

[6] G. J. Rieger, *Elementare Lösung des Waringschen Problems für algebraische Zahlkörper mit der verallgemeinerten Linnickschen Methode*, Math. Annalen 148 (1962), pp. 83-88.

[7] C. L. Siegel, *Generalisation of Waring's problem to algebraic number fields*, Amer. J. Math. 66 (1944), pp. 122-136.

[8] — *Sums of  $m$ -th powers of algebraic integers*, Ann. Math. 46 (1945), pp. 313-339.

[9] R. M. Stemmmler, *The easier Waring problem in algebraic number fields*, Acta Arithmetica 6 (1961), pp. 447-468.

[10] A. Weil, *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc. 55 (1949), pp. 497-508.

[11] C. P. Ramanujam, *Sums of  $m$ -th powers in  $p$ -adic rings*, Mathematika 10 (1963), pp. 137-146.

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## Tauberian theorems for sum sets

by

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**Introduction.** The sums formed from the set of non-negative powers of 2 are just the non-negative integers. It is easy to obtain "abelian" results to the effect that if a set is distributed like the powers of 2, then the sum set will be distributed like the non-negative integers. We will be concerned here with converse, or "Tauberian" results. The main theme of this paper is the following question: if the set of sums formed from a given set of positive real numbers resembles an arithmetic progression, how much must the original set resemble a set of constant multiples of powers of 2?

If we denote the given set by  $k_0, k_1, k_2, \dots$ , arranged in ascending order, and let  $S(x)$  count the number of those sums of distinct  $k_j$  that do not exceed  $x$ , our problem is, roughly, that of showing that  $k_n$  is close to  $2^n$  if  $S(x)$  is close to  $x$ . Our first result gives sharp bounds for  $\liminf$  and  $\limsup$  of  $2^n/k_n$  in terms of  $\liminf$  and  $\limsup$  of  $S(x)/x$ . In the next section, we show that if  $S(x) - x$  is bounded, then  $k_n - 2^n$  is bounded, and furthermore,  $\sum |k_n - 2^n| < \infty$ , so that if the  $k_n$  are integers, then  $k_n = 2^n$  for all large  $n$ . We extend the method in the succeeding section to obtain estimates for  $k_n - 2^n$  and  $\sum_{n \leq N} |k_n - 2^n|$  in terms of suitable bounds for  $S(x) - x$ , even if  $S(x) - x$  is unbounded. Finally, on a slightly different note, we show that it is not possible for  $S(x)$  to behave too much like  $x^\alpha$  if  $\alpha < 1$ .

**1. Asymptotic behavior.** Let  $K = k_0, k_1, k_2, \dots$ ,  $0 < k_0 \leq k_1 \leq k_2 \leq \dots$ , be any sequence of positive real numbers. Let  $S(x)$  denote the number of choices of  $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$  such that for each  $j = 0, 1, 2, \dots$ , either  $\varepsilon_j = 0$  or  $\varepsilon_j = 1$ , and such that  $\varepsilon_0 k_0 + \varepsilon_1 k_1 + \dots \leq x$ . Let

$$A = \liminf_{x \rightarrow \infty} S(x)/x, \quad \alpha = \liminf_{n \rightarrow \infty} 2^n/k_n,$$

$$B = \limsup_{x \rightarrow \infty} S(x)/x, \quad \beta = \limsup_{n \rightarrow \infty} 2^n/k_n.$$

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A simple estimate shows that

$$(1) \quad \alpha \leq A \quad \text{and} \quad \beta \geq B.$$

We now give sharp inequalities in the opposite direction.

THEOREM 1.  $a = A$  and

$$(2) \quad B \geq \beta \left( 2 \frac{a}{\beta} - \frac{a^2}{\beta^2} \right).$$

Inequality (2) is best possible in the sense that, given any  $a$  and  $\beta$  with  $\frac{1}{2} < a/\beta \leq 1$ , there exists a sequence  $K$  for which equality holds, and given any  $A$  and  $B$  with  $\frac{2}{3} < A/B \leq 1$ , there exists a sequence  $K$  for which equality holds.

Remarks. It follows immediately from the theorem that

$$\text{if } \lim_{x \rightarrow \infty} S(x)/x = \theta \neq 0, \text{ then } \lim_{n \rightarrow \infty} 2^n/k_n = \theta.$$

This result was proved by a different method in [1]. The question was raised in [1] whether the statement remains true for  $\theta = 0$ . The answer is no, as the following example shows. Let  $k_{2^n} = 2^{2^n}$ , and let  $k_{2^n-r} = k_{2^n}$  for  $0 \leq r < 2^{n-1}$ . It is easy to see that  $S(x) = o(x)$ . On the other hand,  $2^n/k_n = 1$  for infinitely many  $n$ . It is easy to modify the example so that the  $k_n$  are distinct, but it seems difficult to satisfy the additional condition, described in [1], that the sums of the  $k_n$  are all distinct.

