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Groups with minimum condition

by

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Dedicated to L. J. Mordell
on the occasion of his 75-th birthday

If the minimum condition is satisfied by the subgroups of a group G, then a fortiori

(1) the minimum condition is satisfied by the normal subgroups of G;

and it is fairly obvious that the converse of this statement is false. A further immediate consequence of the minimum condition for subgroups of G is the following condition:

(2) if $H \neq 1$ is an epimorphic image of G, then there exists a normal subgroup $N \neq 1$ of H such that the minimum condition is satisfied by the subgroups of the group of automorphisms, induced in N by H.

It is, however, not difficult to construct examples of groups G, meeting requirements (1) and (2), whereas the minimum condition is not satisfied by the subgroups of G. Thus there arises the problem to complete these conditions (1) and (2) into a criterion for the validity of the minimum condition for subgroups; and solutions of this problem are offered in our Corollary 3.3. It is an open problem whether groups with minimum condition always possess an abelian subgroup of finite index. It is therefore of interest to characterize the groups with minimum condition and an abelian subgroup of finite index among the groups, meeting requirements (1) and (2); and such characterizations are given in Theorem 4.2. The investigation leading to Corollary 3.3 and Theorem 4.2 is conducted within a general framework leading to our principal criterion: the Theorem 3.2. Of this criterion Corollary 3.3 and Theorem 4.2 are more or less obvious special cases.

Notations.

$\mathfrak{z}G$ = center of G;

$\mathfrak{h}G$ = hypercenter of G = intersection of all normal subgroups X of G with $\mathfrak{z}(G/X) = 1$;

$\mathfrak{c}X = \mathfrak{c}_Y X$ = centralizer of the subgroup X of Y [in Y];

$x \circ y = x^{-1}y^{-1}xy = x^{-1}x^y$.

The group G is

nilpotent, if $G = \mathfrak{h}G$;

an $\mathfrak{m}$-*group*, if the minimum condition is satisfied by its subgroups;

an $\mathfrak{mn}$-*group*, if the minimum condition is satisfied by its normal subgroups;

an $\mathfrak{aa}$- [or *almost abelian*] *group*, if it possesses an abelian subgroup of finite index;

of *finite rank*, if there exists a positive integer n such that every finitely generated subgroup of G may be generated by n [or fewer] elements.

The prime p is *G-relevant*, if the group G contains elements of order p.

1. The following probably well-known criterion will prove useful in the sequel.

LEMMA 1.1. *The subgroup U of the torsion group G is part of the center $\mathfrak{z}G$ of G if and only if*

(a) *U is a normal subgroup of G,*

(b) *$U \subseteq \mathfrak{h}G$ and*

(c) *$G/\mathfrak{c}U$ is free of finite epimorphic images $\neq 1$.*

Terminological notes. The *hypercenter* $\mathfrak{h}G$ of the group G is the intersection of all the normal subgroups X of G with $\mathfrak{z}(G/X) = 1$.

$\mathfrak{c}U = \mathfrak{c}_G U$ is for every subset U of G the *centralizer* of U in G. If in particular U happens to be a normal subgroup of G, then $\mathfrak{c}U$ is likewise a normal subgroup of G and $G/\mathfrak{c}U$ is essentially the same as the group of automorphisms, induced in U by the elements in G.

Proof. If U is part of the center of G, then $U \subseteq \mathfrak{z}G \subseteq \mathfrak{h}G$ and $G = \mathfrak{c}U$, putting into evidence the necessity of our conditions (a)-(c).

We assume conversely the validity of our conditions (a)-(c); and we assume by way of contradiction that $U \nsubseteq \mathfrak{z}G$. Then $U \cdot \mathfrak{z}G/\mathfrak{z}G$ is a normal subgroup, not 1, of $G/\mathfrak{z}G$ which is a part of $\mathfrak{h}G/\mathfrak{z}G = \mathfrak{h}(G/\mathfrak{z}G)$. Application of the well-known properties of the hypercenter shows

$$1 \neq \mathfrak{z}(G/\mathfrak{z}G) \cap (U \cdot \mathfrak{z}G/\mathfrak{z}G);$$

cf., for instance, Specht [7], p. 327, Satz 14. Since G is a torsion group, there exists a subgroup $V/\mathfrak{z}G$ of $\mathfrak{z}(G/\mathfrak{z}G) \cap (U \cdot \mathfrak{z}G/\mathfrak{z}G)$ whose order is a prime p. It is clear that V is a normal subgroup of G and that the group Γ of automorphisms, induced by G in V, induces the 1-automorphism both in $\mathfrak{z}G$ and in $V/\mathfrak{z}G$ and is consequently part of the group of stability of the normal subgroup $\mathfrak{z}G$ of V. Application of Specht (p. 88, Satz 19) shows the commutativity of Γ.

If v is an element in V and g an element in G, then v^g belongs to V and both v^p and $v \circ g$ belong to $\mathfrak{z}G$. Hence

$$v^p = (v^p)^g = (v^g)^p = [v(v \circ g)]^p = v^p (v \circ g)^p$$

so that $(v \circ g)^p = 1$. Since $v \circ g$ belongs to $\mathfrak{z}G$, we deduce inductively from $v^g = v(v \circ g)$ the validity of

$$v^{g^i} = v(v \circ g)^i \quad \text{for every positive integer } i.$$

Hence, in particular,

$$v^{g^p} = v(v \circ g)^p = v \quad \text{for every } v \text{ in } V \text{ and every } g \text{ in } G;$$

and thus we have shown that Γ is an elementary abelian p-group.

Now, Γ is essentially the same as $G/\mathfrak{c}V$; and from $V \subseteq U \cdot \mathfrak{z}G$ we deduce $\mathfrak{c}U \subseteq \mathfrak{c}V$. Thus the elementary abelian p-group Γ is an epimorphic image of $G/\mathfrak{c}U$. Elementary abelian p-groups, not 1, possess finite epimorphic images, not 1. Application of condition (c) shows consequently $\Gamma = 1$; and this is equivalent to the desired contradiction $\mathfrak{z}G \subset V \subseteq \mathfrak{z}G$.

