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A local criterion for the covering of space by convex bodies

by

J. H. H. CHALK (Toronto)

1. A point of Euclidean n-space R_n will be denoted by $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are real numbers and Λ_0 will denote the lattice of all such points with integral $x_1, x_2, \ldots, x_n$. If K is a closed convex body in R_n, symmetrical about the origin $\boldsymbol{O}$ and with non-zero content V, we denote its distance function by $F(\boldsymbol{x})$; so that K may be defined by the inequality $F(\boldsymbol{x}) \leqslant 1$, where $F(\boldsymbol{x})$ has the usual properties

(i) $F(\boldsymbol{O}) = 0$, $F(\boldsymbol{x}) > 0$ unless $\boldsymbol{x} = \boldsymbol{O}$,

(ii) $F(t\boldsymbol{x}) = |t| F(\boldsymbol{x})$, for all real t and any $\boldsymbol{x}$,

(iii) $F(\boldsymbol{x}_1 + \boldsymbol{x}_2) \leqslant F(\boldsymbol{x}_1) + F(\boldsymbol{x}_2)$.

Also, for any real t and any fixed point $\boldsymbol{a} \in R_n$, the set of all points of the form $t\boldsymbol{x} + \boldsymbol{a}$, where $\boldsymbol{x} \in K$, will be denoted by $tK + \boldsymbol{a}$. Then the inhomogeneous minimum μ of K, relative to Λ_0, may be defined as the least t for which the set of all convex bodies $tK + \boldsymbol{u}$, where $\boldsymbol{u} \in \Lambda_0$, completely cover R_n. Similarly, the successive minima $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ of K, relative to Λ_0, may be defined by taking λ_k to be the least t for which the body tK contains k linearly independent points of Λ_0. Under the general heading of transference theorems in the geometry of numbers (c.f. J. W. S. Cassels [1], Ch. XI, § 3), there are various inequalities relating μ to one or more of the numbers $\lambda_1, \ldots, \lambda_n$. Of those which involve just one of the successive minima, perhaps the simplest and best known is

$$\mu \leqslant \tfrac{1}{2} n \lambda_n. \tag{1}$$

Without modification, this is the best inequality of its type, for in the special case when K is the generalized octahedron K_0,

$$F(\boldsymbol{x}) \equiv |x_1| + \ldots + |x_n| \leqslant 1 \tag{2}$$

we have $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$ and

$$(3) \qquad \mu = \max_{\xi \in R_n} \min_{g \in \Lambda_0} F(\xi - g)$$

$$(4) \qquad \geqslant \min_{g \in \Lambda_0} F(h - g), \quad h = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right),$$

$$= \frac{1}{2} n,$$

$$(5) \qquad = \frac{1}{2} n \lambda_n.$$

However, more interest has attached to the problem of relating μ to the homogeneous minimum λ_1 and the best known bounds in this direction,

$$(6) \qquad \mu \leqslant \tfrac{1}{2}(q + \{q\}^{1/2})\lambda_1,$$

$$(7) \qquad \mu \leqslant \tfrac{1}{2} q \lambda_1 \ (\text{if } q \geqslant n), \qquad q = (\tfrac{1}{2}\lambda_1)^{-n} V^{-1},$$

follow from applications (by M. Kneser and B. J. Birch) of Macbeath's "Sum-theorem" in the quotient space R_n/Λ_0 (for the history of these and similar inequalities, see [1], Ch. 3, pp. 310-315). So far as q is concerned, the classical theorem of Minkowski on convex bodies gives $q \geqslant 1$ but in special cases q can be exponentially large, e.g. in the case of the hypersphere when $F(x) = (x_1^2 + \ldots + x_n^2)^{1/2}$ we note that $\lambda_1 = \lambda_2 = \ldots = \lambda_n$ and

$$q = 2^n \pi^{-n/2} \Gamma(1 + \tfrac{1}{2}n)^{-1} \sim \left(\frac{2n}{\pi e}\right)^{n/2}, \quad \text{as} \quad n \to \infty.$$