It seems likely that our methods, if carried out in greater detail, would yield an estimate similar to (2), but taking account of the integral part of  $\log_2 \beta/a$ , and that such an estimate would be best possible for any range of  $a/\beta$ , and not merely for  $a/\beta > \frac{1}{2}$ . Finally, if we permit  $\varepsilon_j = 0, 1, \dots, N-1$ , then it seems likely that our methods will yield analogous results for the limsup and liminf of  $N^n/k_n$ .

Proof of the estimates. First,  $S(k_n-1) \leq 2^n$ , since if  $\varepsilon_0 k_0 + \varepsilon_1 k_1 + \dots + \varepsilon_n k_n + \dots \leq k_n - 1$ , then  $\varepsilon_n = \varepsilon_{n+1} = \dots = 0$ , so that there are at most  $2^n$  suitable choices of  $\{\varepsilon_j\}$ . Hence

$$\frac{S(k_n-1)}{k_n-1} \leq \frac{2^n}{k_n-1} = \frac{2^n}{k_n} \cdot \frac{k_n}{k_n-1},$$

and on letting  $n \rightarrow \infty$ , we get  $A \leq a$ .

To obtain the estimate (2), we may suppose that  $a > 0$ , since if  $a = 0$  then (2) is trivially true. We now choose any  $a > 1/a$ , so that  $k_n \leq 2^n a$  for all sufficiently large  $n$ . Without loss of generality, we shall suppose that  $k_n \leq 2^n a$  for all  $n = 0, 1, 2, \dots$ , because for any two sequences  $K$  and  $K'$  with  $k_n = k'_n$  for  $n \geq n_0$ , it is easy to show that  $A = A'$ ,  $B =$

$= B'$ ,  $a = a'$ ,  $\beta = \beta'$ . And given any  $b$  with  $b > 1/\beta$  we will have  $k_n \leq 2^n b$  for infinitely many  $n$ .

We now choose  $n$  large, with  $k_n \leq 2^n b$ , and estimate  $S(2^n a)$ . Clearly,  $S(2^n a) \geq N_1 + N_2$ , where  $N_1$  is the number of choices of  $\{\varepsilon_j\}$ ,  $j = 0, 1, \dots, n-1$ , such that

$$(3) \quad \varepsilon_0 k_0 + \dots + \varepsilon_{n-1} k_{n-1} \leq 2^n a$$

and  $N_2$  is the number of choices of  $\{\varepsilon_j\}$ ,  $j = 0, 1, \dots, n-1$ , such that

$$(4) \quad \varepsilon_0 k_0 + \dots + \varepsilon_{n-1} k_{n-1} + k_n \leq 2^n a.$$

But if  $\varepsilon_0 2^0 a + \dots + \varepsilon_{n-1} 2^{n-1} a \leq 2^n a$ , then (3) holds, and therefore  $N_1 \geq 2^n$ . And if  $\varepsilon_0 2^0 a + \dots + \varepsilon_{n-1} 2^{n-1} a \leq 2^n(a-b)$ , then (4) holds, so that

$$N_2 \geq \left[ 2^n \left( 1 - \frac{b}{a} \right) \right] \geq 2^n \left( 1 - \frac{b}{a} \right) - 1.$$

Hence

$$S(2^n a) \geq 2^n + 2^n(1-b/a) - 1, \quad \frac{S(2^n a)}{2^n a} \geq \frac{1}{a} \left( 2 - \frac{b}{a} \right) - \frac{1}{2^n a}.$$

On letting  $n \rightarrow \infty$  through a suitable sequence, we get

$$B \geq \frac{1}{a} \left( 2 - \frac{b}{a} \right) = \frac{1}{b} \cdot \frac{b}{a} \left( 2 - \frac{b}{a} \right).$$

We may now let  $a \rightarrow 1/a$  and  $b \rightarrow 1/\beta$  to obtain (2).

The estimate is best possible. To show that (2) is best possible, we prove the first part, that given any  $a$  and  $\beta$  with  $\frac{1}{2} < a/\beta < 1$ , there exists a  $K$  such that  $B = \beta(2a/\beta - a^2/\beta^2)$ . The second part then follows since  $\varphi(\beta) = \beta(2A/\beta - A^2/\beta^2)$  is a continuous function of  $\beta$ , with  $\varphi(A) = A$  and  $\varphi(2A) = 3A/2$ , so that if we are given  $A$  and  $B$  with  $1 \leq B/A < 3/2$ , we may apply the first part with  $a = A$  and  $\beta$  such that  $\varphi(\beta) = B$ . For the construction of  $K$ , let  $n_m$  be a sequence of positive integers that increases very rapidly to  $\infty$ . Let  $a = 1/a$  and  $b = 1/\beta$  and define  $k_n$  by  $k_n = 2^n a$  unless  $n = n_m$  for some  $m$ , and  $k_n = 2^n b$  if  $n = n_m$  for some  $m$ . The point of the restriction  $a/\beta > \frac{1}{2}$  now appears; for the sequence  $K$  to be suitably defined, we need  $2^n b \geq 2^{n-1} a$ , or  $b/a \geq \frac{1}{2}$ .