LEMMA 1.2. *If G is a nilpotent torsion group of finite rank and if the number of G-relevant primes is finite, then there exists an abelian subgroup of finite index in G and the minimum condition is satisfied by the subgroups of G.*

Terminological notes. Here we term — following Specht — a group *nilpotent*, if its epimorphic images, not 1, possess centers, not 1. Thus the group G is nilpotent if and only if $G = \mathfrak{h}G$.— The *rank of G is finite*, if there exists a positive integer n such that every finitely generated subgroup of G may be generated by n [or fewer] elements.— The prime p is *G-relevant*, if G contains elements of order p.

Proof. Denote by J the intersection of all the subgroups X of G with finite index $[G:X]$. It is clear that J is a characteristic subgroup of G so that we may form the epimorphic image $H = G/J$ of G. Denote by S the totality of elements of squarefree order in the center $\mathfrak{z}H$ of H. This is an abelian characteristic subgroup of H; and it follows from our hypotheses that the rank of S and the number of S-relevant primes are finite. Consequently S is the direct product of finitely many primary elementary abelian groups and these primary components are finite so that S itself is finite.

The intersection of all the subgroups of finite index in H is 1, as follows from our definition of J and $H = G/J$. To every element $x \neq 1$

in S there exists consequently a subgroup x' of H with finite $[H:x']$ which does not contain x. As x' possesses but a finite number of conjugates in H, there exists a normal subgroup x'' of H with $x'' \subseteq x'$ and finite H/x''. Clearly x is not contained in x'' either. If L is the intersection of all the normal subgroups x'' for $x \neq 1$ in S, then L is a normal subgroup of H and H/L is finite, since S is finite. Furthermore $L \cap S = 1$. If $L \neq 1$, then $L \cap \mathfrak{z}H \neq 1$, since the epimorphic image H of G is nilpotent (cf. for instance Baer [3], p. 192, Lemma 2.1). But H is a torsion group. Hence $L \cap \mathfrak{z}H \neq 1$ contains elements of order a prime; and this implies the contradiction $1 \subset L \cap S = 1$. Hence $L = 1$ so that $G/J = H = H/L$ is finite.

From the finiteness of G/J and the construction of J we deduce that J is free of finite epimorphic images, not 1. The subgroup J of the nilpotent group G is likewise nilpotent; (cf. Baer [3], p. 192, Lemma 2.2). Application of Lemma 1.1 shows the commutativity of J.

Clearly J is an abelian torsion group and its subgroup of elements of squarefree order is finite, since the rank of J and the number of J-relevant primes is finite. It follows that the minimum condition is satisfied by the subgroups of J (cf. Fuchs [6], pp. 68, 19). From the finiteness of G/J we deduce finally that the minimum condition is satisfied by the subgroups of G.

Remark 1.3. It is worth noting that nilpotent groups with minimum condition always possess an abelian subgroup of finite index (cf. Baer [4], p. 7/8, Satz 2.1).

2. It will be convenient to term the group G an $\mathfrak{m}$-*group*, if the minimum condition is satisfied by the subgroups of G; and we shall term G an $\mathfrak{mn}$-*group*, if the minimum condition is satisfied by the normal subgroups of G. — If $\mathfrak{e}$ is any group theoretical property, then the group G is termed an $\mathfrak{em}$-*group*, if it is both an $\mathfrak{e}$-group and an $\mathfrak{m}$-group; and similarly we term G an $\mathfrak{emn}$-*group*, if G is at the same time an $\mathfrak{e}$-group and an $\mathfrak{mn}$-group.

The properties $\mathfrak{e}$ that we admit to our discussion will be subject to [some or all of] the following requirements:

I. Subgroups and epimorphic images of $\mathfrak{e}$-groups are $\mathfrak{e}$-groups.

II. Direct products of two $\mathfrak{e}$-groups are $\mathfrak{e}$-groups.

III. Extensions of $\mathfrak{em}$-groups by finite groups are $\mathfrak{e}$-groups [and hence $\mathfrak{em}$-groups].

IV. Extensions of finite groups by $\mathfrak{em}$-groups are $\mathfrak{e}$-groups [and hence $\mathfrak{em}$-groups].

V. Central extensions of abelian $\mathfrak{m}$-groups by $\mathfrak{em}$-groups are $\mathfrak{e}$-groups [and hence $\mathfrak{em}$-groups].

If the property $\mathfrak{e}$ meets requirements I-V, then by V every abelian $\mathfrak{m}$-group is an $\mathfrak{e}$-group; and it is a consequence of III that every $\mathfrak{m}$-group possessing an abelian subgroup of finite index [the so-called almost abelian $\mathfrak{m}$-groups] are $\mathfrak{em}$-groups. It seems to be an open question whether there exist any further $\mathfrak{m}$-groups; and thus it is conceivable that in every case all $\mathfrak{m}$-groups are $\mathfrak{e}$-groups.

If $\mathfrak{x}$ is any group theoretical property, then $\mathfrak{x}G$ is for any group G the intersection of all normal subgroups X of G such that G/X is an $\mathfrak{x}$-group. Clearly this is a group theoretical functor which attaches to every group G a characteristic subgroup $\mathfrak{x}G$. Furthermore we term the group H an $\mathfrak{x}^*$-*group*, if every epimorphic image, not 1, of H possesses a normal subgroup, not 1, in which an $\mathfrak{x}$-group of automorphisms is induced by H. This derived group theoretical property $\mathfrak{x}^*$ and the derived group theoretical functor $\mathfrak{x}$ are connected by the following relations.

Lemma 2.1. *If the group theoretical property $\mathfrak{x}$ is inherited by subgroups and epimorphic images, then*

(a) *$\mathfrak{x}^*$ is inherited by subgroups and epimorphic images;*

(b) *to every normal subgroup $N \neq 1$ of an $\mathfrak{x}^*$-group G, there exists a normal subgroup J of G with $1 \subset J \subseteq N$ and $\mathfrak{x}$-quotient group $G/\mathfrak{c}J$;*

(c) *$\mathfrak{x}G$ is nilpotent for every $\mathfrak{x}^*$-group G.*

For proofs cf. Baer [3], p. 192/193, Lemmata 2.1, 2.2 and 2.5.

