we have

$$E_q \leqslant 4 \sum_{h=1}^{q} \sum_{\substack{a=1 \\ a\equiv h \pmod q}}^{N} q^{-1} \sum_{j=0}^{q-1} \{D_{q,h}(1, a+j)\}^2 + 2Q_1 V_q(N)$$

$$\leqslant 4q^{-1} \sum_{m=1}^{N} V_q(m) + (2Q_1+4)\, V_q(N)$$

$$\ll q^{-1} \sum_{m=1}^{N} V_q(m) + Q V_q(N).$$

In view of (15) we see that this estimate in conjunction with (12) yields (3).

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Rational zeros of two quadratic forms

by

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1. Let f, g be homogeneous quadratic forms in 13 variables, defined over the rationals. Mordell [3] has shown that f and g have a common non-trivial rational zero, provided that they satisfy certain conditions of a non-number-theoretic nature. In this paper I prove the corresponding result for forms in 11 variables:

THEOREM. *Let f, g be homogeneous quadratic forms in* 11 *variables, defined over the rationals; and suppose that for all real λ, μ not both zero the form $\lambda f + \mu g$ is indefinite. Then f and g have a non-trivial common rational zero.*

We shall see in § 2 that the condition of the theorem is the natural one. Henceforth, in discussing functions homogeneous in a set of variables, we shall implicitly assume that the variables are not all zero; in fact it will be convenient to state part of the argument in the language of projective geometry.

The idea of Mordell's proof is as follows. We arrange that f is non-singular and has signature between -3 and 3 inclusive; then by a change of variables it can be written in the form

$$f = \sum_{i=1}^{5} x_i x_{i+5} + f_1(x_{11}, x_{12}, x_{13}). \tag{1}$$

By putting $x_i = 0$ for $6 \leqslant i \leqslant 13$ we ensure that $f = 0$ and we reduce g to a form $g_1(x_1, \ldots, x_5)$ in five variables. We can certainly find a rational zero of g_1 — and thereby a common rational zero of f and g — if g_1 is indefinite. But the possibility of making g_1 indefinite depends only on real and not on rational conditions; for if we have any real transformation of variables which takes f into the form (1) then we can find a rational transformation as close as we like to it which also takes f into the form (1).

If we apply the analogous argument to a pair of forms in 11 variables, we arrive at a form g_1 in only four variables. This may not have a zero even when it is indefinite; but one expects that an indefinite quadratic form in four variables will usually have a zero, and hence that the argument can be carried through if the reduction of f is chosen with sufficient care. This turns out to be so. With the natural modifications of the previous notation, consider the situation at the stage when we have split off three products from f, thereby reducing it to the form

$$f = x_1x_5 + x_2x_6 + x_3x_7 + f_2(x_4', x_8', x_9', x_{10}', x_{11}').$$

We must next split off a term x_4x_8 from f_2; and the way in which we do this determines the quadratic form $g_1(x_1, x_2, x_3, x_4)$, obtained from g by setting the other seven variables equal to zero. We must ensure that g_1 represents zero; and for this it is sufficient that it should do so in every p-adic field, including the reals. But the form $g_1(x_1, x_2, x_3, 0)$ already represents zero in almost all p-adic fields; and it does not depend on the way in which we split off x_4x_8 from f_2. The problem of doing this in a satisfactory way turns out therefore to be local rather than global; and this makes it much easier. Even so, it can only be done under further conditions. One is that the earlier stages of the reduction should be 'general' in a sense which is defined later. The other is that certain p-adic fields are excluded; these are the reals and those p-adic fields (with p finite) for which the kernel of f has dimension 3. There are only finitely many of these fields; and we can choose the first two stages of the reduction of f so that already $g_1(x_1, x_2, 0, 0)$ represents zero in these fields.

It is natural to ask whether the theorem proved in this paper is best possible; and if not, whether it can be improved without using a radically new idea. For the local case, Demyanov [2] has shown that two quadratic forms in 9 variables have a common zero in any p-adic field; this result is best possible. (A simpler proof has been given by Birch, Lewis and Murphy [1]). With a suitable reality condition, one expects this result to hold even over the rationals. Indeed we may cut down the number of variables even further by posing the problem in a local-to-global form. Let f, g be two quadratic forms in n variables, and suppose that $f = g = 0$ is soluble in every p-adic field — with if necessary a stronger condition over the reals. For what value of n do these conditions imply that $f = g = 0$ is soluble over the rationals? The most interesting case is when $n = 5$. The variety $f = g = 0$ is now the surface given by the intersection of two quadrics in four dimensions. It is birationally equivalent to a plane, though in general only with the help of a field extension. For a detailed account of its geometry, see Segre [5] or Rao [4].