A simple argument now shows that  $B \geq \limsup_{n \rightarrow \infty} B_n$ , where  $B_n$  is defined as follows. Let  $K^n$  be the sequence  $\{k_j\}$ ,  $j = 0, 1, 2, \dots$ , where  $k_j = 2^j a$  for  $j \neq n$  and  $k_j = 2^j b$  if  $j = n$ . Let  $S_n(x) = S(x; K^n)$  and let  $B_n = \sup(S_n(x)/x)$ , where the supremum is over all values of  $x \geq x_0(n)$ , where  $x_0(n)$  is a function of  $n$  that tends very slowly to  $+\infty$  as  $n$  tends to  $+\infty$ .

To determine  $S_n(x)$ , we must count those  $\{\varepsilon_j\}$  for which

$$(5) \quad \varepsilon_0 2^0 a + \dots + \varepsilon_{n-1} 2^{n-1} a + \varepsilon_n 2^n b + \varepsilon_{n+1} 2^{n+1} a + \dots \leq x.$$

We now define  $f_n(t)$  as the number of choices of  $\{\varepsilon_j\}$  for which

$$(6) \quad \varepsilon_0 2^0 + \varepsilon_1 2^1 + \dots + \varepsilon_{n-1} 2^{n-1} + \varepsilon_{n+1} 2^{n+1} + \dots \leq t.$$

Now, considering in (5) the two cases  $\varepsilon_n = 0$  and  $\varepsilon_n = 1$ , we see that

$$(7) \quad S_n(x) = f_n\left(\frac{x}{a}\right) + f_n\left(\frac{x - 2^n b}{a}\right).$$

If we write  $y = x/a$ , then from (7) we get

$$(8) \quad \frac{S_n(ay)}{ay} = \frac{1}{a} \left\{ \frac{f_n(y)}{y} + \frac{f_n(y - 2^n b/a)}{y} \right\},$$

so that

$$(9) \quad B_n = \frac{1}{a} \sup \left\{ \frac{f_n(y) + f_n(y - 2^n b/a)}{y} \right\},$$

where the supremum is over the range  $y \geq y_0$ , where  $y_0 = y_0(n) = x_0(n)/a$ . A computation shows that, writing  $[t]$  for the integral part of  $t$ ,

$$(10) \quad f_n(t) = 2^n \left[ \frac{t}{2^{n+1}} \right] + \min \left( 2^n, 1 + [t] - 2^{n+1} \left[ \frac{t}{2^{n+1}} \right] \right) \quad \text{for } t \geq 0,$$

and, of course,  $f_n(t) = 0$  for  $t \leq 0$ . For we may write  $t = k2^{n+1} + s$ , where  $k$  is a non-negative integer, and  $0 \leq s < 2^{n+1}$ . And  $\varepsilon_0 2^0 + \varepsilon_1 2^1 + \dots + \varepsilon_{n-1} 2^{n-1}$  may be any non-negative integer  $p < 2^n$ , while  $\varepsilon_{n+1} 2^{n+1} + \dots$  may be any number  $2^{n+1}q$ , where  $q$  is any non-negative integer. Thus, we may rewrite (6) as

$$(11) \quad p + 2^{n+1}q \leq k2^{n+1} + s$$

and  $f_n(t) = f_n(k2^{n+1} + s)$  is the number of choices of  $p$  and  $q$  that make (11) valid. Now for  $q = 0, 1, 2, \dots, k-1$  there are exactly  $2^n$  choices of  $p$  that make (11) hold. So far we have accounted for  $k \cdot 2^n$  choices. For  $q = k+1, k+2, \dots$ , there are no acceptable values of  $p$ . For  $q = k$ , if  $s \geq 2^n - 1$  then there are  $2^n$  choices of  $p$ , while if  $s < 2^n - 1$ , then there are  $[s+1]$  choices of  $p$ . Thus, we have

$$(12) \quad f_n(k \cdot 2^{n+1} + s) = 2^n k + \min(2^n, [s+1]),$$

which is equivalent to (10).

