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Some remarks on a method of Mordell in the Geometry of Numbers

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1. Some years ago Lekkerkerker [1] gave a short analysis of a method of Mordell in the Geometry of Numbers, by which sometimes an estimate can be obtained for the critical determinant of an n-dimensional star body by reducing the problem to an $(n-1)$-dimensional one. In this note we add a few remarks that may lead to further elucidation of the method.

2. We consider two distance functions F and G, defining two star bodies K_F and K_G, both of the finite type, in an n-dimensional (Euclidean) space X. We suppose there is a group Ω of automorphs of K_F, all having the property that the contragredient transformation is an automorph of K_G, i.e. we suppose there is a group of non-singular $n\times n$-matrices A, such that $F(A.x) = F(x)$ and $G(\tilde{A}.x) = G(x)$ for all $x \in X$, $\tilde{A}$ denoting the transposed inverse of A.

It is not difficult to prove that, if R is the k-dimensional linear subspace of X generated by $k+1$ linearly independent points, including the origin o, of the lattice

$$\Lambda = \{x \mid x = L.u,\ u \in U\},$$

where L denotes a non-singular $n\times n$-matrix and U the set of all points of X with integral coordinates, then the $(n-k)$-dimensional subspace S of X through o and perpendicular to R is generated by $n-k+1$ linearly independent points of the contragredient lattice

$$\tilde{\Lambda} = \{x \mid x = \tilde{L}.u,\ u \in U\},$$

where $\tilde{L}$ is the transposed inverse of L. Further, denoting the k- and $(n-k)$-dimensional lattices $R \cap \Lambda$ and $S \cap \tilde{\Lambda}$ by λ and $\tilde{\lambda}$, respectively, we have for the determinants

$$d(\Lambda) = \frac{d(\lambda)}{d(\tilde{\lambda})} = \frac{1}{d(\tilde{\Lambda})}. \tag{1}$$

It is also clear that, if A is a non-singular $n\times n$-matrix, and the transformation $x\to A.x$ transforms the lattice Λ into Λ', then the contragredient transformation $x\to \tilde{A}.x$ transforms $\tilde{\Lambda}$ into $\tilde{\Lambda}'$.

We prove the following

THEOREM 1. *If there are in X a k-dimensional linear subspace R and an $(n-k)$-dimensional subspace S, perpendicular to each other, both containing o, with the following properties:*

(a) *The k- and $(n-k)$-dimensional star bodies $R\cap K_F$ and $S\cap K_G$ are of the finite type in R and S, respectively.*

(b) *To any k-dimensional linear subspace R' of X containing o, such that the k-dimensional star body $R'\cap K_F$ is of the finite type in R', corresponds an automorph of K_F in Ω transforming R' into R.*

Then, if $k=n-1$,

$$(2)\qquad (\Delta_{K_F})^k \geqslant \left(\frac{\Delta_{R\cap K_F}}{\Delta_{S\cap K_G}}\right)^n\cdot(\Delta_{K_G})^{n-k},$$

where Δ_{K_F} denotes the critical determinant of K, etc.

3. Proof of the theorem. Obviously, if $k=n-1$, then S is a straight line through o, $S\cap K_G$ is a segment of that line, and $\Delta_{S\cap K_G}$ is half the length of that segment.

We need a

LEMMA. *If Λ is a K_F-admissible lattice and x any point other than o of the contragredient lattice $\tilde{\Lambda}$, then there is an automorph A of K_F in Ω, such that the contragredient automorph $\tilde{A}$ of K_G transforms x into a point of S.*

Proof. Let S' be the line through x and o. Then the $(n-1)$-dimensional subspace R' through o and perpendicular to S' is generated by n linearly independent points of Λ. Hence $R'\cap\Lambda$ is an $(n-1)$-dimensional sublattice of Λ, which is $(R'\cap K_F)$-admissible, since Λ is K_F-admissible. Therefore, $R'\cap K_F$ is a star body of the finite type in R' and, consequently, there exists an automorph A of K_F in Ω transforming R' into R. But then, the contragredient automorph $\tilde{A}$ of K_G transforms S' into S. In particular $\tilde{A}.x\in S$.

From this lemma it follows that $G(x)>0$ for any point x other than o of $\tilde{\Lambda}$, if Λ is K_F-admissible.