There seems no hope of modifying the argument of the present paper so as to use a form g in only three variables. The only other way to save a variable, without an entirely new approach, is to waste only two variables instead of three in the analogue of the function f_1 in (1). We are therefore led to the following problem:

Let f, g be quadratic forms in 10 variables. Is there a form in the pencil defined by f, g whose kernel has dimension two or zero?

(The kernel of a quadratic form is the part left when we have split off as many products as possible; it is well defined, by Witt's Lemma.) Certainly this is not always true; but if the pencils for which it is false form a small enough exceptional class one could hope to deal with them by special arguments. In particular, suppose that no form in the pencil has rank less than 9, that is, that the ten roots of the equation $\det(f+\lambda g) = 0$ are all distict. Then the problem above becomes a purely local problem, for there is a finite set $\mathscr{S}$ of primes p with the following property: if $p \notin \mathscr{S}$ then no form of the pencil has kernel of dimension greater than two over the p-adic numbers. But I have been unable to find any plausible line of attack on the local problem.

2. Before embarking on the proof of the theorem, it is convenient to obtain some preliminary results. We first put the reality condition into the form in which we shall actually use it. It is the natural condition for the problem; for if there is a form of the pencil generated by f, g which is definite, then the manifold given by $f = g = 0$ can have no real points. Again, suppose that the form $\lambda f + \mu g$ is semi-definite for some real λ, μ not both zero. Then the real points on $\lambda f + \mu g = 0$ are just those of a real linear subspace, and the rational points are those of the maximal rational linear subspace contained in it. The problem of finding rational solutions of $f = g = 0$ therefore reduces, in this case, to that of finding zeros of a single quadratic form in fewer variables.

LEMMA 1. *Let f, g be real quadratic forms in n variables. Then*

(i) *the manifold $f = g = 0$ contains real points if and only if $\lambda f + \mu g$ is never definite for any real λ, μ;*

(ii) *if f is indefinite there exist points on $f = 0$ giving either sign to g if and only if $\lambda f + g$ is indefinite for all real λ.*

Since the degree of g is even, it makes sense to talk about the sign of g at a point of projective space. Now suppose that h is any quadratic form in n variables. The real manifold $h = 0$ is not empty if and only if h is not positive definite; and in this case it is connected. Moreover it separates the projective space into two disjoint parts, defined by $h > 0$ and $h < 0$, if and only if h is indefinite.

We can now prove the Lemma. The necessity in (i) is trivial; so we may assume that $f = g = 0$ contains no real points and $\lambda f + \mu g$ is never definite, and obtain a contradiction. For given λ, μ with $\mu \neq 0$, the manifold $\lambda f + \mu g = 0$ does not meet $f = 0$ in real points; it therefore lies entirely in $f>0$ or entirely in $f<0$. Suppose (λ,μ) runs through the points of the open semicircle $\mathscr{C}$: $\lambda^2+\mu^2=1$, $\mu>0$; and let M_1, M_2 be the sets of those (λ,μ) in $\mathscr{C}$ for which the real manifold $\lambda f+\mu g=0$ lies in $f>0$, $f<0$ respectively. Then M_1, M_2 are disjoint sets, closed in $\mathscr{C}$, whose union covers $\mathscr{C}$. Since this is impossible, we have a contradiction; and this proves (i).

The proof of (ii) is similar. Again the necessity is trivial; so we assume, say, that $g\geqslant 0$ whenever $f=0$, and that $\lambda f+g$ is indefinite for all real λ, and obtain a contradiction. For any fixed λ, the set of real points at which $\lambda f+g<0$ is open, connected and not empty; and it does not meet $f=0$. It therefore lies either entirely in $f>0$ or entirely in $f<0$; for if it met both these we would obtain a representation of it as the union of two disjoint open sets, which is impossible. Let Λ_1, Λ_2 be the set of values of λ for which the region $\lambda f+g<0$ lies in $f>0, f<0$, respectively. Since f is indefinite, Λ_1 and Λ_2 are neither of them empty; and they are disjoint open sets whose union is the real numbers. This is impossible, and the contradiction completes the proof of the Lemma.