Now, writing  $y = 2^{n+1}k + s$ , with  $k$  a non-negative integer and  $0 \leq s < 2^{n+1}$  as before, we get

$$(13) \quad f_n(y) = 2^n k + \min(2^n, [s+1]),$$

$$(14) \quad f_n(y - 2^n b/a) = \max\{0, 2^n k + 2^n [*] + \min(2^n, [s - 2^n b/a] + 1 - 2^{n+1}[*])\},$$

where

$$(15) \quad [*] = \frac{s - 2^n b/a}{2^{n+1}},$$

and we remark that  $[*] = 0$  or  $-1$  according as  $s \geq 2^n b/a$  or  $s < 2^n b/a$ , respectively.

We now let

$$(16) \quad g(y) = g_n(y) = 2^n k + \min(2^n, s),$$

$$(17) \quad h(y) = h_n(y) = \max\{0, 2^n k + 2^n [*] + \min(2^n, s - 2^n b/a - 2^{n+1}[*])\},$$

and let

$$(18) \quad B'_n = \frac{1}{a} \sup_{y \geq y_0(n)} \psi(y)$$

where

$$(19) \quad \psi(y) = \psi_n(y) = \frac{g_n(y) + h_n(y)}{y}.$$

Since  $|B'_n - B_n| \leq \frac{2}{y_0(n)}$ , we see that  $B = \limsup_{n \rightarrow \infty} B'_n$ . We now compute  $B'_n$ .

Case 1.  $s < 2^n b/a$ . Here  $[*] = -1$ ,  $g(y) = 2^n k + s$ , and  $h(y) = \max\{0, 2^n k - 2^n + \min(2^n, s - 2^n b/a + 2^{n+1})\}$ , but  $2^{n+1} + s - 2^n b/a \geq 2^{n+1} - 2^n b/a = 2^n(2 - b/a) \geq 2^n$  since  $b/a = \alpha/\beta \leq 1$ , so that  $h(y) = 2^n k$ , and  $\sup \psi(y) = (2^{n+1}k + s)/(2^{n+1}k + s) = 1$ . There are three more cases, in all of which  $[*] = 0$  since  $s \geq 2^n b/a$ .

Case 2.  $2^n b/a \leq s \leq 2^n$ . Here  $g(y) = 2^n k + s$  and  $h(y) = 2^n k + \min(2^n, s - 2^n b/a) = 2^n k + s - 2^n b/a$  since  $s - 2^n b/a \leq 2^n$ . Hence

$$\psi(y) = \frac{2^{n+1}k + 2s - 2^n b/a}{2^{n+1}k + s},$$

and an elementary computation shows that

$$\sup \psi(y) = \frac{g(2^n) + h(2^n)}{2^n} = 2 - \frac{b}{a}.$$

Case 3.  $2^n \leq s \leq 2^n + 2^n b/a$ . Here,  $g(y) = 2^n k + 2^n$  and  $h(y) = 2^n k + \min(2^n, s - 2^n b/a) = 2^n k + s - 2^n b/a$ , and hence

$$\psi(y) = \frac{2^{n+1}k + 2^n + s - 2^n b/a}{2^{n+1}k + s} = 1 + \frac{2^n - 2^n b/a}{2^{n+1}k + s},$$

and an elementary computation shows that

$$\sup_3 \psi(y) = \frac{g(2^n) + h(2^n)}{2^n} = 2 - \frac{b}{a}.$$

Case 4.  $2^n + 2^n b/a < s < 2^{n+1}$ . Here  $g(y) = 2^n k + 2^n$  and  $h(y) = 2^n k + \min(2^n, s - 2^n b/a) = 2^n k + 2^n$  so that

$$\psi(y) = \frac{2^{n+1}k + 2^{n+1}}{2^{n+1}k + s},$$

and an elementary computation shows that

$$\sup_4 \psi(y) = \frac{g(2^n + 2^n b/a) + h(2^n + 2^n b/a)}{2^n + 2^n b/a} = \frac{2}{1 + b/a}.$$

So we must compare the three numbers  $2 - b/a$ ,  $1$ ,  $2(1 + b/a)^{-1}$ . Now each of them is  $\geq 1$ , and  $2 - b/a \geq 2(1 + b/a)^{-1}$ , as an elementary estimate shows. Hence  $B'_n = (2 - b/a)/a = \beta(2a/\beta - a^2/\beta^2)$ , and the result follows, on letting  $n \rightarrow \infty$ .

