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Universal graphs and universal functions

by

R. Rado (Reading)

In a very general sense, a class K of mathematical structures is said to *contain a universal element* u^* if every element u of K can be embedded in u^* in some specified sense. The first three sections of this note are concerned with universality in certain classes of denumerable structures, and by means of elementary number theory we shall construct universal elements. Theorem 2, due to N.G. de Bruijn, asserts that a certain class, although rather similar to that considered in Theorem 1, nevertheless does not contain a universal element. In section 4 we consider analogous problems for higher cardinals.

1. Graphs.

1.1. In this note a *graph* is a pair $\Gamma = (A, B)$ of sets A, B such that every element of B is a subset of A of cardinal 2. Geometrically, the elements of A are thought of as *points* or *nodes*, and those of B as *lines* or *edges*. The intersection of two distinct edges is either empty or consists of exactly one point. The graph $\Gamma = (A, B)$ is *weakly embedded* in the graph $\Gamma^* = (A^*, B^*)$ if there is a one-one map φ from A into A^* such that $\{x, y\} \epsilon B$ implies $\{\varphi(x), \varphi(y)\} \epsilon B^*$. The graph Γ is *embedded* in Γ^* if such a map φ exists for which, in addition, for $x, y \epsilon A$, the relation $\{x, y\} \notin B$ implies $\{\varphi(x), \varphi(y)\} \notin B^*$. Weak embeddibility does not imply embeddibility as is seen by considering the case

$$A = A^* = \{0, 1\}; \quad B = \emptyset; \quad B^* = \{\{0, 1\}\}.$$

The graph Γ^* is said to be (*weakly*) *universal* in the class K of graphs if $\Gamma^* \epsilon K$ and if every graph Γ of K is (weakly) embeddible in Γ^*.

THEOREM 1. *The following graph $\Gamma^* = (A^*, B^*)$ is universal in the class K of all graphs with at most denumerably many nodes. We put $A^* = \{0, 1, \ldots\}$ and take as B^* the set of all subsets $\{x, y\}$ of A^* for which*

$$(1) \qquad y = \sum (0 \leqslant \nu < \infty)\ 2^{\nu} a(\nu),$$

where $a(\nu) \epsilon \{0, 1\}$ for all $\nu \geqslant 0$, and $a(x) = 1$.

Proof. Let $\Gamma = (A, B) \epsilon K$. We may assume $A = A^*$. We define inductively an embedding function φ of Γ in Γ^*. Let $\xi \epsilon A$, and suppose that $\varphi(x)$ has been defined as non-negative integer for all integral x in $0 \leqslant x < \xi$. We now define $\varphi(\xi)$. For $x \epsilon A$, put $a_0(x) = 1$ if [1] $\{x, \xi\}_< \epsilon B$, and $a_0(x) = 0$ otherwise. Choose a non-negative integer $\psi(x)$ such that $\varphi(x) < \psi(x)$ for $0 \leqslant x < \xi$, and put

$$\varphi(\xi) = \sum (0 \leqslant x < \xi)\ 2^{\varphi(x)} a_0(x) + 2^{\psi(x)}.$$

This defines a one-one map φ from A into A^*. If $\{x, \xi\}_< \epsilon B$ then $a_0(x) = 1$ and therefore, by definition of B^*, $\{\varphi(x), \varphi(\xi)\} \epsilon B^*$. If, on the other hand, $x \epsilon A$ and $\{x, \xi\}_< \notin B$, then $a_0(x) = 0$ and therefore $\{\varphi(x), \varphi(\xi)\} \notin B^*$. This proves that φ is an embedding function of Γ in Γ^* and proves Theorem 1.

1.2. The graph $\Gamma = (A, B)$ is *locally finite* if for every $x \epsilon A$ the set $\{y\colon \{x, y\} \epsilon B\}$ is finite. The graph Γ is a *tree* if Γ contains no circuit, i.e. if for no $n \geqslant 3$ and no nodes $x_0, \ldots, x_n$ we have $x_0 = x_n$ and $x_\mu \neq x_\nu$ for $0 \leqslant \mu < \nu < n$, and $\{x_\nu, x_{\nu+1}\} \epsilon B$ for $0 \leqslant \nu < n$.

The next theorem, and its proof, was kindly communicated to the author by N. G. de Bruijn.

THEOREM 2. *The class K of all locally finite graphs with denumerably many nodes does not contain a weakly universal element.*

Proof. We shall in fact prove more than is required. We shall show that given any graph Γ^* of K there always is a tree in K which cannot be weakly embedded in Γ^*.

