{|\Gamma(s)|} = \frac{2\Gamma(\sigma)}{k^{1/2}|\Gamma(s)|}\sigma_{-1}(n)e^{-2\pi kn},$$

provided $\frac{1}{2} \leqslant \sigma = \operatorname{Re} s \leqslant 1$. In view of (4), this proves the first part of Theorem 2. To prove the second part, note that if $n \geqslant 2$, we have

$$\sigma_{-1}(n) = \sum_{d|n}d^{-1} \leqslant 1+\tfrac{1}{2}(n-1) \leqslant \tfrac{3}{2}(n-1),$$

with equality only when $n = 2$. Thus

$$\sum_{n=2}^{\infty}\sigma_{-1}(n)e^{-2\pi kn} < \frac{3}{2}\sum_{n=2}^{\infty}(n-1)e^{-2\pi kn} = \frac{3e^{-4\pi k}}{2(1-e^{-2\pi k})^2}.$$

Using (19) for $n = 1$, we get the second part of Theorem 2.

Finally we turn to Theorem 3. Setting $s = \frac{1}{2}$ in (7) and using Lemma 3 and the preceding inequality, we have

$$\left|H\left(\frac{1}{2}\right)\right| \leqslant 4K_0(2\pi k)+\frac{3e^{-4\pi k}}{k^{1/2}(1-e^{-2\pi k})^2} \tag{20}$$

$$\leqslant \frac{2e^{-2\pi k}}{k^{1/2}}\left\{1-\frac{1}{16\pi k}+\frac{9}{512\pi^2k^2}+\frac{3e^{-2\pi k}}{2(1-e^{-2\pi k})^2}\right\} < \frac{2e^{-2\pi k}}{k^{1/2}},$$

provided $k \geqslant \sqrt{3}/2$, say. Thus it remains to evaluate the sum of the first two terms on the right-hand side of (3) or (5) at the point $s = \frac{1}{2}$. Let us rewrite (5) as

$$k^{-1/2}(ak/\pi)^s\Gamma(s)Z(s) = f(s)+f(1-s)+H(s), \tag{21}$$

where

$$f(s) = k^{-1/2}(k/\pi)^s \Gamma(s)\,\zeta(2s) = \zeta(2s)\exp\left\{\left(s-\frac{1}{2}\right)\log\frac{k}{\pi}+\log\frac{\Gamma(s)}{\Gamma(\frac{1}{2})}\right\}.$$

Now for $|s-\frac{1}{2}|<\frac{1}{2}$ we have the power series

$$\log\frac{\Gamma(s)}{\Gamma(\frac{1}{2})} = \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})}\left(s-\frac{1}{2}\right)+\sum_{j=2}^{\infty}\frac{(s-\frac{1}{2})^j}{j!}\sum_{l=0}^{\infty}\frac{(-1)^j(j-1)!}{(l+\frac{1}{2})^j}$$

$$= (\gamma+\log 4)\left(\frac{1}{2}-s\right)+\sum_{j=2}^{\infty}\frac{(2^j-1)\,\zeta(j)}{j}\left(\frac{1}{2}-s\right)^j.$$

Since the Laurent series for $\zeta(2s)$ around $\frac{1}{2}$ is

$$\zeta(2s) = \frac{1}{2(s-\frac{1}{2})}+\gamma+\ldots,$$

$f(s)$ has the Laurent series

$$f(s) = \left\{\frac{1}{2(s-\frac{1}{2})}+\gamma+\ldots\right\}\left\{1+\left(-\gamma+\log\frac{k}{4\pi}\right)\left(s-\frac{1}{2}\right)+\ldots\right\}$$

$$= \frac{1}{2(s-\frac{1}{2})}+\frac{1}{2}\left(\gamma+\log\frac{k}{4\pi}\right)+\ldots,$$

valid for $|s-\frac{1}{2}|<\frac{1}{2}$. Thus the "value" of $f(s)+f(1-s)$ at the removable singularity at $s=\frac{1}{2}$ is

$$\gamma+\log k-\log 4\pi.$$

Substituting $s=\frac{1}{2}$ in (21) and using (20), we obtain Theorem 3.

Added in proof (Sept. 1, 1964):

Recently R. A. Rankin has called our attention to the fact that the main assertion of Theorem 1, namely the validity of formulas (3) and (4), was actually proved on p. 157 of his paper, *A minimum problem for the Epstein zeta function*, Proc. Glasgow Math. Assoc. 1 (1953), pp. 149-158.

In a forthcoming paper Marc E. Low has proved the result envisaged in the last paragraph of § 1, that is, he has shown that if $k \leqslant 7.0554$, then $Z(s)$ is negative throughout $(0, 1)$. He did this by showing that if $k < 4\pi e^{-\gamma}$, then (in the notation introduced above) the series for $f(s)+f(1-s)$ in powers of $(s-\frac{1}{2})^2$ has negative coefficients and so takes its maximum on $(0, 1)$ at $s=\frac{1}{2}$. Thus, as mentioned in § 1, it follows by superposition that the Dedekind zeta function of any imaginary quadratic field with discriminant between -3 and -199 inclusive is negative throughout $(0, 1)$. By using similar methods in conjunction with a high-speed computer Low has extended this last assertion to all imaginary quadratic fields

with discriminants between -3 and -400000. A particularly interesting case is the imaginary quadratic fields with discriminant $-115147 = -113\cdot 1019$, for in this case the Dedekind zeta function takes a negative value very close to zero when $s=\frac{1}{2}$.

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