It is convenient to recall here some elementary facts about quadratic forms and quadrics which will be needed below. Let f be a quadratic form in n variables, with rank r; and let Q be the quadric in $(n-1)$-dimensional projective space whose equation is $f=0$. If H is any hyperplane, then $Q\cap H$ can be regarded as a quadric in $(n-2)$-dimensional space. Let its rank be r_1; we shall need to know the relation between r_1 and r. Suppose first that Q is non-singular, so that $n=r$. Then $Q\cap H$ has rank $r-1$ in general, so that it too is non-singular; but it has rank $r-2$ if H is a tangent hyperplane to Q. If $n>r$ then the singular points of Q form a linear space L of dimension $n-r-1$. If H does not contain L, then $Q\cap H$ has rank r; and if $r<n-1$ the singularities of $Q\cap H$ are just the points of $H\cap L$. If H contains L then in general $Q\cap H$ has rank $r-1$; but it has rank $r-2$ if H touches Q at a non-singular point of Q.

Suppose now that f, g are two quadratic forms in n variables, and that the general form of the pencil generated by f, g is non-singular. As (λ,μ) moves on the circle $\lambda^2+\mu^2=1$, say, the form $\lambda f+\mu g$ varies in the pencil. Of the forms thus obtained, at most $2n$ are singular. The signature of $\lambda f+\mu g$ only changes as we pass through a singular form; and it changes by at most twice the nullity of the form. (The nullity of a quadratic form is the number of variables minus the rank.) Moreover the nullity of the form is at most the multiplicity of the corresponding value of λ/μ as a root of the equation $\det(\lambda f+\mu g)=0$. That we can have strict inequality here is shown by the example $f=x_1^2$, $g=x_1x_2$. All these results are classical; and they can easily be proved by diagonalization.

It is natural to deal at this point with the proof of the theorem in the case to which the remarks above do not apply.

LEMMA 2. *Let f, g be rational quadratic forms in n variables, such that every form of the pencil generated by f, g is singular. Then f, g have a common rational zero.*

Let $r<n$ be the maximum of the ranks of the forms in the pencil. That the form $\lambda f+\mu g$ should have rank less than r involves a set of non-trivial algebraic conditions on λ,μ; hence these fail almost always, and so for some rational λ,μ. By a rational change of basis for the pencil we may assume that f has rank r; and by a rational change of variables we can put it into the form

$$f=a_1y_1^2+\ldots+a_ry_r^2,$$

where the $a_i\neq 0$. Let us write also $g=\sum\sum b_{ij}y_iy_j$, and consider the determinant of the form obtained from $f+\varepsilon g$ by setting all but the first $r+1$ variables to zero. If we assume that ε is small then the determinant is

$$\varepsilon b_{r+1,r+1}\prod a_{r+1}+O(\varepsilon^2)=0$$

since $f+\varepsilon g$ has rank at most r. Hence $b_{r+1,r+1}=0$, and the point at which $y_{r+1}=1$ and all other $y_i=0$ is a common zero of f, g. This proves the Lemma.

For later use, we translate into geometric terms the process of splitting off a product x_1x_2 from a quadratic form f in n variables. We assume that the rank of f is at least 2 and that $f=0$ has non-singular rational solutions. Let P_1 be a non-singular point on $f=0$, and let P_2 be any non-singular point on $f=0$ not on the tangent at P_1. Such a point P_2 exists, for rational points are everywhere dense on the real part of $f=0$; moreover the relation between P_1 and P_2 is symmetric. If we choose a coordinate system such that the tangent planes at P_1, P_2 are $x_1=0$, $x_2=0$ respectively and all the other coordinates hyperplanes pass through P_1P_2, then we shall have

$$f=cx_1x_2+f_1(x_3,\ldots,x_n)$$

for some non-zero c.