For $\Gamma = (A, B) \epsilon K$ and $x \epsilon A$, and for every positive integer m we denote by $k(\Gamma, x, m)$ the number of nodes y such that, for suitable nodes $x_0, \ldots, x_m$, we have

$$x_0 = x; \quad \{x_\nu, x_{\nu+1}\} \epsilon B \quad \text{for} \quad 0 \leqslant \nu < m; \quad x_m = y.$$

Clearly, $k(\Gamma, x, m)$ is a non-negative integer.

Let $\Gamma^* = (A^*, B^*) \epsilon K$. We may assume $A^* = \{1, 2, \ldots\}$. Choose non-negative integers $p_1, p_2, \ldots$ and consider the following graph $\Gamma = (A, B)$. Put [2]

$$A = \{x_1, x_2, \ldots\} + \sum (1 \leqslant n < \infty)\ \{y_{n\nu}\colon 1 \leqslant \nu \leqslant p_n\}$$

where the x_m, $y_{n\nu}$ are mutually distinct;

$$B = \big\{\{x_m, x_{m+1}\}\colon 1 \leqslant m < \infty\big\} + \sum (1 \leqslant n < \infty)\ \big\{\{x_n, y_{n\nu}\}\colon 1 \leqslant \nu \leqslant p_n\big\}.$$

Then $\Gamma \epsilon K$. In fact, Γ is a tree of a very simple structure. Suppose that

(1) The symbol $\{x_0, x_1, \ldots, x_{n-1}\}_<$ denotes the set $\{x_0, \ldots, x_{n-1}\}$ and at the same time expresses the fact that $x_0 < x_1 < \ldots < x_{n-1}$.

(2) The operators "+" and "Σ" denote set union.

φ is a weak embedding function of Γ in Γ^*. Then, for $m \geqslant 1$, we have

$$k(\Gamma, x_1, m) \leqslant k\big(\Gamma^*, \varphi(x_1), m\big).$$

Also, by definition of Γ,

$$k(\Gamma, x_1, m) > p_m$$

since each of the $p_m + 1$ nodes $y_{m1}, y_{m2}, \ldots, y_{m,p_m}, x_{m+1}$ can be reached from x_1 in m steps along suitable edges of Γ. Hence $p_m < k\big(\Gamma^*, \varphi(x_1), m\big)$ for $m \geqslant 1$ and, in particular,

$$p_{\varphi(x_1)} < k\big(\Gamma^*, \varphi(x_1), \varphi(x_1)\big).$$

We obtain a contradiction if in the original definition of Γ we take $p_m = k(\Gamma^*, m, m)$ for $m = 1, 2, \ldots$

2. Complexes.

2.1. Graphs are one-dimensional simplicial complexes. Let l be a fixed positive [3] integer. A *l-dimensional complex* is a pair of sets, (A, B), where the elements of B are subsets of A of cardinal $l+1$. The notions of *embedding* and *universality* carry over in the obvious way from one dimension to l dimensions. The l-dimensional complex (A, B) is *embedded* in the l-dimensional complex (A^*, B^*) if there is a one-one map φ of A into A^* such that, for $X \subset A$, the relations $X \epsilon B$ and $\varphi(X) \epsilon B^*$ are equivalent.

THEOREM 3. *The following l-dimensional complex*

$$C^{*l} = (A^{*l}, B^{*l})$$

is universal in the class K^l of all l-dimensional complexes with denumerably many nodes. We put $A^{*l} = \{0, 1, \ldots\}$, and as B^{*l} we take the set of all sets $\{z_0, \ldots, z_l\}_< \subset A^{*l}$ such that there is a representation

$$z_l = \sum (0 \leqslant x_0 < \ldots < x_{l-1})\, 2^{\binom{x_0}{1} + \binom{x_1}{2} + \ldots + \binom{x_{l-1}}{l}}\, a(x_0, \ldots, x_{l-1}),$$

where all $a(x_0, \ldots, x_{l-1}) \epsilon \{0, 1\}$, and $a(z_0, \ldots, z_{l-1}) = 1$.

In proving the theorem we shall omit in our notation the index l, and we put $l-1 = m$. We need a simple lemma.

2.2. LEMMA. *The function*

$$\tau(x_0, \ldots, x_m) = \sum (0 \leqslant \lambda < l) \binom{x_\lambda}{\lambda+1}$$

defines a one-one map of the set S of all systems $(x_0, \ldots, x_m)$ of l integers with $0 \leqslant x_0 < \ldots < x_m$ onto [4] *the set $\{0, 1, \ldots\}$.*

(3) Theorem 3 also holds, with obvious changes, for $l = 0$.