Lemma 2.2. *If the group theoretical property $\mathfrak{x}$ meets requirements* I *and* II, *then*

(a) *$G/\mathfrak{x}G$ is an $\mathfrak{x}$-group for every $\mathfrak{mn}$-group G;*

(b) *the $\mathfrak{mn}$-group G is an $\mathfrak{x}^*$-group if and only if $\mathfrak{x}G$ is nilpotent.*

Proof. If G is an $\mathfrak{mn}$-group, then there exists among the normal subgroups X of G with $\mathfrak{x}$-quotient group G/X a minimal one, say M. Clearly $\mathfrak{x}G \subseteq M$. If X is any normal subgroup of G with $\mathfrak{x}$-quotient group G/X, then $G/(M \cap X)$ is isomorphic to a subgroup of the direct product of the $\mathfrak{x}$-groups G/M and G/X so that $G/(M \cap X)$ is by I and II itself an $\mathfrak{x}$-group. Application of the minimality of M shows $M = M \cap X \subseteq X$, proving $M \subseteq \mathfrak{x}G$ and (a). — (b) is a fairly immediate consequence of (a) and Baer [3], p. 194, Satz 2.6.

Proposition 2.3. *If the group theoretical property $\mathfrak{e}$ meets the requirements* I-V, *then the group G is an $\mathfrak{em}$-group if and only if*

(a) *G is an extension of an $\mathfrak{em}$-group by an $\mathfrak{em}$-group and*

(b) *G is an $\mathfrak{e}^*$-group.*

Proof. Only the sufficiency of the conditions (a), (b) needs verification: by (a) there exists a normal subgroup N of G such that N and G/N are $\mathfrak{em}$-groups. As extensions of $\mathfrak{m}$-groups by $\mathfrak{m}$-groups are $\mathfrak{m}$-groups, we find that

(1) G is an $\mathfrak{m}$-group.

Application of Lemma 2.2 [and (1)] shows that

(2) $\mathfrak{e}G$ is nilpotent and $G/\mathfrak{e}G$ is an $\mathfrak{e}$-group.

Since G/N is an $\mathfrak{e}$-group, we have $\mathfrak{e}G \subseteq N$; and since N is an $\mathfrak{e}$-group, we deduce from I that

(3) $\mathfrak{e}G$ is an $\mathfrak{e}$-group.

Because of (1) and Lemma 2.2, (a) there exists a subgroup M of $\mathfrak{e}G$ with the property:

The subgroup S of $\mathfrak{e}G$ contains M if and only if $[\mathfrak{e}G:S]$ is finite.

It follows that M is the intersection of all the subgroups of finite index in $\mathfrak{e}G$; and as such M is a characteristic subgroup of the characteristic subgroup $\mathfrak{e}G$ of G and hence a characteristic subgroup of G. The group G/M is an extension of the finite group $\mathfrak{e}G/M$ by the $\mathfrak{em}$-group $G/\mathfrak{e}G$ — see (1) and (2). It follows from IV that G/M is an $\mathfrak{e}$-group. Hence $M \subseteq \mathfrak{e}G \subseteq M$ so that $M = \mathfrak{e}G$. As M has been shown to be the intersection of all the subgroups of finite index in $\mathfrak{e}G$, we have shown

(4) $\mathfrak{e}G$ is free of proper subgroups of finite index.

It follows from (1) and (2) that $\mathfrak{e}G$ is a nilpotent torsion group. Because of (4) we may apply Lemma 1.1 on the normal subgroup $\mathfrak{e}G$ of $\mathfrak{e}G$. It follows that

(5) $\mathfrak{e}G$ is abelian.

Because of (1) and Lemma 2.2, (a) there exists a subgroup $\mathfrak{f}G$ of G with the property:

(6) A subgroup S of G contains $\mathfrak{f}G$ if and only if $[G:S]$ is finite.

Because of (1) and (5) the group $\mathfrak{e}G$ is an abelian $\mathfrak{m}$-group. Thus $\mathfrak{e}G$ contains to every positive integer n only a finite number of solutions of the equation $x^n = 1$ (see Fuchs, [6], p. 65, Theorem 19.2). If t is any element in the abelian torsion group $\mathfrak{e}G$, there exists consequently a finite characteristic subgroup T of $\mathfrak{e}G$ which contains t. Then T is a finite characteristic subgroup of G so that $\mathfrak{c}_G T$ is a characteristic subgroup of finite index in G. It follows that $\mathfrak{f}G \subseteq \mathfrak{c}_G T$; and this implies that t is centralized by $\mathfrak{f}G$. Hence $\mathfrak{e}G$ is centralized by $\mathfrak{f}G$. By IV finite groups are $\mathfrak{e}$-groups. The finite group $G/\mathfrak{f}G$ is consequently an $\mathfrak{e}$-group so that $\mathfrak{e}G \subseteq \mathfrak{f}G$. Thus we have shown:

(7) $\mathfrak{e}G \subseteq \mathfrak{z}\mathfrak{f}G$.

The subgroup $\mathfrak{f}G/\mathfrak{e}G$ of $G/\mathfrak{e}G$ is by (2) and (1) an $\mathfrak{em}$-group. It follows from (7) that $\mathfrak{f}G$ is a central extension of the abelian $\mathfrak{m}$-group $\mathfrak{e}G$ by the $\mathfrak{em}$-group $\mathfrak{f}G/\mathfrak{e}G$. Application of V shows that $\mathfrak{f}G$ is an $\mathfrak{em}$-group. Consequently G is an extension of the $\mathfrak{em}$-group $\mathfrak{f}G$ by the finite group $G/\mathfrak{f}G$; and application of III shows that G itself is an $\mathfrak{em}$-group.