Retaining the same notation, suppose also that f is non-singular. Let g be a fixed non-singular quadratic form not a multiple of f; and let h, which need not be rational or even real, be a singular form in the pencil generated by f and g. Let $\xi_1=0$ be the polar hyperplane of P_1 with respect to $g=0$. Since the rational points on $f=0$ are dense in the Zariski topology, we can choose P_1 so that it is not on $g=0$ and $\xi_1=0$ does not contain all the singular points of $h=0$; for since g is non-singular there is no point common to all possible hyperplanes $\xi_1=0$. Similarly we can choose P_2, for fixed P_1, so that $x_2=0$ does not touch the restriction of $g=0$ to $\xi_1=0$ nor coincide with $\xi_1=0$, and P_2 is not on $x_1=0$; and if the rank of h is less than $n-1$ then $x_2=0$ does not contain all those singular points of $h=0$ which lie on $\xi_1=0$. Moreover, it is obvious that all these restrictions are compatible with P_1, P_2 being as close as we like to assigned real or p-adic points on $f=0$, for finitely many p.

It may clarify things to restate this in purely algebraic terms. Then we can split off x_1x_2 from f and find a

$$\xi_1 = x_1+c_2x_2+\ldots+c_nx_n$$

such that

$$g = b_1\xi_1^2+g_1(x_2, \ldots, x_n),$$

where $b_1\neq 0$. Moreover the restriction of g to $\xi_1=x_2=0$, or of g_1 to $x_2=0$, is non-singular; and if ν, ν_1 are the nullities of h and of its restriction to $\xi_1=x_2=0$, then $\nu_1\leqslant\nu$ always and $\nu_1=\nu-2$ if $\nu>1$. Since there are effectively only finitely many possible h, we can ensure that this last condition holds for all of them.

3. In this section we prove two Lemmas which will be needed for the proof of the main theorem. The first deals with a special case which requires an *ad hoc* argument; it has been separated off in order not to confuse the main line of the proof.

LEMMA 3. *Under the conditions of the theorem, suppose that there exist λ, μ not both zero such that the form $\lambda f+\mu g$ has rank at most 6. Then f, g have a non-trivial common rational zero.*

Note that in the statement of this Lemma we do not assume that λ, μ are real.

Suppose first that the pencil contains a rational form of rank at most 6. By a rational change of basis of the pencil we may suppose that this is f; and by a rational change of variables we can write it in the form $f=f(x_1, \ldots, x_6)$. If the coefficient of x_{11}^2 in g were 0 we would have a common rational zero of f and g; so without loss of generality we can assume that it is positive. By Lemma 1 (ii) there are real points — and so rational points — on $f=0$ which make g negative; and by a rational change of variables on $x_1, \ldots, x_6$ only we can assume that one of them lies on $x_1=\ldots=x_5=0$. Thus, in particular, the coefficient of x_6^2 in f is zero. Now let g_1 be the form obtained from g by setting $x_1=\ldots=x_5=0$; g_1 is an indefinite quadratic form in 6 variables and therefore has a non-trivial rational zero. This extends in an obvious way to the common zero of f and g that we are seeking.

Thus we may assume that the forms of rank 6 or less in the pencil are irrational. After Lemma 2 we may also take f to be non-singular. Thus the roots of the equation $\det(\lambda f+g)=0$ are one simple rational one and two conjugate ones α and α', each with multiplicity 5; each of α and α'

generates a quadratic extension k of the rationals, and the corresponding forms $\alpha f+g$ and $\alpha' f+g$ have rank 6. Let L, L' be the linear spaces of their singular points; they are defined over k, conjugate over the rationals, and have dimension 4. Moreover, they do not intersect; for their intersection would be a singular point of every form of the pencil, and f is non-singular. Hence the least linear space containing both of them is a rational space of dimension 9, which by a change of variables we can take to be $x_1=0$. Now we can find a linear transformation defined over k which has $z_1\equiv x_1$ and takes $\alpha f+g$ into the form $h(z_1, \ldots, z_6)$ with coefficients in k. Suppose that this transformation implies

$$z_i\equiv y_i+\alpha y_{i+5}\quad (i=2, \ldots, 6),$$

where the y_i are rational linear forms in the x_i; and for convenience write $y_1\equiv x_1$. The y_i are linearly independent; for otherwise we could find a rational point at which they all vanished, and this would be a common point of L, L'. Hence it is enough to show that we can choose them rational and not all zero so that $h(z)=0$. But we can find a zero of $h(z)$ in k, by Hasse's theorem; for if k is a real field then $h(z)$ is indefinite for both injections of k into the real numbers, by hypothesis. By scaling we obtain a solution of $h(z)=0$ with $z_1=0$ or 1; and this gives us what we want. This completes the proof of the Lemma.