(4) In our application of the Lemma we shall only need "into".

Proof of the Lemma. Since $\tau(0,1,\ldots,m)=0$ it suffices to show that if S is ordered by last differences then neighbouring elements of S are taken into neighbouring integers. Let $(x_0,\ldots,x_m)$, $(y_0,\ldots,y_m)$ be neighbouring elements of S, the first preceding the second. Then

$$(x_0,\ldots,x_m)=(a,a+1,\ldots,a+r-1,\ a+r,b_{r+1},\ldots,b_m),$$

$$(y_0,\ldots,y_m)=(0,1,\ldots,r-1,a+r+1,b_{r+1},\ldots,b_m),$$

for some r, $0\leqslant r\leqslant m$. Hence

$$\tau(x_0,\ldots,x_m)-\tau(y_0,\ldots,y_m)$$
$$=\sum(0\leqslant\lambda\leqslant r)\binom{a+\lambda}{\lambda+1}-\sum(0\leqslant\lambda<r)\binom{\lambda}{\lambda+1}-\binom{a+r+1}{r+1}$$
$$=\sum(0\leqslant\lambda\leqslant r)\left(\binom{a+\lambda+1}{\lambda+1}-\binom{a+\lambda}{\lambda}\right)-\binom{a+r+1}{r+1}=-1,$$

and the Lemma follows.

2.3. Proof of Theorem 3. Let $C=(A,B)\epsilon K$. We may assume $A=A^*$. We define an embedding φ of C in C^*. Let $\xi\epsilon A$, and suppose that $\varphi(x)$ has been defined for $\{x,\xi\}_<\subset A$, and that $\varphi(x)\epsilon A^*$. We now define $\varphi(\xi)$. For $0\leqslant x_0<\ldots<x_m<\xi$ put

$$b(x_0,\ldots,x_m)=\begin{cases}1 & \text{if } \{x_0,\ldots,x_m,\xi\}\epsilon B,\\ 0 & \text{otherwise.}\end{cases}$$

Choose a non-negative integer $\psi(\xi)$ such that $\varphi(x)<\psi(\xi)$ for $0\leqslant x<\xi$, and whenever $0\leqslant x_0<\ldots<x_m<\xi$ then $\tau(\varphi(x_0),\ldots,\varphi(x_m))<\psi(\xi)$. Then we put

$$(2)\quad \varphi(\xi)=\sum(0\leqslant x_0<\ldots<x_m<\xi)\,2^{\tau(\varphi(x_0),\ldots,\varphi(x_m))}b(x_0,\ldots,x_m)+2^{\psi(\xi)}.$$

This defines an order-preserving map φ from A into A^*.

Now let $0\leqslant y_0<\ldots<y_m<\xi$. We want to show that the two relations

$$(3)\quad \{y_0,\ldots,y_m,\xi\}\epsilon B,$$

$$(4)\quad \{\varphi(y_0),\ldots,\varphi(y_m),\varphi(\xi)\}\epsilon B^*$$

are equivalent.

(i) If (3) holds then $b(y_0,\ldots,y_m)=1$ and hence, by (2) and the definition of B^*, (4) follows.

(ii) If (4) holds then, by definition of B^*,

$$(5)\quad \varphi(\xi)=\sum(0\leqslant x_0<\ldots<x_m)\,2^{\tau(x_0,\ldots,x_m)}a(x_0,\ldots,x_m)$$

for some $a(x_0,\ldots,x_m)\epsilon\{0,1\}$ such that $a(\varphi(x_0),\ldots,\varphi(x_m))=1$. By comparing (2) and (5) and using the Lemma we find

$$1=a(\varphi(y_0),\ldots,\varphi(y_m))=b(y_0,\ldots,y_m).$$

Hence (3) follows by definition of $b(y_0,\ldots,y_m)$. This proves Theorem 3.

2.4. We briefly consider *mixed complexes*, i.e. complexes whoses simplexes are not necessarily of the same dimension. For any one of these complexes these dimensions need not be bounded, and the set of its nodes is denumerable.