## 2. Bounded error terms.

**THEOREM 2.** *If there are constants  $c_1$  and  $c_2$  so that for all  $x > 0$  we have*

$$(20) \quad x - c_1 \leq S(x) \leq x + c_2,$$

*then*

$$(21) \quad k_n \leq 2^n + c_1 \quad \text{for all } n$$

*and*

$$(22) \quad k_n \geq 2^n - (c_1 + c_2) \quad \text{if} \quad 2^{n-1} > nc_1 + c_2.$$

*Finally, we have*

$$(23) \quad \sum_n |k_n - 2^n| < \infty$$

*so that if the  $k_n$  are integers, then  $k_n = 2^n$  for all sufficiently large  $n$ .*

**Proof.** As before, if  $x < k_n$ , then  $S(x) \leq 2^n$ . Thus  $k_n - c_1 \leq 2^n$  and (21) is established. Now let

$$K_n = k_0 + k_1 + \dots + k_{n-1}.$$

Then

$$(24) \quad K_n + c_2 \geq S(K_n) \geq 2^n.$$

We next prove that

$$2k_n > K_n$$

for all  $n$  satisfying  $2^{n-1} - nc_1 - c_2 > 0$ . For suppose that  $2k_n \leq K_n$ . Then for each choice of  $\varepsilon_0, \dots, \varepsilon_{n-1}$ , at least one of the sums

$$k_n + \sum_{j=0}^{n-1} \varepsilon_j k_j \quad \text{or} \quad k_n + \sum_{j=0}^{n-1} (1 - \varepsilon_j) k_j$$

is less than  $K_n$ , so that in this case we would have  $S(K_n) \geq 2^n + 2^{n-1}$ . According to (21) and (23), we would have

$$(25) \quad 2^n + nc_1 + c_2 \geq K_n + c_2 \geq S(K_n) \geq 2^n + 2^{n-1},$$

and the assertion is proved. Under the hypothesis of (22), we have  $2k_n > K_n$ . Now for each  $y$  with

$$0 \leq y < 2k_n - K_n$$

we have

$$(26) \quad K_n + y + c_2 \geq S(K_n + y) \geq 2^n + S(K_n + y - k_n),$$

where the second term on the right counts the number of  $\varepsilon_0, \dots, \varepsilon_{n-1}$  for which

$$k_n + \sum_{j=0}^{n-1} \varepsilon_j k_j \leq K_n + y.$$

Hence

$$(27) \quad K_n + y + c_2 \geq 2^n + K_n + y - k_n - c_1,$$

and (22) is established.

Now we choose  $p$  so that  $2^{p-1} > c_1 + c_2$ . Then

$$(28) \quad K_n \leq 2^n + K_p \quad \text{for all large } n.$$

For, assume that  $n$  is so large that (22) holds, that  $n > p$ , and that  $k_{n+1} > 2^n$ . Then if (28) fails, there would exist at least  $2^p$  choices of  $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}$  for which

$$\sum_{j=0}^{n-1} \varepsilon_j k_j > 2^n,$$

namely all choices with  $\varepsilon_{p+1} = \dots = \varepsilon_{n-1} = 1$ . Since there are at most  $S(c_1 + c_2)$  sums not exceeding  $2^n$  in which one of the summands is  $k_n$ , and no such sums in which one of the summands is  $k_{n+1}$  or larger, we obtain

$$(29) \quad 2^n - c_1 \leq S(2^n) \leq 2^n - 2^p + S(c_1 + c_2) \leq 2^n - 2^p + c_1 + c_2 + c_2,$$

which is contrary to the hypothesis that  $2^p > 2c_1 + 2c_2$ .

If  $\sum |k_n - 2^n| = \infty$ , then according to (24),

$$(30) \quad \sum_{k_n > 2^n} (k_n - 2^n) = \infty.$$

Thus, we could choose  $n_1 < n_2 < \dots < n_r$ , with  $n_1$  so large that (22) and (28) hold for  $n \geq n_1$ , with

$$K_{n_1} > K_{n_1} + c_1 + c_2 \quad \text{and} \quad k_{n_j} > 2^{n_j}$$

and such that

$$(31) \quad A = \sum_{j=1}^r k_{n_j} > \sum_{j=1}^r 2^{n_j} + c_1 + 1 = B + 1 + c_1.$$

We now show that

$$\sum 2^{n_j} > B \quad \text{implies} \quad \sum k_{n_j} > A.$$

This is obvious if  $\{n_j\}$  is a subset of  $\{m_j\}$ . If not, let  $n_s$  be the largest  $n$  not contained in  $\{m_j\}$ . It follows that

$$(32) \quad \sum k_{m_j} \geq A + k_{n_s+1} - \sum_{j=1}^s k_{n_j} \geq A + k_{n_s+1} - K_{n_s+1} + K_{n_1} \\ \geq A + 2^{n_s+1} - c_1 - c_2 - 2^{n_s+1} - K_p + K_{n_1} > A.$$

Hence  $S(A)$  is no greater than the number of sums of powers of 2 that do not exceed  $B$ , and this number is at most  $B+1$ . Hence  $S(A) \leq B+1 < A - c_1$ , contrary to hypothesis.