This example also serves to illustrate the fact that when q is much larger than n and the successive minima are comparable in size, the classical inequality (1) is more effective. For such cases, it is of interest to enquire whether (1) can be modified and improved. For this purpose, I introduce a modified set of successive minima [1]

$$(8) \qquad \lambda_1^* \leqslant \lambda_2^* \leqslant \ldots \leqslant \lambda_n^*$$

of K, relative to Λ_0, and define λ_k^* $(k = 1, 2, \ldots, n)$ to be the least t for which there is a $d = d(t)$ such that the "displaced" body $tK + d$ contains a k-dimensional set of $k+1$ points of Λ_0. It is evident that λ_k and λ_k^* are related by the inequalities

$$(9) \qquad \tfrac{1}{2}\lambda_k \leqslant \lambda_k^* \leqslant \lambda_k \quad (k = 1, 2, \ldots, n),$$

[1] More precisely, it should be noted that, since K is closed and bounded, μ, λ_k λ_k^* are attained bounds.

for $\lambda_k K$ contains k linearly independent points of Λ_0, which together with O give a set of the type required, while the difference body $D(\lambda_k^* K + d) = 2\lambda_k^* K$ contains k linearly independent points of Λ_0. Moreover, the inequality on the left of (9) is exact [2], for in the case when K is the "box" $\max(|x_1|, \ldots, |x_n|) \leqslant \frac{1}{2}$, we have $\lambda_k = 2$ immediately, while $\lambda_k^* = 1$ follows on considering $K + (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Thus there is the possibility of improving the classical inequality (1) by a factor, depending on K and lying between $\frac{1}{2}$ and 1, in replacing λ_n by λ_n^*. In fact, I conjecture that [3]

$$(10) \qquad \mu \leqslant \tfrac{1}{2} n \lambda_n^*,$$

for all n. This is known to be true for $n = 2$, (see [2], Lemma 2), but the proof for this special case does not seem to extend to higher dimensions without considerable wastage. In this note, I give a purely elementary argument to secure the slightly weaker result

$$(11) \qquad \mu \leqslant \tfrac{1}{2} p \lambda_n^*$$

with

$$(12) \qquad p = \begin{cases} n+1, & \text{if } n \text{ is odd}, \\ n\left(1 + \dfrac{1}{n+1}\right), & \text{if } n \text{ is even}. \end{cases}$$

In any event, by considering the octahedron K_0 we know that any such constant p satisfies $p \geqslant n$. In § 2, we give a proof of the intuitively obvious result $\lambda_n^* = 1$ for K_0 and in § 3 we calculate μ for the modified octahedron:

$$(13) \qquad K_1: \ |x_1| + \ldots + |x_n| + |x_1 + \ldots + x_n| \leqslant 1,$$

which plays a special role in the proof of (11), (12) following in § 4.

2. Lemma 1. *For any set $(a_1, a_1, \ldots, a_n) \not\equiv (0, 0, \ldots, 0) \bmod 1$, the inequality*

$$(14) \qquad |x_1 + a_1| + \ldots + |x_n + a_n| < 1$$

is satisfied by at most n distinct sets of integers $(x_1, x_2, \ldots, x_n)$.

Proof. Suppose, if possible, that there exist $n+1$ distinct integral sets $(x_1, x_2, \ldots, x_n)$ satisfying (14). By means of the operations (i) $a_i \to u_i + a_i$, $x_i \to -u_i + x_i$, (ii) $a_i \to -a_i$, $x_i \to -x_i$, where u_i is an integer, we may assume that $(0, 0, \ldots, 0)$ is one of these sets, that

$$0 \leqslant a_i < 1, \quad a_1 + a_2 + \ldots + a_n < 1,$$

[2] On the right of (9), equality occurs when $k = n$, $K = K_0$; but if $k < n$, the question is unsettled.

[3] I.e., if K can be translated to cover a proper simplex with vertices in Λ_0, then $\frac{1}{2}nK + u$, $u \in \Lambda_0$ constitute a covering of space.

and that there are n distinct sets of integers $(x_1, x_2, \ldots, x_n)$, other than $(0, 0, \ldots, 0)$, which satisfy (14). Hence $x_i = 0$ or -1, for all i. But if $x_i = x_j = -1$ for some $i \neq j$

$$(1-a_i)+(1-a_j) = 2-(a_i+a_j) \geqslant 2-(a_1+\ldots+a_n) > 1.$$

Hence the only possible values for the left of (14) are

$$a_1+a_2+\ldots+(1-a_k)+\ldots+a_n \quad (k = 1, 2, \ldots, n).$$

Each of these is <1, but their sum is $n+(n-2)(a_1+\ldots+a_n) \geqslant n$; a contradiction.