THEOREM 4. *The following mixed complex $C^{*\infty}=(A^{*\infty},B^{*\infty})$ is universal in the class K of all mixed complexes with denumerably many nodes. We put $A^{*\infty}=\{0,1,\ldots\}$ and as $B^{*\infty}$ we take the set of all sets $\{z_0,\ldots,z_\lambda\}_<\subset A^{*\infty}$ for varying $\lambda\geqslant 0$, such that there is a representation*

$$z_\lambda=\sum(\mu\geqslant 0;\ 0\leqslant x_0<\ldots<x_{\mu-1})\,2^{2^{x_0}+\ldots+2^{x_{\mu-1}}}a(x_0,\ldots,x_{\mu-1}),$$

where all $a(x_0,\ldots,x_{\mu-1})\epsilon\{0,1\}$ and $a(z_0,\ldots,z_{\lambda-1})=1$.

Concerning the proof of Theorem 4 we only need to remark that the analogue of the Lemma is trivial and that the proof of Theorem 4 is closely parallel to that of Theorem 3.

3. Functions: the finite case. The Theorems 1, 3, 4 are most conveniently expressed in terms of functions rather than graphs and complexes. For $k\geqslant 2$ and $l\geqslant 0$ we denote by K_{kl} the class of all functions $f(x_0,\ldots,x_l)$ with range in $\{0,1,\ldots,k-1\}$ which are defined for integral $x_0,\ldots,x_l$ such that $0\leqslant x_0<\ldots<x_l$. The function $f^*\epsilon K_{kl}$ is *universal* in K_{kl} if, given any $f\epsilon K_{kl}$, there is a one-one map φ of the set $\{0,1,\ldots\}$ into itself such that

$$f(x_0,\ldots,x_l)=f^*(\varphi(x_0),\ldots,\varphi(x_l))$$

for $0\leqslant x_0<\ldots<x_l$. By K_k we denote the class of all functions $f(x_0,\ldots,x_\lambda)$ with range in $\{0,\ldots,k-1\}$ which are defined for any $\lambda\geqslant 0$ and $0\leqslant x_0<\ldots<x_\lambda$. The definition of a universal member of K_k is obvious. Theorem 1 is concerned with K_{21}, Theorem 3 with K_{2l}, and Theorem 4 with K_2. The following theorem is a generalization of Theorems 1, 3 and 4. It shows that these theorems remain valid if the simplexes are oriented and the complexes have coefficients from a ring of integers mod k.

THEOREM 5. (a) *Let $k\geqslant 2$ and $l\geqslant 1$. The following function f_{kl}^* is universal in K_{kl}. Every $x\geqslant 0$ has a unique representation*

$$(6)\quad x=\sum(0\leqslant x_0<\ldots<x_{l-1})\,k^{\binom{x_0}{1}+\ldots+\binom{x_{l-1}}{l}}f_0(x_0,\ldots,x_{l-1},x)$$

where $f_0\epsilon\{0,\ldots,k-1\}$. We put, for $0\leqslant x_0<\ldots<x_{l-1}<x$,

$$f_{kl}^*(x_0,\ldots,x_{l-1},x)=f_0(x_0,\ldots,x_{l-1},x).$$

(b) *Let $k \geqslant 2$. The following function f_k^* is universal in K_k. The range of f_k^* is in $\{0, \ldots, k-1\}$, and for every $x \geqslant 0$ we have*

$$(7)\qquad x = \sum (\lambda \geqslant 0; 0 \leqslant x_0 < \ldots < x_{\lambda-1} < x)\ k^{2^{x_0}+\ldots+2^{x_{\lambda-1}}} f_k^*(x_0, \ldots, x_{\lambda-1}, x).$$

Clearly f_k^* is uniquely defined by (7).

Proof of (a). Put $\tau(x_0, \ldots, x_{l-1}) = \sum (0 \leqslant \lambda < l) \binom{x_\lambda}{\lambda+1}$. Let $f \in K_{kl}$. We define $\varphi(x)$ as follows. Let $\xi \geqslant 0$, and let $\varphi(x)$ be defined as non-negative integer, for integral x, $0 \leqslant x < \xi$. We can choose an integer $\psi(\xi) \geqslant 0$ such that

$$\varphi(x) < \psi(\xi) \quad \text{for} \quad 0 \leqslant x < \xi,$$

and

$$\tau(\varphi(y_0), \ldots, \varphi(y_{l-1})) < \psi(\xi) \quad \text{for} \quad 0 \leqslant y_0 < \ldots < y_{l-1} < \xi.$$

Put

(8)

$$\varphi(\xi) = \sum (0 \leqslant y_0 < \ldots < y_{l-1} < \xi)\ k^{\tau(\varphi(y_0), \ldots, \varphi(y_{l-1}))} f(y_0, \ldots, y_{l-1}, \xi) + k^{\psi(\xi)}.$$