**COROLLARY.** If  $-c_1 \leq S(x) - \lambda x \leq c_2$  for some positive constant  $\lambda$  and all  $x \geq 0$ , then

$$(33) \quad k_n \leq \lambda^{-1} 2^n + c_1 \quad \text{for all } n,$$

$$(34) \quad k_n \geq \lambda^{-1} 2^n - (c_1 + c_2) \quad \text{if} \quad \lambda^{-1} 2^{n-1} > nc_1 - c_2,$$

and

$$(35) \quad \sum |k_n - \lambda^{-1} 2^n| < \infty.$$

This result follows by applying Theorem 2 to the sequence  $\{\lambda k_n\}$ .

**COROLLARY.** If the  $k_n$  are integers, then the only constants  $\lambda$  that can occur above have the form  $\lambda = 2^N/M$ , where  $N \geq 0$  and  $M > 0$  are integers, and then  $k_n = \lambda^{-1} 2^n$  for all sufficiently large  $n$ .

The proof is a simple application of (35), and we omit it.

**3. Unbounded error terms.** The methods of the preceding section can be extended to the case where  $S(x) - x$  is unbounded.

**THEOREM 3.** Suppose that

$$x - f_1(x) \leq S(x) \leq x + f_2(x) \quad \text{for all } x \geq 0,$$

where the  $f_i$  are continuous, positive, non-decreasing functions, not both

bounded, such that  $f_i(x)/x \rightarrow 0$  as  $x \rightarrow \infty$ , and such that  $x - f_1(x)$  and  $x + f_2(x)$  are strictly increasing. Let  $\varphi_1$  and  $\varphi_2$  be the inverse functions defined by

$$x = y - f_1(y) \Leftrightarrow y = x + \varphi_1(x),$$

$$x = y + f_2(y) \Leftrightarrow y = x - \varphi_2(x),$$

so that the  $\varphi_i$  are non-decreasing,  $\varphi_i(x)/x \rightarrow 0$  as  $x \rightarrow \infty$ , and  $x + \varphi_1(x)$ ,  $x - \varphi_2(x)$  are strictly increasing for sufficiently large  $x$ . Then

$$(37) \quad k_n \leq 2^n + \varphi_1(2^n) \quad \text{for all } n,$$

$$(38) \quad k_n \geq 2^n - \varphi_2(2^n) \quad \text{for all large } n.$$

Let  $\varphi_3$  be the inverse function defined by

$$x = y + f_2(y) \Leftrightarrow y = x - \varphi_3(x).$$

Then

$$(39) \quad K_n \geq 2^n - \varphi_3(2^n) \quad \text{for all } n$$

and

$$(40) \quad K_n \leq 2^n + \varphi_4(2^n) \quad \text{for all large } n,$$

where

$$\varphi_4(x) = 2f_1(x) + 2f_2(x) + 2\varphi_2(x).$$

Finally, if we set

$$\psi(x) = \max\{f_1(2^{x+1}), \varphi_2(2^x) + \varphi_3(2^x) + \varphi_4(2^x)\},$$

then

$$(41) \quad \sum_0^N |k_n - 2^n| = O(\psi(N)).$$

For example, if  $f_1(x)$  and  $f_2(x)$  are both  $x^a$  for large  $x$ , then the  $\varphi_i(x)$  are each asymptotic to a suitable constant multiple of  $x^a$ .

To prove (37), use the inequality  $S(x) < 2^n$  if  $x < k_n$ , as before. To prove (39), use the inequality

$$K_n + f_2(K_n) \geq S(K_n) \geq 2^n,$$

also as before. By a method entirely analogous to that of the preceding section, it follows that  $2k_n > K_n$  for all sufficiently large  $n$ . And for such  $n$ , proceeding again as before, we have, for each  $y$  with  $0 \leq y < 2k_n - K_n$ ,

$$(42) \quad K_n + y + f_2(K_n + y) \geq S(K_n + y) \geq 2^n + S(K_n + y - k_n) \\ \geq 2^n + K_n + y - k_n - f_1(K_n + y - k_n),$$

so that

$$k_n \geq 2^n - f_1(K_n + y - k_n) - f_2(K_n + y) \geq 2^n - f_1(k_n) - f_2(k_n),$$

which implies (38).