Remark 2.4. We have not been able to decide whether condition (b) is indispensable. The construction of an example showing the indispensability of (b) should prove difficult in view of the fact that it would have to be an $\mathfrak{m}$-group without abelian subgroups of finite index; see the remarks appended to the statement of the postulates I-V.

3. We are now ready to prove our principal result.

THEOREM 3.1. *If the property $\mathfrak{e}$ meets requirements* I-V, *then the following properties of the group G are equivalent*:

(i) *G is an $\mathfrak{em}$-group.*

(ii)
- (a) *G is an $\mathfrak{mn}$-group.*
- (b) *G is an $\mathfrak{e}^*$-group.*
- (c) *If M is an abelian minimal normal subgroup of the epimorphic image H of G, and if $H/\mathfrak{c}_H M$ is an $\mathfrak{e}$-group, then M is finite.*
- (d) *If H is an epimorphic image of G with $\mathfrak{em}H/\mathfrak{e}H \neq 1$, then there exists a normal subgroup $N \neq 1$ of H with finite N'.*

(iii)
- (a) *G is an $\mathfrak{mn}$-group.*
- (b) *G is an $(\mathfrak{em})^*$-group.*
- (c) *If M is a minimal normal subgroup of the epimorphic image H of G, and if $M \subseteq \mathfrak{z}(\mathfrak{em}H)$, then M is finite.*

(iv)
- (a) *$G/\mathfrak{em}G$ and $\mathfrak{em}G$ are $\mathfrak{mn}$-groups.*
- (b) *G is an $(\mathfrak{em})^*$-group.*

(v)
- (a) *$G/\mathfrak{em}G$ is an $\mathfrak{mn}$-group.*
- (b) *G is an $(\mathfrak{em})^*$-group.*
- (c) *$\mathfrak{em}G$ is a torsion group of finite rank and the number of $\mathfrak{em}G$-relevant primes is finite.*

Proof. Assume first that G is an $\mathfrak{em}$-group. Then G is an $\mathfrak{m}$-group and a fortiori an $\mathfrak{mn}$-group. Since G is an $\mathfrak{e}$-group, and since $\mathfrak{e}$ is inherited by epimorphic images, G is an $\mathfrak{e}^*$-group. The abelian minimal normal subgroups of epimorphic images of G are characteristically simple abelian $\mathfrak{m}$-groups and these are finite; cf. Fuchs ([6], p. 65, Theorem 19.2). Furthermore, $\mathfrak{em}G = 1$; and this implies $\mathfrak{em}H = 1$ for every epimorphic image H of G. Thus we have shown that (ii) is a consequence of (i).

Assume next the validity of (ii). Then we note the identity of (ii.a) and (iii.a). Assume next that M is a minimal normal subgroup of the epimorphic image H of G with $M \subseteq \mathfrak{z}(\mathfrak{em}H)$. Since H is by (ii.a) an $\mathfrak{mn}$-group, we deduce from Lemma 2.2, (a) that $H/\mathfrak{em}H$ is an $\mathfrak{em}$-group. Furthermore, $\mathfrak{em}H \subseteq \mathfrak{c}_H M$ so that the epimorphic image $H/\mathfrak{c}_H M$ of $H/\mathfrak{em}H$ is likewise an $\mathfrak{em}$-group. Application of (ii.c) shows the finiteness of M and the validity of (iii.c). Assume by way of contradiction that G is not an $(\mathfrak{em})^*$-Group. Then we deduce from (ii.a) and Lemma 2.2, (b) that $\mathfrak{em}G$ is not nilpotent; and this, by definition, is equivalent to $\mathfrak{em}G \neq \mathfrak{h}(\mathfrak{em}G)$. We form the epimorphic image $H = G/\mathfrak{h}(\mathfrak{em}G)$. Then $\mathfrak{em}H = \mathfrak{em}G/\mathfrak{h}(\mathfrak{em}G) \neq 1$; and recalling the definition of the hypercenter and the construction of H we deduce $\mathfrak{z}(\mathfrak{em}H) = 1$. Consider a finite normal subgroup F of H with $F \subseteq \mathfrak{em}H$. Then $L = \mathfrak{em}H \cap \mathfrak{c}_H F$ is a normal subgroup of H; and $\mathfrak{em}H/L$ is essentially the same as the group of automorphisms, induced in F by elements in $\mathfrak{em}H$. It follows that $\mathfrak{em}H/L$ is finite. The extension H/L of the finite group $\mathfrak{em}H/L$ by the $\mathfrak{em}$-group $H/\mathfrak{em}H$ is (by IV) an $\mathfrak{em}$-group. Hence $\mathfrak{em}H \subseteq L \subseteq \mathfrak{c}_H F$ so that F is centralized by $\mathfrak{em}H$. It follows that $F \subseteq \mathfrak{z}(\mathfrak{em}H) = 1$; and we have shown:

(+) 1 is the only finite normal subgroup of H which is part of $\mathfrak{em}H$.

Assume by way of contradiction that $\mathfrak{e}H \neq 1$. Then $\mathfrak{e}H$ is nilpotent by (ii.b) and Lemma 2.1, (c). Hence $\mathfrak{z}(\mathfrak{e}H) \neq 1$; and we deduce from (ii.a) the existence of a minimal normal subgroup K of H with $K \subseteq \mathfrak{z}(\mathfrak{e}H)$. Then K is abelian and $\mathfrak{e}H \subseteq \mathfrak{c}_H K$. Now $H/\mathfrak{e}H$ is an $\mathfrak{e}$-group by (ii.a) and Lemma 2.2, (a) so that the epimorphic image $H/\mathfrak{c}_H K$ of $H/\mathfrak{e}H$ is likewise an $\mathfrak{e}$-group. Application of (ii.c) shows the finiteness of $K \neq 1$ in contradiction to (+). Hence $\mathfrak{e}H = 1$.