The next Lemma is purely local; it is needed to ensure that the final stage of the reduction is satisfactory. In its statement, 0 is to be regarded as a p-adic square; thus in applying it we can treat the squares as a closed set. The condition on the kernel of f' is vital; without it the Lemma would be false for every p.

LEMMA 4. *Let p be a finite prime and let f', g' be linearly independent quadratic forms in 5 variables defined over the p-adic numbers. Suppose that f' is non-singular and its kernel has dimension 1, and that the value of g' at each p-adic zero of f' is a square; then there is a form of rank 1 in the pencil generated by f', g'.*

Since we can add a multiple of f' to g' without altering hypothesis or conclusion, we may assume g' non-singular. Let L be any line on $f'=0$ defined over the p-adic numbers; then the restriction of g' to L is singular since any form of rank 2 represents some non-squares. In geometric terms, L touches or lies in $g'=0$. But we find L by choosing any p-adic point on $f'=0$ and joining it to any other p-adic point on the intersection of $f'=0$ with the tangent plane at the first point — and such L exist because the kernel of f' has dimension 1. Thus the set of lines L is, in the Zariski topology, dense in the set of all lines on $f'=0$; and so any such line touches $g'=0$. In particular, it follows that the tangent hyperplanes to $f'=0$ and $g'=0$ at any common point are the same. Since the coordinates of the tangent hyperplane are linear in those of the original point, we deduce either that $f' = 0$ and $g' = 0$ are the same or that their intersection lies entirely in a hyperplane. The first possibility has been explicitly ruled out; the second implies that there is a form of rank 1 in the pencil generated by f', g'. For let the hyperplane be $y_1 = 0$, where y_1 is a linear form in the original variables whose coefficients may be irrational. The restrictions of f', g' to $y_1 = 0$ are, after scaling, identical. Hence $f' - g' \equiv y_1 y_2$ for some y_2. But the intersection of $y_2 = 0$ with $f' = 0$ must lie on $y_1 = 0$; and this can only happen if y_2 is a multiple of y_1. This proves the Lemma.

4. We can now embark on the proof of the theorem. After Lemmas 2 and 3 we may assume that $\lambda f + \mu g$ is non-singular in general and has rank at least 7 if λ, μ are not both zero. We consider the signature of $\lambda f + \mu g$ as (λ, μ) moves round the circle $\lambda^2 + \mu^2 = 1$. This is an odd integer except perhaps at the finitely many points for which $\lambda f + \mu g$ is singular; these points are its only discontinuities and it has a jump of at most 8 at them. Since it is an odd function of (λ, μ), it takes both positive and negative values. Hence there is an interval on the unit circle on which it is absolutely bounded by 3; and this interval contains rational points (λ, μ). Thus by a rational change of basis of the pencil, we may henceforth assume that f is non-singular and has signature absolutely bounded by 3, and g is non-singular.

Now let $\mathscr{S}$ be the set of those finite primes p for which the kernel of f over the p-adic numbers has dimension 3. The set $\mathscr{S}$ is finite, for if we make f integral any odd prime in $\mathscr{S}$ must divide the determinant of f. Our next step will be to split off two products from f, taking it into the form

$$f = x_1 x_2 + x_3 x_4 + f'(x_5, \ldots, x_{11});$$

and we must do this in such a way that the binary quadratic form obtained from g by putting $x_i = 0$ for $i \neq 1, 3$ is indefinite and represents zero in each p-adic field for which $p \in \mathscr{S}$.

We choose real points $P_{1\infty}, P_{2\infty}$ of $f = 0$ such that $g(P_{1\infty}) > 0$ and $g(P_{2\infty}) < 0$, which is possible by Lemma 1 (ii); and by a small change in $P_{2\infty}$ we further ensure that neither point is on the tangent hyperplane at the other. The restrictions of f, g to any 8-dimensional linear subspace are distinct, since the forms of the pencil which these restrictions generate have rank at least 3; thus we can find a real point $P_{3\infty}$ which is on $f = 0$ and on the tangent hyperplanes to it at $P_{1\infty}, P_{2\infty}$ but not on $g = 0$. Without loss of generality we may assume that $g(P_{3\infty}) < 0$; for otherwise we interchange $P_{1\infty}, P_{2\infty}$ and change the sign of g.