In order to prove (40), we suppose that

$$K_n > 2^n + K_p,$$

where  $n$  is so large that  $k_{n+1} > 2^n$  and  $k_n > 2^{n-1}$ . Then, as in the proof of Theorem 2, we get

$$\begin{aligned} 2^n - f_1(2^n) &\leq S(2^n) \leq 2^n - 2^p + S(2^n - k_n) \\ &\leq 2^n - 2^p + 2^n - 2^n + \varphi_2(2^n) + f_2(2^n) \end{aligned}$$

or

$$(43) \quad 2^p \leq f_1(2^n) + f_2(2^n) + \varphi_2(2^n)$$

so that in view of (37), we have

$$(44) \quad K_n \leq 2^n + K_p \leq 2^n + 2^{p+1} \leq 2^n + 2f_1(2^n) + 2f_2(2^n) + 2\varphi_2(2^n)$$

for all sufficiently large  $n$ .

Now assume that

$$\limsup \sum_{k=0}^N |k_n - 2^n| / \psi(N) = \infty.$$

In view of (39), this implies that

$$\limsup \sum_{k=0}^N (k_n - 2^n)^+ / \psi(N) = \infty,$$

where

$$(k_n - 2^n)^+ = \max(0, k_n - 2^n).$$

We first prove that if  $\chi(N) = \log_2(\varphi_2(2^N) + \varphi_3(2^N) + \varphi_4(2^N))$ , then

$$(45) \quad \sum_{\chi(N) < n < N} (k_n - 2^n)^+ / (\psi(N) + 1) \leq 1.$$

For, let  $n_1, n_2, \dots, n_r$  be the values of  $n$  for which

$$k_n > 2^n, \quad \chi(N) < n_1 < n_2 < \dots < n_r < N.$$

Let

$$A = \sum_{j=1}^r k_{n_j} > \sum_{j=1}^r 2^{n_j} + \psi(N) + 1 = B + \psi(N) + 1.$$

Then, as in the proof of Theorem 2,  $\sum 2^{n_j} > B$  implies that  $\sum k_{n_j} > A$ .

This is obvious if  $\{n_j\}$  is a subset of  $\{m_i\}$ . If not, let  $n_s$  be the largest element of  $\{n_i\}$  not contained in  $\{m_i\}$ , so that

$$\begin{aligned} \sum k_{m_j} &\geq A + k_{n_s+1} - K_{n_s+1} + K_{n_1} \\ &\geq A + 2^{n_s+1} - \varphi_2(2^{n_s+1}) - 2^{n_s+1} - \varphi_4(2^{n_s+1}) + 2^{n_1} - \varphi_3(2^{n_1}) \\ &> A + 2^{\chi(N)} - \varphi_2(2^N) - \varphi_3(2^N) - \varphi_4(2^N) > A. \end{aligned}$$

Hence  $S(A)$  is no greater than the number of sums of powers of 2 that do not exceed  $B$ , so that  $S(A) \leq B + 1$ . It follows that

$$A - f_1(A) \leq S(A) \leq B + 1 < A - \psi(N),$$

which leads to a contradiction, since  $A \leq K_N < 2^{N+1}$  for all large  $N$ .

Now we have  $1 + \psi(x) < \frac{1}{2} 2^x$  for all  $x > x_0$ , and therefore, from (45), if we let

$$\chi_0(N) = N \quad \text{and} \quad \chi_{m+1}(N) = \chi(\chi_m(N)),$$

then, provided  $\chi_{m+1}(N) > x_0$ , we have

$$\sum_{\chi_{m+1}(N) < n < \chi_m(N)} (k_n - 2^n)^+ \leq 1 + \psi(\chi_m(N)) < \frac{1}{2} 2^{\chi_m(N)}.$$

But

$$2^{\chi_m(N)} \leq \psi(\chi_{m-1}(N)),$$

so that if  $\chi_{m+1}(N) > x_0$ , we have

$$(46) \quad \sum_{\chi_{m+1}(N) < n < \chi_m(N)} (k_n - 2^n)^+ \leq \frac{1}{2^m} \psi(N).$$

On adding the inequalities (46) for all suitable  $m$ , we get

$$\sum_{n=0}^N (k_n - 2^n)^+ \leq 2\psi(N) + O(1),$$

which proves (41) by contradiction.

**4. Irregularity of  $S(x)$ .** We say that a function  $f$  is *slowly oscillating* to mean that for each positive constant  $a$ ,  $f(ax)/f(x) \rightarrow 1$  as  $x \rightarrow \infty$ .