It follows that $1 \neq \mathfrak{em}H = \mathfrak{em}H/\mathfrak{e}H$. Application of (ii.a) shows the existence of a minimal normal subgroup J of H with $J \subseteq \mathfrak{em}H$. Among the normal subgroups X of H with $J \cap X = 1$ there exists a maximal one, say R. The epimorphic image $H^* = H/R$ of H has the following properties:

$$\mathfrak{e}H^* = R\cdot\mathfrak{e}H/R = 1, \qquad \mathfrak{em}H^* = R\cdot\mathfrak{em}H/R \supseteq RJ/R \simeq J \neq 1$$

[because of $J \cap R = 1$]. Application of (ii.d) shows the existence of a normal subgroup $N/R \neq 1$ of H^* with finite $(N/R)'$. If N' were not part of R, then $R \subset RN'$; and we would deduce $J \cap RN' \neq 1$ from the maximality of R. From the minimality of J we deduce now $J = J \cap RN' \subseteq RN'$, since R, N and N' are normal subgroups of H. Hence

$$J = J/(R \cap J) \simeq RJ/R \subseteq RN'/R = (N/R)'$$

is finite, contradicting $1 \subset J \subseteq \mathfrak{em}H$ and (+). Hence $N' \subseteq R$. Because of $R \subset N$ and the maximality of R we have $J \cap N \neq 1$ which implies $J \subseteq N$ because of the minimality of J. We have furthermore

$$J' \subseteq J \cap N' \subseteq J \cap R = 1$$

so that J is abelian. From $\mathfrak{e}H = 1$ [and I] we deduce that $H/\mathfrak{c}_H M$ is an $\mathfrak{e}$-group. Application of (ii. c) shows the finiteness of J; and this contradicts $1 \subset J \subseteq \mathfrak{em}H$ and (+). This contradiction shows that G is an $(\mathfrak{em})^*$-group; and thus we have deduced (iii) from (ii).

Assume next the validity of (iii). If X is any group, then we denote by $\mathfrak{F}X$ the set of all the elements x in X such that the class x^X of all the elements, conjugate to x in X, is finite. It is well known, and easily verified, that the set $\mathfrak{F}X$ is actually a characteristic subgroup of X; cf. Baer [1], p. 1023, Proposition 1, or [2], p. 22, (A). Denote by $\mathfrak{F}^*X$ the uniquely determined characteristic subgroup of X with

$$\mathfrak{F}X \subseteq \mathfrak{F}^*X \quad \text{and} \quad \mathfrak{F}^*X/\mathfrak{F}X = \mathfrak{F}(X/\mathfrak{F}X).$$

Application of Baer [2], p. 11, Theorem 1 and Corollary 1 and (iii.a) shows that

(*) $\quad \mathfrak{F}^*G/\mathfrak{F}G$ is finite; $\mathfrak{F}(G/\mathfrak{F}^*G) = 1$; $\mathfrak{F}^*G$ is an $\mathfrak{m}$-group.

Assume now by way of contradiction that $\mathfrak{em}G \nsubseteq \mathfrak{F}^*G$. The characteristic subgroup $L = \mathfrak{em}G \cap \mathfrak{F}^*G$ of G is then a proper subgroup of $\mathfrak{em}G$. We form the epimorphic image $H = G/L$ of G. Then $\mathfrak{em}H = \mathfrak{em}G/L \neq 1$. From (iii.b) and Lemma 2.1, (c) we deduce that $\mathfrak{em}H$ is nilpotent. Hence $\mathfrak{z}(\mathfrak{em}H) \neq 1$; and we deduce from (iii.a) the existence of a minimal normal subgroup M of H with $M \subseteq \mathfrak{z}(\mathfrak{em}H)$. Application of (iii.c) shows the finiteness of the normal subgroup M of H. There exists a normal subgroup N of G with $L \subseteq N$ and $N/L = M$. Then

$$N/L = M \subseteq \mathfrak{em}H = \mathfrak{em}G/L$$

so that $N \subseteq \mathfrak{em}G$. Hence

$$(N\mathfrak{F}^*G)/\mathfrak{F}^*G \simeq N/(N \cap \mathfrak{F}^*G) = N/(N \cap \mathfrak{em}G \cap \mathfrak{F}^*G) = N/(N \cap L) = M$$

is finite so that

$$(N\mathfrak{F}^*G)/\mathfrak{F}^*G \subseteq \mathfrak{F}(G/\mathfrak{F}^*G) = 1$$

by (*). Hence $N \subseteq \mathfrak{F}^*G \cap \mathfrak{em}G = L$ so that $1 \neq M = N/L = 1$, a contradiction proving $\mathfrak{em}G \subseteq \mathfrak{F}^*G$. Application of (*) shows that $\mathfrak{em}G$ is an $\mathfrak{m}$-group; and that $G/\mathfrak{em}G$ is an $\mathfrak{mn}$-group, is an immediate consequence of (iii.a). Now it is clear that (iv) is a consequence of (iii).

Assume next the validity of (iv). Applying Lemma 2.2, (a) on the $\mathfrak{mn}$-group $G/\mathfrak{em}G$ we see that $G/\mathfrak{em}G$ is an $\mathfrak{em}$-group [since clearly $\mathfrak{em}(G/\mathfrak{em}G) = 1$]. Application of Lemma 2.1, (c) shows the nilpotency of $\mathfrak{em}G$. The nilpotent $\mathfrak{mn}$-group $\mathfrak{em}G$ is an $\mathfrak{m}$-group by a Theorem of Duguid and McLain ([5], p. 396, Lemma 3.3). Application of Lemma 2.2,

(a) on the $\mathfrak{m}$-group $\mathfrak{em}G$ shows the existence of a characteristic subgroup C of $\mathfrak{em}G$ with the property:

(++) The subgroup S of $\mathfrak{em}G$ contains C if and only if $[\mathfrak{em}G : S]$ is finite.