COROLLARY. *If $K = K_0$, then*

$$\lambda_n^* = 1. \tag{15}$$

Proof. By the right of (9), $\lambda_n^* \leqslant \lambda_n$ and $\lambda_n = 1$. But, for any $\lambda < 1$, there are, by Lemma 1, at most n points of Λ_0 in the set $\lambda K + \boldsymbol{d}$, unless $\boldsymbol{d} \equiv \boldsymbol{O} \pmod{\Lambda_0}$, when $\boldsymbol{d} = \boldsymbol{O}$ is the only point of Λ_0 in the set. Hence $\lambda_n^* \geqslant 1$, and so $\lambda_n^* = 1$.

3. LEMMA 2. *Let $a_1, a_2, \ldots, a_n$ be n real numbers. Then there exist integers $x_1, x_2, \ldots, x_n$ such that*

$$|x_1+a_1|+\ldots+|x_n+a_n|+|(x_1+a_1)+\ldots+(x_n+a_n)| \tag{16}$$

$$\leqslant \begin{cases} \tfrac{1}{2}(n+1), & \text{if } n \text{ is odd,} \\ \tfrac{1}{2}n\left(1+\dfrac{1}{n+1}\right), & \text{if } n \text{ is even} \end{cases}$$

with strict inequality, unless

$$\begin{aligned} &a_1 \equiv \ldots \equiv a_n \equiv \frac{1}{2} \pmod 1, && \text{when } n \text{ is odd,} \\ &a_1 \equiv \ldots \equiv a_n \equiv \pm\frac{1}{2}\cdot\frac{n}{n+1} \pmod 1, && \text{when } n \text{ is even.} \end{aligned} \tag{17}$$

Proof. We may suppose that

$$1 \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0, \quad a_1+a_2+\ldots+a_n \leqslant \tfrac{1}{2}n,$$

by applying one or more of the operations

(i) $a_i \to u_i + a_i$, where u_i is an integer,

(ii) $(a_1, \ldots, a_n) \to (1-a_1, \ldots, 1-a_n)$,

(iii) permutation of $a_1, \ldots, a_n$.

Put

$$a_1+a_2+\ldots+a_n = k+\Theta, \quad \text{where} \quad 0 \leqslant \Theta < 1$$

and k is an integer $\geqslant 0$. If $k = 0$, the values $x_1 = \ldots = x_n = 0$ give $2(a_1+\ldots+a_n) = 2\Theta < 2$; which satisfies (16) unless $n = 2$. In this special case, the 3 values

$$2(a_1+a_2) = 2\Theta, \quad a_1-a_2+2-\Theta, \quad -a_1+a_2+2-\Theta,$$

corresponding to $(x_1, x_2) = (0, 0), (0, -1), (-1, 0)$ have a sum $= 4$, and hence at least one of them is $\leqslant \frac{4}{3}$. If $k \geqslant 1$, we make the two selections

$$x_1 = \ldots = x_k = -1, \quad x_{k+1} = \ldots = x_n = 0$$

and

$$x_1 = \ldots = x_{k+1} = -1, \quad x_{k+2} = \ldots = x_n = 0$$

and denote the corresponding values by

$$\psi_k = (1-a_1)+\ldots+(1-a_k)+a_{k+1}+\ldots+a_n+\Theta,$$
$$\psi_{k+1} = (1-a_1)+\ldots+(1-a_{k+1})+a_{k+2}+\ldots+a_n+1-\Theta.$$

Note that

$$\begin{aligned} a_1+\ldots+a_k-a_{k+1}-\ldots-a_n &= 2(a_1+\ldots+a_k)-k-\Theta \\ &\geqslant 2\frac{k}{n}(a_1+\ldots+a_n)-k-\Theta \\ &= \frac{2}{n}k(k+\Theta)-k-\Theta = (k+\Theta)\left[\frac{2k}{n}-1\right], \end{aligned}$$

with strict inequality, unless $a_1 = \ldots = a_n$, and

$$\begin{aligned} a_1+\ldots+a_{k+1}-a_{k+2}-\ldots-a_n &= 2(a_1+\ldots+a_{k+1})-k-\Theta \\ &\geqslant 2\left(\frac{k+1}{n}\right)(a_1+\ldots+a_n)-k-\Theta = (k+\Theta)\left[\frac{2}{n}(k+1)-1\right], \end{aligned}$$

with strict inequality, unless $a_1 = \ldots = a_n$. Hence

$$\psi_k \leqslant k-(k+\Theta)\left[\frac{2k}{n}-1\right]+\Theta = \frac{2k(n-k)}{n}+\frac{2(n-k)}{n}\cdot\Theta$$

and

$$\begin{aligned} \psi_{k+1} &\leqslant (k+1)+(1-\Theta)-(k+\Theta)\left[\frac{2}{n}(k+1)-1\right] \\ &= \frac{2}{n}(n-k)(k+1)-\frac{2}{n}(k+1)\Theta. \end{aligned}$$