Again, for each $p \in \mathscr{S}$ we choose a p-adic point P_{0p} on $f = g = 0$ at which the tangent hyperplanes to $f = 0$ and $g = 0$ are distinct. That there is a p-adic point on $f = g = 0$ we know from Demyanov's theorem; if the tangent hyperplanes were the same we could by a p-adic change of variables take the point to be $(1, 0, 0, \ldots)$ and the coincident hyperplanes to be $y_2 = 0$. Then we should have

$$f \equiv a y_1 y_2 + f_0(y_2, \ldots, y_{11}), \qquad g \equiv b y_1 y_2 + g_0(y_2, \ldots, y_{11})$$

for some non-zero a, b; and the intersection $f = g = 0$ would be given by

$$b f_0 - a g_0 = 0, \qquad y_1 = -a^{-1} y_2^{-1} f_0.$$

Hence p-adic points would be everywhere dense on this intersection in the Zariski topology; and since $f = 0$ and $g = 0$ do not touch at a general point of the intersection, we can choose P_{0p} to satisfy the conditions stated. Given P_{0p} we choose P_{1p} on $f = 0$ and on the tangent hyperplane at P_{0p} to $f = 0$, but not on that to $g = 0$; and we then choose P_{2p} on $f = 0$ and not on the tangent hyperplane to it at P_{1p}. The line $P_{0p}P_{1p}$ lies entirely in $f = 0$; let P_{3p} be the point where it meets the tangent hyperplane to $f = 0$ at P_{2p}, so that P_{3p} is distinct from P_{1p}. The line $P_{0p}P_{1p}$ meets $g = 0$ in two distinct p-adic points, and so the restriction of g to it, in any convenient coordinates, will have determinant minus a non-zero p-adic square. Hence any line near enough to it will have the same properties.

We now choose a rational point P_1 on $f = 0$ near to $P_{1\infty}$ and each P_{1p}, and similarly for P_2; and in addition we impose on P_1, P_2 the conditions stated at the end of § 2. With the same notation as there we have

$$f = x_1 x_2 + f_2(x_3, \ldots, x_{11}), \qquad g = b_1 \xi^2 + g_1(x_2, \ldots, x_{11}),$$

where $b_1 > 0$, $\xi_1 = x_1 + c_{12} x_2 + \ldots$; and if g_2 is the restriction of g_1 to $x_2 = 0$ then f_2, g_2 are non-singular and no form in the pencil generated by them has rank less than 7. Now let P_3 be a rational point on $f_2 = 0$ near to $P_{3\infty}$ and the P_{3p}; we can clearly make this as near as we need provided we make P_1, P_2 near enough to the corresponding points. P_3 is initially defined in the space of $x_3, \ldots, x_{11}$; but we may extend it to the original space by putting $x_1 = x_2 = 0$. We also define P_4 on $f_2 = 0$ but not on the tangent hyperplane at P_3 to it; and as before we impose the additional constraints given at the end of § 2. Then with the obvious notation we have

$$f_2 = x_3 x_4 + f_4(x_5, \ldots, x_{11}), \qquad g_2 = b_3 \xi_3^2 + g_3(x_4, \ldots, x_{11})$$

where $\xi_3 = x_3 + c_{34} x_4 \ldots$; and $b_1 c_{13}^2 + b_3 < 0$ whence $b_3 < 0$. Moreover, if g_4 is the restriction of g_3 to $x_4 = 0$ then f_4, g_4 are non-singular and no form in the pencil generated by them has rank less than 5. Finally, P_1P_3 is near $P_{1p}P_{3p}$ and so meets $g = 0$ in two points defined over the p-adic numbers; in other words $b_1 \xi_1^2 + b_3 \xi_3^2$ represents zero over the p-adic numbers for each $p \in \mathscr{S}$.