For, if $G(x)=0$ and $\tilde{A}$ is the automorph of K_G for which $\tilde{A}.x\in S$, then also $G(\tilde{A}.x)=0$ and, hence, $G(x)=0$ for all $x\in S$, contrary to our assumption that $S\cap K_G$ should be of the finite type, i.e. of finite length.

It also follows that

$$G(\tilde{\Lambda}) = \inf\{G(x)\mid x\in\tilde{\Lambda},\ x\neq o\}>0.$$

For, let x be a primitive point of $\tilde{\Lambda}$, which is, according to the lemma, transformed into a point of S. Then, after the transformation, putting $R\cap\Lambda=\lambda$ and $S\cap\tilde{\Lambda}=\tilde{\lambda}$, we have $d(\tilde{\lambda})=|x|$. Further, since $\Delta_{S\cap K_G}$ is the distance to o of the point x_0 on S with $G(x_0)=1$, we have also $|x| = G(x).\Delta_{S\cap K_G}$. Hence

$$d(\tilde{\lambda}) = G(x).\Delta_{S\cap K_G}.$$

And so, by (1),

$$d(\Lambda)=\frac{d(\lambda)}{G(x).\Delta_{S\cap K_G}}.$$

But, since Λ is K_F-admissible, λ is $(R\cap K_F)$-admissible in R and, therefore,

$$d(\lambda)\geqslant \Delta_{R\cap K_F}.$$

Hence,

$$d(\Lambda)\geqslant\frac{\Delta_{R\cap K_F}}{G(x).\Delta_{S\cap K_G}}.$$

As the point x can be chosen arbitrarily, this implies $G(\tilde{\Lambda})>0$. Also

$$(3)\qquad d(\Lambda)\geqslant\frac{\Delta_{R\cap K_F}}{G(\tilde{\Lambda}).\Delta_{S\cap K_G}}.$$

Now, putting $G(\tilde{\Lambda})=1/c$, we see immediately that the lattice

$$\{x\mid x=cy,\ y\in\tilde{\Lambda}\},$$

which has determinant $c^n.d(\tilde{\Lambda})$, is K_G-admissible. Hence

$$\frac{1}{G(\tilde{\Lambda})^n}=c^n\geqslant\frac{\Delta_{K_G}}{d(\tilde{\Lambda})}=\Delta_{K_G}.d(\Lambda).$$

Substituting this result in (3), we obtain

$$d(\Lambda)^{n-1}\geqslant\left(\frac{\Delta_{R\cap K_F}}{\Delta_{S\cap K_G}}\right)^n.\Delta_{K_G}.$$

As this is true for any K_F-admissible lattice, the theorem immediately follows.

4. We finish with a few remarks.

i. The fundamental ideas of the theorem and of the proof are, of course, due to Mordell. Our general approach is somewhat inspired by Armitage [2]. However, the proof by Armitage is not complete, since no attention is given to the possibility that $G(x)=0$ for the point x, which is transformed into a point of S. To avoid this difficulty, Lekkerkerker makes a condition concerning the subspaces R' through o with the property that the star body $R' \cap K_F$ is not of the finite type: There should not be to many of these. As we have shown, such a condition is not necessary.

ii. The use of the integer k in the enunciation of the theorem is, to show, whether generalisations of the method are possible.

Clearly, it is much more difficult to satisfy the condition (b), if $1 < k < n-1$.

However, even if the condition (b) can be satisfied for such k, as is the case when $F(x) = G(x) = |x|$ for all $x \in X$, then the argument still breaks down. For, it is only possible to obtain an estimate for $d(\tilde{\lambda})$ in the way we have done, if $k = n-1$, because only for 1-dimensional lattices it is true that an arbitrary lattice is always proportional to a critical lattice.

Yet, taking $F(x) = G(x) = |x|$, we may get a result for arbitrary k by the use of a generalisation of the notion of the critical determinant: Let Δ_n^k be the infinum of the determinants of the n-dimensional lattices with the property, that no k-dimensional sublattices have determinant less than 1. Then one can prove

THEOREM 2.

$$(\Delta_n^1)^k \geqslant (\Delta_k^1)^n \cdot (\Delta_n^{n-k})^{n-k}.$$

However, since it does not seem easier to find an estimate for the generalised determinant Δ_n^{n-k} than for the ordinary Δ_n^1, it is hardly worth elaborating on it.

iii. In his paper Lekkerkerker states two theorems, from which all the results can be derived, which so far have been obtained by the application of Mordell's method. His first theorem is contained in our Theorem 1. It is not necessary to restate his second theorem, since no new aspects would come into view.

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