Then C is a characteristic subgroup of G and G/C is an extension of the finite group $\mathfrak{em}G/C$ by the $\mathfrak{em}$-group $G/\mathfrak{em}G$. Application of IV shows that G/C is an $\mathfrak{em}$-group. Hence $\mathfrak{em}G = C$ so that $\mathfrak{em}G$ is free of proper subgroups of finite index. Thus we have shown that $\mathfrak{em}G$ is a nilpotent $\mathfrak{m}$-group which is free of finite epimorphic images, not 1. We deduce from Lemma 1.1 the commutativity of $\mathfrak{em}G$. The abelian $\mathfrak{m}$-group $\mathfrak{em}G$ is by V an $\mathfrak{em}$-group. Thus we have shown that G is an extension of the $\mathfrak{em}$-group $\mathfrak{em}G$ by the $\mathfrak{em}$-group $G/\mathfrak{em}G$. The group G is an $\mathfrak{e}^*$-group, since it is, by (iv.b), an $(\mathfrak{em})^*$-group.

Application of Proposition 2.3 shows that G is an $\mathfrak{em}$-group. Hence (i) is a consequence of (iv).

If the equivalent properties (i)-(iv) are satisfied by G, then it is clear that $G/\mathfrak{em}G$ is an $\mathfrak{mn}$-group, that G is an $(\mathfrak{em})^*$-group and that $\mathfrak{em}G = 1$. Thus (v) is a consequence of the equivalent conditions (i)-(iv).

Assume conversely the validity of (v). Then it is a consequence of (v.a) and Lemma 2.2, (a) that

(1) $$G/\mathfrak{em}G \text{ is an } \mathfrak{em}\text{-group;}$$

and we deduce from (v.b) and Lemma 2.1, (c) that

(2) $$\mathfrak{em}G \text{ is nilpotent.}$$

It is a consequence of (v.c), (2) and Lemma 1.2 that

(3) $$\mathfrak{em}G \text{ is an } \mathfrak{m}\text{-group.}$$

Combining (1), (3) and (v.b) we see that (iv) is a consequence of (v), proving the equivalence of (i)-(v).

Remark 3.2. The reader should observe that the condition:

$$G \text{ is an } \mathfrak{mn}\text{-group}$$

has been needed only to assure the applicability of Lemma 2.2 and the existence of minimal normal subgroups contained in certain normal subgroups, not 1.— It is evident that condition (iv.a) is considerably stronger than the condition that G be an $\mathfrak{mn}$-group.— When deriving (iii) from (ii) it has become evident that (iii.c) is just a weak form of (ii.c) whereas (ii.b) is considerably weaker than (iii.b). A more detailed analysis of the various conditions may be found in section 5.

Corollary 3.3. *The following properties of the group G are equivalent:*

(i) *G is an $\mathfrak{m}$-group.*

(ii)
- (a) *G is an $\mathfrak{mn}$-group.*
- (b) *Abelian minimal normal subgroups of epimorphic images of G are finite.*
- (c) *If H is an epimorphic image of G with $\mathfrak{m}H \neq 1$, then there exists a normal subgroup $N \neq 1$ of H with finite N'.*

(iii)
- (a) *G is an $\mathfrak{mn}$-group.*
- (b) *G is an $\mathfrak{m}^*$-group.*
- (c) *If M is a minimal normal subgroup of the epimorphic image H of G, and if $M \subseteq \mathfrak{z}(\mathfrak{m}H)$, then M is finite.*

(iv)
- (a) *$G/\mathfrak{m}G$ and $\mathfrak{m}G$ are $\mathfrak{mn}$-groups.*
- (b) *G is an $\mathfrak{m}^*$-group.*

(v)
- (a) *$G/\mathfrak{m}G$ is an $\mathfrak{mn}$-group.*
- (b) *G is an $\mathfrak{m}^*$-group.*
- (c) *$\mathfrak{m}G$ is a torsion group of finite rank and the number of $\mathfrak{m}G$-relevant primes is finite.*

This is the special case of Theorem 3.1, obtained by letting $\mathfrak{e}$ be the universal property of just being a group.

4. When applying Theorem 3.1 two extreme special cases present themselves for inspection: the case dealt with in Corollary 3.3 where $\mathfrak{e}$ has been selected as the universal property of just being a group and at the other extreme the "minimal" property which, according to an observation made in section 2, is the property of being an $\mathfrak{m}$-group possessing an abelian subgroup of finite index. In the present section we shall concern ourselves with just this case.

Proposition 4.1. *The following properties of the almost abelian group G are equivalent:*

(i) *G is an $\mathfrak{m}$-group.*

(ii) *G is an $\mathfrak{mn}$-group.*

(iii) *G is a torsion group of finite rank and the number of G-relevant primes is finite.*

Terminological note. Groups possessing abelian subgroups of finite index are termed *almost abelian*.

Proof. By hypothesis there exists an abelian subgroup of finite index; and it is a fairly immediate consequence of Poincaré's Theorem that this abelian subgroup of finite index in G contains a necessarily abelian normal subgroup A of G with finite G/A.

It is clear that (ii) is a consequence of (i).— If (ii) is satisfied by G, then G induces in its abelian normal subgroup A a finite group Γ of automorphisms. Since the normal subgroups of G, contained in A, are just the Γ-admissible subgroups of A, the minimum condition is satisfied by the Γ-admissible subgroups of A. Application of Baer ([2], p. 4/5, Lemma 1) shows that the abelian torsion group A is an $\mathfrak{m}$-group. Hence A is the direct product of finitely many primary groups of rank 1; see Fuchs ([6], p. 65, Theorem 19.2). Thus A is a torsion group of finite rank and the number of A-relevant primes is finite. Since G/A is finite, it follows that G is a torsion group of finite rank and that the number of G-relevant primes is finite. Hence (iii) is a consequence of (ii).

If (iii) is satisfied by G, then A is an abelian torsion group of finite rank and the number of A-relevant primes is finite. Hence A is the direct product of finitely many primary groups A_p; and the number of elements of order p in A_p is finite. This implies that A is an $\mathfrak{m}$-group (see Fuchs [6], p. 68, 19). The extension of the $\mathfrak{m}$-group A by the finite group G/A is an $\mathfrak{m}$-group so that (i) is a consequence of (iii) proving the equivalence of (i)-(iii).