This defines φ, and we have

$$\varphi(x) < \psi(\xi) < k^{\psi(\xi)} \leqslant \varphi(\xi) \quad \text{for} \quad 0 \leqslant x < \xi.$$

By (6),

$$(9)\qquad \varphi(\xi) = \sum (0 \leqslant y_0 \ldots < y_{l-1})\ k^{\tau(y_0, \ldots, y_{l-1})} f_0(y_0, \ldots, y_{l-1}, \varphi(\xi)).$$

If $0 \leqslant x_0 < \ldots < x_{l-1} < \varphi(\xi)$, then

$$f_{kl}^*(x_0, \ldots, x_{l-1}, \varphi(\xi)) = f_0(x_0, \ldots, x_{l-1}, \varphi(\xi)).$$

Now let $0 \leqslant x_0 < \ldots < x_{l-1} < \xi$. We compare the coefficients of $k^{\tau(\varphi(x_0), \ldots, \varphi(x_{l-1}))}$ in (8) and (9) and obtain

$$f(x_0, \ldots, x_{l-1}, \xi) = f_0(\varphi(x_0), \ldots, \varphi(x_{l-1}), \varphi(\xi))$$
$$= f_{kl}^*(\varphi(x_0), \ldots, \varphi(x_{l-1}), \varphi(\xi)).$$

This proves the assertion.

Proof of (b). Put $\varrho(x_0, \ldots, x_{\lambda-1}) = 2^{x_0} + \ldots + 2^{x_{\lambda-1}}$ for $\lambda \geqslant 0$ and $0 \leqslant x_0 < \ldots < x_{\lambda-1}$. Let $f \in K_k$. We define φ. Let $\xi \geqslant 0$, and let $\varphi(x)$ be defined for $0 \leqslant x < \xi$. There is $\psi(\xi) \geqslant 0$ such that $\varphi(x) < \psi(\xi)$ for $0 \leqslant x < \xi$, and

$$\varrho(\varphi(y_0), \ldots, \varphi(y_{\lambda-1})) < \psi(\xi) \quad \text{for} \quad \lambda \geqslant 0 \text{ and } 0 \leqslant y_0 < \ldots < y_{\lambda-1} < \xi.$$

Put

(10) $\varphi(\xi) =$

$$\sum (\lambda \geqslant 0; 0 \leqslant y_0 < \ldots < y_{\lambda-1} < \xi)\ k^{\varrho(\varphi(y_0), \ldots, \varphi(y_{\lambda-1}))} f(y_0, \ldots, y_{\lambda-1}, \xi) + k^{\psi(\xi)}.$$

This defines $\varphi(x)$ for all $x \geqslant 0$, and we have

$$\varphi(x) < \psi(\xi) < k^{\psi(\xi)} \leqslant \varphi(\xi) \quad \text{for} \quad 0 \leqslant x < \xi.$$

By (7),

(11) $\varphi(\xi) =$

$$\sum (\lambda \geqslant 0;\ 0 \leqslant y_0 < \ldots < y_{\lambda-1} < \varphi(\xi))\ k^{\varrho(y_0, \ldots, y_{\lambda-1})} f_k^*(y_0, \ldots, y_{\lambda-1}, \varphi(\xi)).$$

Let $0 \leqslant x_0 < \ldots < x_{\lambda-1} < \xi$. We compare the coefficients of $k^{\varrho(\varphi(x_0), \ldots, \varphi(x_{\lambda-1}))}$ in (10) and (11) and obtain

$$f(x_0, \ldots, x_{\lambda-1}, \xi) = f_k^*(\varphi(x_0), \ldots, \varphi(x_{\lambda-1}), \varphi(\xi)).$$

This completes the proof of Theorem 5.