Thus,

$$\begin{aligned} (k+1)\psi_k+(n-k)\psi_{k+1} &\leqslant 2(n-k)(k+1) \\ &\leqslant \begin{cases} \tfrac{1}{2}(n+1), & \text{if } n \text{ is odd } \left(k \leqslant \tfrac{1}{2}(n-1)\right), \\ \dfrac{1}{2}n\left(\dfrac{n+2}{n+1}\right), & \text{if } n \text{ is even } (k \leqslant \tfrac{1}{2}n). \end{cases} \end{aligned}$$

4. THEOREM.

$$\mu \leqslant \tfrac{1}{2}\lambda_n^* \begin{cases} n+1, & \textit{if } n \textit{ is odd}, \\ n\left(1+\dfrac{1}{n+1}\right), & \textit{if } n \textit{ is even}. \end{cases} \tag{18}$$

Proof. Let $\lambda > 0$ be any number large enough to ensure the existence of a point $\boldsymbol{d} = \boldsymbol{d}(\lambda)$ such that the body λK contains an n-dimensional set of $n+1$ points $\boldsymbol{a}_0-\boldsymbol{d}, \boldsymbol{a}_1-\boldsymbol{d}, \ldots, \boldsymbol{a}_n-\boldsymbol{d}$, where $\boldsymbol{a}_i \in \Lambda_0$ $(i = 0, 1, \ldots, n)$. Then it is sufficient to prove that, for any given point $\xi \in R_n$, there is a point $\boldsymbol{x} \in \Lambda_0$ of the form

$$\boldsymbol{x} = \sum_1^n u_i(\boldsymbol{a}_i-\boldsymbol{a}_0), \qquad u_i \equiv 0 \pmod 1 \tag{19}$$

satisfying

$$F(\xi-\boldsymbol{x}) \leqslant \tfrac{1}{2}p\lambda, \tag{20}$$

where $p = n+1$, if n is odd and $= n\left(1+\dfrac{1}{1+n}\right)$, if n is even. On putting $\boldsymbol{b}_i = \boldsymbol{a}_i-\boldsymbol{d}$ $(i = 0, 1, 2, \ldots, n)$, we have

$$F(\boldsymbol{b}_i) \leqslant \lambda \tag{21}$$

and, from (19), $\boldsymbol{x} \in \Lambda_0$ may be taken in the form

$$\boldsymbol{x} = \sum_1^n u_i(\boldsymbol{b}_i - \boldsymbol{b}_0), \qquad u_i \equiv 0 \pmod 1.$$

Since $\boldsymbol{b}_i-\boldsymbol{b}_0$ $(i = 1, 2, \ldots, n)$ are linearly independent, by hypothesis, we may express $\xi \in R_n$ as

$$\xi = \sum_1^n \xi_i(\boldsymbol{b}_i-\boldsymbol{b}_0).$$

Hence, by properties (i), (ii), (iii) of the convex distance function $F(\boldsymbol{x})$, we have

$$\begin{aligned} F(\xi-\boldsymbol{x}) &= F\Big\{\sum_1^n (\xi_i-u_i)\boldsymbol{b}_i-\Big(\sum_1^n (\xi_i-u_i)\Big)\boldsymbol{b}_0\Big\} \\ &\leqslant \sum_1^n |\xi_i-u_i| F(\boldsymbol{b}_i)+\Big|\sum_1^n (\xi_i-u_i)\Big| F(\boldsymbol{b}_0) \\ &\leqslant \Big\{\sum_1^n |\xi_i-u_i|+\Big|\sum_1^n (\xi_i-u_i)\Big|\Big\}\lambda, \end{aligned}$$

and the result (20) now follows from Lemma 2.

References

[1] J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Springer 1959, Ch XI, pp. 310-315.

[2] J. H. H. Chalk and C. A. Rogers, *The critical determinant of a convex cylinder*, Journal London Math. Society 23 (1948), pp. 178-187.

UNIVERSITY OF TORONTO, TORONTO, CANADA

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