In the sequel we shall denote the *property of being almost abelian* by $\mathfrak{aa}$. It is trivial that this property meets the basic requirements I-III. Beyond these it satisfies:

E. *An extension of an* $\mathfrak{aam}$*-group by an* $\mathfrak{aam}$*-group is an* $\mathfrak{aam}$*-group.* (See Baer [4], p. 14, Folgerung 2.6.) It is clear that the properties IV and V are consequences of E. Thus $\mathfrak{aa}$ meets all the requirements I-V. We note furthermore that

$$\mathfrak{aam} = \mathfrak{aamn}$$

is a consequence of Proposition 4.1 — the equivalence of (i) and (ii).

THEOREM 4.2. *The following properties of the group G are equivalent:*

(i) *G is an $\mathfrak{aam}$-group.*

(ii)
- (a) *G is an $\mathfrak{mn}$-group.*
- (b) *G is an $(\mathfrak{aa})^*$-group.*
- (c) *If M is a minimal normal subgroup of the epimorphic image H of G, and if $M \subseteq \mathfrak{z}(\mathfrak{aa}H)$, then M is finite.*

(iii)
- (a) *$G/\mathfrak{aam}G$ and $\mathfrak{aam}G$ are $\mathfrak{mn}$-groups.*
- (b) *G is an $(\mathfrak{aa})^*$-group.*

(iv)
- (a) *$G/\mathfrak{aam}G$ is an $\mathfrak{mn}$-group.*
- (b) *G is an $(\mathfrak{aa})^*$-group.*
- (c) *G is a torsion group of finite rank and the number of G-relevant primes is finite.*

Proof. If we let $\mathfrak{e} = \mathfrak{aa}$ in Theorem 3.1, then the condition (i) of Theorem 3.1 and our present condition (i) are identical whereas our present conditions (ii) and (iii) are consequences of the conditions (iii) and (iv) of Theorem 3.1, respectively. Thus our present conditions (ii) and (iii) are consequences of (i).

If (ii) is true, then we deduce from (ii.a) that every epimorphic image H of G which is an $\mathfrak{aa}$-group is an $\mathfrak{aamn}$-group and hence an $\mathfrak{aam}$-group. Thus G is an $(\mathfrak{aam})^*$-group and we have $\mathfrak{aa}H = \mathfrak{aam}H$ for every epimorphic image H of G. Hence condition (iii) of Theorem 3.1 is true and this implies that G is an $\mathfrak{aam}$-group.

If (iii) is true, then we deduce from (iii.a) that G is an $\mathfrak{mn}$-group. It follows from (iii.b) and Proposition 4.1 that G is an $(\mathfrak{aam})^*$-group. Consequently condition (iv) of Theorem 3.1 is satisfied by G; and it follows that G is an $\mathfrak{aam}$-group. Thus we have shown the equivalence of our conditions (i)-(iii).

If the equivalent conditions (i)-(iii) are satisfied by G, then the conditions (iv.a) and (iv.b) are likewise true. The validity of (iv.c) is a consequence of Proposition 4.1. Assume conversely the validity of condition (iv). If the epimorphic image J of G is an $\mathfrak{aa}$-group, then J is by (iv.c) a torsion group of finite rank and the number of J-relevant primes is finite. Application of Proposition 4.1 shows that J is an $\mathfrak{aam}$-group. From (iv.b) we deduce that G is an $(\mathfrak{aam})^*$-group. Now it is clear that our present condition (iv) implies the validity of condition (v) of Theorem 3.1; and this proves the equivalence of our conditions (i)-(iv).

Remark 4.3. If $G/\mathfrak{aam}G$ is an $\mathfrak{mn}$-group, then we deduce from Lemma 2.2, (a) that $G/\mathfrak{aam}G$ is an $\mathfrak{aam}$-group; and this implies:

(iv.a*) $G/\mathfrak{aam}G$ is an $\mathfrak{aa}$-group.

It is clear that (iv.a*) is a weaker condition than (iv.a). But (iv.a) is a consequence of (iv.a*) and (iv.d), as follows from Proposition 4.1. Thus we may substitute for (iv.a) the weaker condition (iv.a*). Whether it is possible to dispense altogether with condition (iv.a), we have not been able to decide.

Remark 4.4. We have pointed out before that it is an open question whether or not all $\mathfrak{m}$-groups are $\mathfrak{aa}$-groups. If this should be true, then the conditions (iii) and (iv) of Corollary 3.3 would appear to be stronger than the conditions (ii) and (iii) of Theorem 4.2, respectively, whereas conditions (i) of Corollary 3.3 and (i) of Theorem 4.2 would be identical.

5. In this section we are going to construct some examples showing the indispensability of certain conditions appearing in our principal theorems.

EXAMPLE 5.1. Suppose that Σ is the group of all permutations of some countably infinite set $\mathfrak{S}$. The permutations in Σ which move only a finite number of elements and effect an even permutation in the [finite] subset of $\mathfrak{S}$ actually moved form a normal subgroup Λ of Σ, the so-called alternating group on $\mathfrak{S}$.

It is a well-known Theorem of Schreier-Ulam that

(1) Λ is a simple group.

For a proof, see Specht [7], p. 65, Satz 36.

If p is a prime, then we may divide $\mathfrak{S}$ into mutually disjoint $2p$-element subsets $\mathfrak{P}_i$. To every i there exists a permutation in Λ which leaves invariant every element, not in $\mathfrak{P}_i$, and which induces an even permutation of order p in $\mathfrak{P}_i$. These permutations commute pairwise and generate therefore an infinite elementary abelian p-group. Thus we have shown:

(2) Λ contains infinite elementary abelian p-groups.

Combination of (1) and (2) shows that

(3) Λ is an $\mathfrak{mn}$-group, but not an $\mathfrak{m}$-group.

Combining (1) and (3) with the fact that Λ is obviously not abelian we see that

(4) $\Lambda = \mathfrak{m}\Lambda$ and $\mathfrak{z}(\mathfrak{m}\Lambda) = 1$.