4. Functions: the transfinite case.

4.1. Notation. In this final section every small letter, unless a contrary statement is made, denotes an ordinal number (ordinal). The sequence of transfinite initial ordinals is

$$\omega_0, \omega_1, \ldots$$

and their cardinal numbers are

$$\aleph_0, \aleph_1, \ldots$$

respectively. We recall the definition of Tarski's *cofinality function* $\mathrm{cf}(n)$. For every n the symbol $\mathrm{cf}(n)$ denotes the least ordinal such that $\aleph_n$ can be written as the sum of $\aleph_{\mathrm{cf}(n)}$ cardinals smaller than $\aleph_n$. By $|X|$ and $|n|$ we denote the cardinal of the set X and of the ordinal n respectively. If $n = r+1$ then we put $n \dot{-} 1 = r$, and if n has no immediate predecessor then we put $n \dot{-} 1 = n$. We make frequent use of the *obliteration operator* $\hat{}$ whose effect upon a well ordered sequence consists in removing from that sequence the element upon which it is placed. For $a \leqslant b$ we put

$$[a, b) = \{x\colon a \leqslant x < b\} = \{a, a+1, \ldots, \hat{b}\}.$$

4.2. Let l and m be ordinals and S a set. Denote by $F(l, m, S)$ the set of all functions $f(x_0, \ldots, x_\lambda)$ with range in S which are defined for all $\lambda < l$ and $x_0, \ldots, \hat{x}_\lambda \leqslant x_\lambda < m$. We call f^* a *universal function* of $F(l, m, S) = F$ if $f^* \in F$ and if for every $f \in F$ there exists a one-one map φ of $[0, m)$ into $[0, m)$ such that

$$f(x_0, \ldots, x_\lambda) = f^*(\varphi(x_0), \ldots, \varphi(x_\lambda))$$

for $\lambda < l$ and $x_0, \ldots, \hat{x}_\lambda \leqslant x_\lambda < m$. It follows that if $F(l, m, S)$ contains a universal function then $F(l_0, m, S_0)$ contains a universal function whenever $l_0 \leqslant l$ and $\emptyset \neq S_0 \subset S$. Without loss of generality we may assume m to be an initial ordinal.

THEOREM 6. *Let l and n be ordinals, and suppose that $2^{\aleph_\nu} = \aleph_{\nu+1}$ for $\nu < n$. Then the set $F(l, \omega_n, S)$ contains a universal function if either* (i) $n = n \dot{-} 1 = \mathrm{cf}(n)$; $l < \omega_n$; $1 \leqslant |S| \leqslant \aleph_n$ *or* (ii) $n = r+1$; $l < \omega_{\mathrm{cf}(r)}$; $1 \leqslant |S| \leqslant \aleph_n$ *or* (iii) $n > \mathrm{cf}(n)$; $l < \omega_n$; $1 \leqslant |S| < \aleph_n$.

The author does not know how far in the definition of $F(l, m, S)$ the inequalities imposed on the arguments $x_0, \ldots, x_\lambda$ are essential nor whether the order type $(= \lambda+1)$ of the sequence of arguments $x_0, \ldots, x_\lambda$ necessarily must be of the second kind for the assertion of Theorem 6 to hold. Also, it would be of interest to have non-trivial cases of sets $F(l, m, S)$ which do not contain a universal function.

4.3. Proof of Theorem 6.

Case 1. Let (i) or (ii) hold. We may assume $S = [0, \omega_n)$. Let $\tau < \omega_n$, and let G_τ be the set of all functions $g(x_0, \ldots, \hat{x}_\lambda) \in S$, defined for $\lambda < l$ and $x_0, \ldots, \hat{x}_\lambda \in S$, such that $g(x_0, \ldots, \hat{x}_\lambda) = 0$ if $\sup(\mu < \lambda)\, x_\mu \geqslant \tau$. Put $G = \sum(\tau < \omega_n) G_\tau$. For fixed λ the number of systems $(x_0, \ldots, \hat{x}_\lambda)$ with $x_0, \ldots, \hat{x}_\lambda < \tau$ is $|\tau|^{|\lambda|}$, and for varying λ the total number of such systems is

$$\sum(\lambda < l)\, |\tau|^{|\lambda|} \leqslant |\tau|^{|l|}\, |l|.$$

Hence

$$|G| \leqslant \sum(\tau < \omega_n)\, |S|^{|\tau|^{|l|}|l|}.$$

If (i) holds then

$$|G| \leqslant \sum(\tau < \omega_n)\, \aleph_n^{|\tau|^{|l|}|l|} \leqslant \aleph_n,$$

and if (ii) holds then

$$|G| \leqslant \sum(\tau < \omega_n)\, \aleph_n^{\aleph_r} = \aleph_n.$$

In either case we can write

$$G = \{g_0, \ldots, \hat{g}_{\omega_n}\}.$$

Also, there is a representation $S = \sum(\nu < \omega_n) S_\nu$, where $|S_\nu| = \aleph_n$ for $\nu < \omega_n$ and $S_\mu \cap S_\nu = \emptyset$ for $\mu < \nu < \omega_n$. Define f^* by putting

$$f^*(y_0, \ldots, y_\lambda) = g_\nu(y_0, \ldots, \hat{y}_\lambda) \quad \text{if} \quad \lambda < l;\ y_0, \ldots, \hat{y}_\lambda \leqslant y_\lambda \in S_\nu;\ \nu < \omega_n.$$