**THEOREM 4.** *It is impossible to have  $S(x) \sim x^a f(x)$ , where  $0 < a < 1$ , and  $f(x)$  is a continuous positive slowly oscillating function such that  $x^a f(x)$  is strictly increasing.*

**Proof.** Define the inverse function  $g$  by

$$y = x^a f(x) \iff x = y^{1/a} g(y).$$

Then  $g$  is also a continuous positive slowly oscillating function. From  $S(k_n -) \leq 2^n$ , we get

$$k_n < (1 + \varepsilon) 2^{n/a} g(2^n)$$

for any  $\varepsilon > 0$  and all sufficiently large  $n$ , so that

$$K_n = \sum_{m=0}^{n-1} k_m < (1 + \varepsilon) \sum_{m=0}^{n-1} 2^{m/a} g(2^m).$$

On the other hand, we have  $S(K_n) \geq 2^n$ , so that

$$K_n > (1 - \varepsilon) 2^{n/a} g(2^n),$$

and hence

$$(47) \quad \frac{1 - \varepsilon}{1 + \varepsilon} < \sum_{m=0}^{n-1} 2^{(m-n)/a} \frac{g(2^m)}{g(2^n)}$$

for all sufficiently large  $n$ . We show now that (47) is impossible for small  $\varepsilon$ . We use the result [2] that there is a function  $h(x)$  with  $h(x) \sim cg(x)$  as  $x \rightarrow \infty$ , where  $c$  is a positive constant, such that  $h(x)$  has the representation

$$(48) \quad h(x) = \exp \int_1^x \beta(t) t^{-1} dt$$

where

$$(49) \quad \beta(t) = o(1) \quad \text{as} \quad t \rightarrow \infty.$$

It follows from (49) that  $h(x) \geq x^{-\delta}$  for any  $\delta > 0$ , for all sufficiently large  $x$ , and the same inequality consequently holds for  $g$ . It follows that the values of  $g(x)$  when  $x$  is small do not affect the inequality (47) for large  $n$ , and that to contradict (47), it is enough to contradict the corresponding inequality for  $h$ , which by (48) may be written as

$$(50) \quad \frac{1 - \varepsilon}{1 + \varepsilon} < \sum_{m=0}^{n-1} \exp \left( - \int_{2^m}^{2^n} \left\{ \frac{1}{a} + \beta(t) \right\} \frac{dt}{t} \right).$$

If we now choose  $\gamma$  so that  $1 < \gamma < 1/a$ , then for all sufficiently large  $t$ ,

$$1/a + \beta(t) > \gamma,$$

and by the above remarks, there is no loss in assuming this for all  $t$ . We then have

$$(51) \quad \frac{1 - \varepsilon}{1 + \varepsilon} < \sum_{m=0}^{n-1} 2^{\gamma(m-n)} = \sum_{r=1}^n 2^{-\gamma r} < \frac{1}{2^\gamma - 1},$$

which is a contradiction if  $\varepsilon$  is small enough, and the theorem is proved.

On the other hand, it is possible, for each positive integer  $a$ , to have  $S(x) \sim cx^a$ , where  $c$  is any positive constant. For example, if we let  $K$  consist of  $a$  copies of  $\{2^n\}$ ,  $n = 0, 1, \dots$ , then a simple computation shows that  $S(x) \sim c_a x^a$ . Perhaps, then, it is impossible to have  $S(x) \sim f(x)x^a$ , where  $a > 0$ ,  $f(x)$  is slowly oscillating, and  $x^a f(x)$  is strictly increasing, unless  $a$  is an integer. We outline here a proof of a partial result in this direction, namely that if  $1 < a < a_0$ , for a certain  $a_0$  (with  $1 < a_0 < 2$ ) then  $S(x) \sim f(x)x^a$  is impossible. We treat the case  $f(x) = 1$  for all  $x$ ; the general case is similar.

In this case, we apply arguments like those above to get  $(2 - \delta)k_n > K_n$  for some  $\delta > 0$  and infinitely many  $n$ . But then we have

$$S((\tfrac{1}{2} + \varepsilon)K_n) = 2^n - S((\tfrac{1}{2} - \varepsilon)K_n),$$

provided only that  $\varepsilon$  is chosen so small that

$$k_n > (\tfrac{1}{2} + \varepsilon)K_n,$$

since for every sum

$$\sum_{i=0}^{n-1} \varepsilon_i k_i < (\tfrac{1}{2} - \varepsilon)K_n,$$

we have

$$\sum_{i=0}^{n-1} (1 - \varepsilon_i) k_i > (\tfrac{1}{2} + \varepsilon)K_n.$$

But the asymptotic relation

$$(\tfrac{1}{2} + \varepsilon)^a K_n^a \sim 2^n - (\tfrac{1}{2} - \varepsilon)^a K_n^a$$

cannot hold identically in  $\varepsilon$  unless  $a = 1$ , which is excluded, and the result is proved.

## References

- [1] Basil Gordon and L. A. Rubel, *On the density of sets of integers possessing additive bases*, Illinois J. Math. 4 (1960), pp. 367-369.
- [2] J. Korevaar, T. van Aardenne-Ehrenfest and N. G. de Bruijn, *A note on slowly oscillating functions*, Nieuw. Arch. Wisk. (2), 23 (1949), pp. 77-86.

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