Since every permutation in Λ moves but a finite number of elements, its order is finite too, proving that

(5) Λ is a torsion group.

This example shows the indispensability of conditions (ii.c), (iii.b), (iv.b) of Corollary 3.3 and of conditions (ii.b), (iii.b) of Theorem 4.2.

Example 5.2. Denote by $\mathfrak{P} = \mathfrak{P}_p$ the essentially uniquely determined algebraically closed, absolutely algebraic field of characteristic p, a prime; and denote by $\mathfrak{E} = \mathfrak{E}_p$ the multiplicative group of the elements, not 0, in $\mathfrak{P}$. This is a group of roots of unity whose abstract characterization is given by the following properties:

$\mathfrak{E}_p$ is an abelian torsion group of rank 1; the p-component of $\mathfrak{E}_p$ is 1 and the q-component for $q \neq p$ is a group of Prüfer's type q^∞.

It is a consequence of these properties that

(a) isomorphic subgroups of $\mathfrak{E}_p$ are identical

and that

(b) the group Γ is isomorphic to a subgroup of $\mathfrak{E}_p$ if and only if Γ is an abelian torsion group of rank 1 without elements of order p.

Consider now any subgroup Γ of $\mathfrak{E}_p$. It is a subset of $\mathfrak{P}_p$ which spans a certain subring K of $\mathfrak{P}_p$. Since $\mathfrak{P}$ is an absolutely algebraic field, K is a subfield of $\mathfrak{P}$.

Denote now by A the additive group of the elements in the field K. Then A is an [additively written] elementary abelian p-group which may be finite or infinite. The subgroup Γ of K acts on A as group of automorphisms [by multiplication]; and K is the ring of endomorphisms of A spanned by Γ. It follows that 0 and A are the only Γ-admissible subgroups of A; a fact that we express shortly by saying:

(c) Γ is an irreducible group of automorphisms of the elementary abelian p-group A.

Combining (b) and (c) we obtain the statement:

(d) If Γ is an abelian torsion group of rank 1 without elements of order p, then Γ acts as an irreducible group of automorphisms on an elementary abelian p-group A.

If we consider A again as a multiplicatively written group, then it is a normal subgroup of its holomorph; and Γ is a subgroup of its holomorph. Within the holomorph of A we may form the product $G = A\Gamma$. It is clear that A is its own centralizer in G; and we deduce from (d) that A is a minimal normal subgroup of G. Thus we have translated (d) into the following statement:

(e) If Q is an abelian torsion group of rank 1 without elements of order p, then there exists an extension G of an elementary abelian p-group A by Q such that $A = \mathfrak{c}_G A$ is a minimal normal subgroup of G [and A is contained in every normal subgroup, not 1, of G].

Note. A similar construction, due to Ph. Hall, has been used by Duguid-McLain ([5], p. 398).

If we select in particular the group Q [in (e)] as an infinite $\mathfrak{m}$-group — for instance, as a group of Prüfer's type q^∞ — then we obtain a group G with the following properties:

G is a locally finite $\mathfrak{mn}$-group, but not an $\mathfrak{m}$-group;

$G'' = 1$; $G' = \mathfrak{m}G = \mathfrak{aa}G$;

$G/\mathfrak{m}G$ is an abelian $\mathfrak{m}$-group of rank 1;

G is an $(\mathfrak{aam})^*$-group.

This shows the indispensability of Conditions (ii.b), (iii.c), (iv. a, second half), (v.c) of Corollary 3.3 and of conditions (ii.c), (iii.a, second half), (iv.c) of Theorem 4.2. Note that G is a locally finite metabelian group.

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On certain elliptic functions of order three

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1. The parametrisation of the general plane cubic curve, in the form

$$y^2 = 4x^3 - g_2x - g_3, \tag{1}$$

where g_2, g_3 are constants, by means of the Weierstrassian elliptic functions

$$x = \wp u, \quad y = \wp' u$$

is familiar. There is, however, another canonical form of the equation of the cubic, in terms of homogeneous coordinates (x, y, z)

$$x^3 + y^3 + z^3 + 6mxyz = 0, \tag{2}$$

which from a geometrical point of view is at least as important as (1); and the elliptic functions by which this equation can be parametrised have not, so far as I know, received attention. A brief study of their outstanding properties is the object of this note.

2. We denote by Ω a lattice of complex numbers $\omega = p\omega_1 + q\omega_2$, where p, q range over all integers, and $I(\omega_2/\omega_1) > 0$. ($I(\tau)$ denoting the imaginary part of any complex number τ.) $n\Omega$ will denote the lattice of numbers $n\omega$ for all ω in Ω. ω_1, ω_2 are a basis for Ω. We define also

$$\omega_3 = -\omega_1 - \omega_2, \quad \omega_4 = \omega_1 - \omega_2.$$

Ω has four sublattices $\Omega^{(i)}$ $(i = 1, 2, 3, 4)$ (i.e. subgroups with respect to addition) of index three, with the bases

$$\begin{cases}\omega_1^{(1)} = \omega_1,\\ \omega_2^{(1)} = 3\omega_2,\end{cases} \quad \begin{cases}\omega_1^{(2)} = 3\omega_1,\\ \omega_2^{(2)} = \omega_2,\end{cases} \quad \begin{cases}\omega_1^{(3)} = 2\omega_1 - \omega_2,\\ \omega_2^{(3)} = -\omega_1 + 2\omega_2,\end{cases} \quad \begin{cases}\omega_1^{(4)} = 2\omega_1 + \omega_2,\\ \omega_2^{(4)} = \omega_1 + 2\omega_2,\end{cases} \tag{3}$$

of which $\Omega^{(i)}$ contains ω_i but none of the other three of $\omega_1, \omega_2, \omega_3, \omega_4$. 3Ω is a sublattice of index three in each of these; in fact, with the convention (3) as to their bases

$$3\Omega = (\Omega^{(1)})^{(2)} = (\Omega^{(2)})^{(1)} = (\Omega^{(3)})^{(4)} = (\Omega^{(4)})^{(3)}. \tag{4}$$