Then $f^* \in F(l, \omega_n, S)$. Now let $f \in F(l, \omega_n, S)$. We define a one-one map φ of S into S. Let $\xi \in S$, and suppose that $\varphi(x)$ has been defined for $x < \xi$ and that $\varphi(x) < \varphi(y)$ for $x < y < \xi$. We now define $\varphi(\xi)$. There is an ordinal $\eta(\xi) < \omega_n$ such that $\varphi(x) < \eta(\xi)$ for $x < \xi$. There is $\nu(\xi) < \omega_n$ such that

$$f(x_0, \ldots, \hat{x}_\lambda, \xi) = g_{\nu(\xi)}(\varphi(x_0), \ldots, \hat{\varphi}(x_\lambda))$$

for $\lambda < l$; $x_0, \ldots, \hat{x}_\lambda \leqslant \xi$. We can now choose $\varphi(\xi) \in S_{\nu(\xi)}$ such that $\eta(\xi) \leqslant \varphi(\xi)$. This defines the map φ, and we have $\varphi(x) < \varphi(y) < \omega_n$ for $x < y < \omega_n$.

Now let $\lambda < l$ and $x_0, \ldots, \hat{x}_\lambda \leqslant x_\lambda < \omega_n$. Put $\xi = x_\lambda$. Then

$$f(x_0, \ldots, x_\lambda) = f(x_0, \ldots, \hat{x}_\lambda, \xi) = g_{\nu(\xi)}(\varphi(x_0), \ldots, \hat{\varphi}(x_\lambda)).$$

Since $\varphi(x_0), \ldots, \hat{\varphi}(x_\lambda) \leqslant \varphi(\xi) \in S_{\nu(\xi)}$ we have

$$f^*(\varphi(x_0), \ldots, \varphi(x_\lambda)) = g_{\nu(\xi)}(\varphi(x_0), \ldots, \hat{\varphi}(x_\lambda)) = f(x_0, \ldots, x_\lambda)$$

which shows that f^* is universal.

Case 2. Let (iii) hold. We may assume that $S = [0, s)$, where $1 \leqslant s < \omega_n$. Put

$$M = \{m:\ s, l \leqslant \omega_m < \omega_n\};$$

$$X_m = \{(m, \mu):\ \mu < \omega_{m+2}\} \text{ for } m \in M; \qquad X = \sum(m \in M)\, X_m.$$

We order X lexicographically. Then the order type of X is

$$\sum(s, l \leqslant \omega_m < \omega_n)\, \omega_{m+2} = \omega_n.$$

Instead of $F(l, \omega_n, S)$ we may consider the set F' of all functions $f(x_0, \ldots, x_\lambda) \in S$ defined for $\lambda < l$ and $x_0, \ldots, \hat{x}_\lambda \leqslant x_\lambda \in X$.

Let $m \in M$; $\eta \in X_m$ and consider a function $g(x_0, \ldots, \hat{x}_\lambda) \in S$ defined for $\lambda < l$ and $x_0, \ldots, \hat{x}_\lambda < \eta$. For fixed λ the number of systems $(x_0, \ldots, \hat{x}_\lambda)$ is at most $\aleph_{m+1}^{|\lambda|} \leqslant \aleph_{m+1}^{\aleph_m} = \aleph_{m+1}$. For varying λ the total number of such systems is at most $\aleph_{m+1}|l| \leqslant \aleph_{m+1}\aleph_m = \aleph_{m+1}$. Hence the number of functions g is at most $|s|^{\aleph_{m+1}} \leqslant \aleph_m^{\aleph_{m+1}} = \aleph_{m+2}$. Hence for fixed $m \in M$ the set of all functions g, for all η, can be written as $\{g_{m\mu}:\ \mu < \omega_{m+2}\}$. Also, we can write, for $m \in M$, $X_m = \sum(\mu < \omega_{m+2})\, X_{m\mu}$, where $|X_{m\mu}| = \aleph_{m+2}$ for $\mu < \omega_{m+2}$, and $X_{m\mu} \cap X_{m\lambda} = \emptyset$ for $\mu < \lambda < \omega_{m+2}$. We define f^* by putting

$$f^*(y_0, \ldots, y_\lambda) = g_{m\mu}(y_0, \ldots, \hat{y}_\lambda)$$

if

$$\lambda < l; \quad y_0, \ldots, \hat{y}_\lambda \leqslant y_\lambda \in X_{m\mu}; \quad m \in M; \quad \mu < \omega_{m+2}.$$