The next stage of the reduction is analogous except that we have no local conditions to trouble us. It enables us to write

$$f_4 = x_5x_6+f_6(x_7,\ldots,x_{11}),\qquad g_4 = b_5\xi_5^2+g_5(x_6,\ldots,x_{11})$$

with $\xi_5 = x_5+c_{56}x_6+\ldots$; and if g_6 is the restriction of g_5 to $x_6 = 0$ then f_6, g_6 are non-singular and no form of the pencil generated by them has rank less than 3. Moreover we may assume that $b_5 \neq 0$; for otherwise we can obtain a rational solution of $f = g = 0$ by putting $x_5 = 1$, $\xi_1 = \xi_3 = x_2 = x_4 = x_6 = \ldots = 0$.

Now let $\mathcal{S}'$ be the finite set of these primes p such that $b_1\xi_1^2+b_3\xi_3^2+b_5\xi_5^2$ does not represent zero over the p-adic numbers. We have arranged that $\mathcal{S}'$ contains no prime of $\mathcal{S}$, and we can therefore apply the result of Lemma 4 to the forms f_6 and $b_1b_3b_5g_6$ for each $p\in\mathcal{S}'$. For each of them we obtain a p-adic point P_{7p} on $f_6 = 0$ such that $b_1b_3b_5g_6(P_{7p})$ is not a p-adic square. Let P_7 be a rational point on $f_6 = 0$ so near to each P_{7p} that it has the same properties; by a further change of variables we may take it to be $(1,0,0,0,0)$. Let $b_7 = g_6(P_7) \neq 0$ and consider the linear subspace given by

$$x_2 = x_4 = x_6 = x_8 = x_9 = x_{10} = x_{11} = 0.$$

On this we have $f = 0$ identically; and since ξ_1, ξ_3, ξ_5, x_7 are an acceptable system of homogeneous coordinates we can write the restriction of g in the form

$$b_1\xi_1^2+b_3\xi_3^2+b_5\xi_5^2+b_7x_7^2.$$

But this is an indefinite quadratic form which represents zero in every p-adic field. (The only difficulty is with the $p\in\mathcal{S}'$, for which we appeal to the theorem that a quaternary quadratic form which does not represent zero in a p-adic field must have determinant a p-adic square.) Hence it represents zero over the rationals; and this representation extends in an obvious way to a rational solution of $f = g = 0$. This completes the proof of the theorem.

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Simultaneous representation by adjoint quadratic forms

by

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Dedicated to Professor L. J. Mordell

1. Introduction. Consider an n-ary quadratic form φ with real coefficients, and its adjoint form φ'. Denote their matrices by $A = (a_{ij})$ and $A' = (a'_{ij})$, so that a'_{ij} is the cofactor of the element a_{ij} in the determinant of A. Two real numbers m and m' are said to be *simultaneously represented by φ and φ'* if there exist integers x_i, z'_i $(i = 1,\ldots,n)$ such that

$$(1)\qquad m = \sum_{i,j=1}^{n} a_{ij}x_ix_j,\qquad m' = \sum_{i,j=1}^{n} a'_{ij}z'_iz'_j,\qquad 0 = \sum_{i=1}^{n} x_iz'_i.$$

The pair of column vectors $x = (x_i)$ and $z' = (z'_i)$ is called a *simultaneous representation*. The representation is termed *primitive* if each vector is primitive, that is the n components of each vector are relatively prime.

The notion of simultaneous representation was first introduced by G. Eisenstein [1], as part of an expression for his invariant system for a genus of ternary quadratic forms. The extension of Eisenstein's idea to n-ary quadratic forms, due to H. J. S. Smith [2] and H. Minkowski [3], involved the sequence of leading minor determinants in the matrix of A. It is interesting that the definition we have given above allows a quantitative development, which is the main purpose of this article. An algorithm will be given which produces all the simultaneous representations of given m and m' by φ and φ', each *set* of primitive representations (a set being an aggregate Wx, $W'z'$, W running over the unimodular automorphs of φ) being associated with a unique class of quadratic forms in $n-2$ variables and a certain set of solutions of certain quadratic congruences modulo m and m'. A formula similar to those of Smith, Minkowski, and Siegel [4] for the weighted number of simultaneous representations by a genus, exists for the weighted number of simultaneous representations by the system of classes of a genus and the adjoint genus.

As an example, the number of simultaneous, primitive solutions of

$$(2)\qquad m = x_1^2+x_2^2+x_3^2,\qquad m' = y_1^2+y_2^2+y_3^2,\qquad 0 = x_1y_1+x_2y_2+x_3y_3,$$

where m and m' are coprime positive integers, is $24gg'$, where g and g'