Then $f^* \in F'$. Now let $f \in F'$. We define the map φ. Let $\xi \in X$, and let $\varphi(x)$ be defined for $x < \xi$. Suppose that $\varphi(x) \in X_m$ if $\{x, \xi\}_< \subset X_m$, and also that $\varphi(x) < \varphi(y)$ for $x < y < \xi$. We now define $\varphi(\xi)$. There is $m(\xi) \in M$ such that $\xi \in X_{m(\xi)}$. There is $\eta(\xi) \in X_{m(\xi)}$ such that $\varphi(x) < \eta(\xi)$ for $x < \xi$. There is $\mu(\xi) < \omega_{m(\xi)+2}$ such that

$$f(x_0, \ldots, \hat{x}_\lambda, \xi) = g_{m(\xi),\mu(\xi)}(\varphi(x_0), \ldots, \hat{\varphi}(x_\lambda))$$

for $\lambda < l$; $x_0, \ldots, \hat{x}_\lambda \leqslant \xi$.

We can now choose $\varphi(\xi)\in X_{m(\xi),\mu(\xi)}$ such that $\eta(\xi)\leqslant\varphi(\xi)$. This defines φ, and we have $\varphi(X_m)\subset X_m$ for $m\in M$, and $\varphi(x)<\varphi(y)$ for $x<y$.

Let $\lambda<l$ and $x_0,\ldots,\hat{x}_\lambda\leqslant x_\lambda\in X$. Put $\xi=x_\lambda$. Then, by definition of $m(\xi)$ and $\mu(\xi)$,

$$f(x_0,\ldots,x_\lambda)=f(x_0,\ldots,\hat{x}_\lambda,\xi)=g_{m(\xi),\mu(\xi)}\big(\varphi(x_0),\ldots,\hat{\varphi}(x_\lambda)\big).$$

Since

$$\varphi(x_0),\ldots,\hat{\varphi}(x_\lambda)\leqslant\varphi(x_\lambda)=\varphi(\xi)\in X_{m(\xi),\mu(\xi)}$$

we have

$$f^*\big(\varphi(x_0),\ldots,\varphi(x_\lambda)\big)=g_{m(\xi),\mu(\xi)}\big(\varphi(x_0),\ldots,\hat{\varphi}(x_\lambda)\big)=f(x_0,\ldots,x_\lambda)$$

which completes the proof of Theorem 6.

THE UNIVERSITY, READING, ENGLAND

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Arithmetical Tauberian theorems

by

P. Erdös (Budapest) and A. E. Ingham (Cambridge)

Dedicated to Professor L. J. Mordell
on his 75th birthday

1. Introduction. At successive stages in the development of the proof of the prime number theorem several authors have investigated the relation

$$\sum_{n=1}^{x} f\left(\frac{x}{n}\right)=\sum_{n=1}^{x}\frac{x}{n}+bx+o(x)\qquad(x\to\infty),$$

or the same relation with a stronger error term, and deduced from it, under various supplementary conditions on $f(x)$, that

$$f(x)=x+o(x). \tag{1}$$

The problem is discussed explicitly by Landau ([7], pp. 597-604; [8]), Ingham ([3]), Karamata ([5], [6]), Gordon ([1]), and is implicit in the 'Eratosthenian' summation method introduced by Wintner ([10], [11]).

In this paper we consider the analogous problem in which the sequence $\{n\}$ of all positive integers is replaced by a finite or infinite sequence $1, a_1, a_2, \ldots$ of real numbers for which

$$1<a_1\leqslant a_2\leqslant\ldots,\qquad A=\sum\frac{1}{a_n}<\infty.$$

Initially $f(x)$ is supposed defined for all $x\geqslant 1$, but for formal convenience we extend its definition by putting $f(x)=0$ when $x<1$. We may then write our basic hypothesis in the form

$$f(x)+\sum f\left(\frac{x}{a_n}\right)=\left(1+\sum\frac{1}{a_n}\right)x+o(x), \tag{2}$$

or in the equivalent form

$$f_0(x)+\sum f_0\left(\frac{x}{a_n}\right)=o(x), \tag{2$_0$}$$