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by

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Louis Joel Mordell was born on 28 January, 1888 at Philadelphia, Pennsylvania. He attended the Central High School in that city, and showed his genius for mathematics at an early age. He conceived the ambition of studying at Cambridge, and in December, 1906 he travelled to England and sat for the open scholarship examination of one of the groups of Cambridge Colleges; he was placed highest on the list and was awarded a Major Scholarship at St. John's College. Recently he described his undertaking as "most audacious and hazardous, and justified only by its success". At that time he was very much a self-educated mathematician; what he had learnt at school had been of a comparatively elementary character, but he had read widely for himself. He has always retained an affection for books such as Horner's edition of Euler's *Algebra*, and some of the older American books on algebra, which first awakened his interest in indeterminate equations, the subject which has remained closest to his heart.

Mordell sat for the main mathematical examination, Part II of the Mathematical Tripos, in 1909, and this was the last year in which the names of the most successful candidates ('Wranglers') were published in order of merit. (The old system, in the reform of which Hardy played a leading part, led to excessive concentration on the subjects of examination, and, so directed the students' attention away from modern developments in mathematics.) Mordell was placed 3rd, the 1st being P. J. Daniell (of the 'Daniell integral') and the 2nd being E. H. Neville. Bracketed equal as 4th were W. E. H. Berwick, C. G. (later Sir Charles) Darwin, and G. H. Livens. Mordell is now the last survivor of this group of Cambridge mathematicians.

After a period of postgraduate study, Mordell became a lecturer at Birkbeck College, London, a College of the University of London concerned particularly with part-time students. From 1916 to 1919 he served in the Ministry of Munitions, after which he returned to Birkbeck College until 1920, when he went to a post at the Manchester College of Technology. In 1923 he was appointed Fielden Professor of Pure Mathematics in the University of Manchester, and continued to hold this office until 1945.

The years in Manchester were very fruitful ones, both in respect of Mordell's own researches and in respect of the influence that he exercised through his students and his younger colleagues. Many of these are now well-known mathematicians, established in various parts of the world. Among them may be mentioned R. Baer (Frankfurt), the late G. Billing (Stockholm), P. Erdös (Budapest), Chao Ko (China), K. Mahler (Canberra), B. Segre (Rome), J. A. Todd (Cambridge), P. Du Val (London), L. C. Young (Madison), and the present writer. There were also numerous visitors to Manchester in the 1930's; among them C. Chabauty (Grenoble), H. Heilbronn (Bristol), D. H. Lehmer (Berkeley). When one recalls the very small scale of mathematical activity in that age, both in England and in the world at large, as compared with the activity of today, one realizes that Mordell at Manchester exercised a notable influence.

Those of us who served under him as junior members of staff found him an admirable head of department. He was very conscious of his responsibilities, and made us conscious of ours, but at the same time he did everything possible to encourage us in our researches, and this independently of whether their subject-matter interested him personally or not.

In 1945 Mordell was elected to succeed Hardy as Sadleirian Professor of Pure Mathematics in the University of Cambridge, and was at the same time elected a Fellow of St. John's College. In the years that followed, he built up an active school of research in Cambridge, which specialized mainly, but not entirely, in the geometry of numbers. Among those who came under his influence, in varying degrees, at Cambridge may be mentioned R. P. Bambah (Panjab), E. S. Barnes (Adelaide), B. J. Birch (Manchester), J. W. S. Cassels (Cambridge), J. H. H. Chalk (Toronto), R. F. Churchhouse (Atlas Computer Laboratory) C. S. Davis (Brisbane), S. Knapowski (Poznań), A. M. Macbeath (Birmingham), P. Mullender (Amsterdam), K. Rogers (Los Angeles), P. A. Samet (Southampton), E. S. Selmer (Bergen), H. P. F. Swinnerton-Dyer (Cambridge).

Mordell's eminence as a mathematician has received recognition in many forms. In 1924 he was elected a Fellow of the Royal Society, and was awarded the Society's Sylvester Medal in 1949. He received the De Morgan Medal of the London Mathematical Society in 1941, and the Berwick Prize in 1946, and served as President of the Society from 1943 to 1945. He is a foreign member of the Academies of Science at Bologna, Oslo, and Uppsala, and has received honorary doctorates from the University of Glasgow and from Mount Allison University (New Brunswick).

In 1953 Mordell became Professor Emeritus in Cambridge, and entered upon a new phase in his career. He has since then served as Visiting Professor at the University of Toronto, at the University Colleges

of Ghana and Nigeria, at Mount Allison University, at the University of Colorado, at the University of Notre Dame, and is now at the University of Arizona. In addition he has given occasional lectures at a very large number of Universities and institutions in many countries. There can be few who have done as much in recent years to spread a love of mathematics in the world at large by their personal influence, and it is safe to say that Mordell's zest for his subject, and his love of learning in general, will have served as an inspiration to many young people, especially those in places that are remote from important scientific centres.

It is not easy to convey a true picture of Mordell's personality to those who have not known him. Among his leading qualities are a simplicity of outlook, and an absolute honesty of thought and purpose, which have made it easy for him to give advice and guidance to others without impinging on their freedom or dominating them in any way. He is very much an individualist himself (only one of his papers is a joint paper), and as such he has an innate respect for the individuality of others. Another of his qualities is a warmhearted generosity, which shows itself in an intense appreciation of the achievements of others — whether they are students, colleagues, or strangers to him. Many of us owe more than we can say to the stimulus given to us by his encouragement and by the deep interest he has taken in our attempted researches.

Mordell has thought much about some of the wider questions which inevitably concern us as mathematicians: questions of education and of the organization of scientific research, for instance. This is apparent from his *Reflections of a mathematician* [167] and from his recent article [195] on the Putnam Prize competition. For a mathematician of his standing, he has written an unusually large number of expository articles, and of book reviews which are in effect essays on various mathematical subjects.

Mordell has a natural genius for friendship, and his friends include men and women, both eminent and obscure, in every walk of life. In his social life he has had the invaluable help and support of Mrs. Mordell, who has endeared herself to all his friends, and has made their homes into centres of generous hospitality to students, colleagues and visitors. The Mordells have one son and one daughter, and it may be of interest to mention that the father's migration to England has been followed by the son's migration in the opposite direction: Donald Mordell is now Dean of the Faculty of Engineering at McGill University.

Mordell's intense devotion to mathematics has not absorbed all his time and energy. Among his favourite recreations are bridge, swimming, and walking — especially in the mountains — and formerly rock-climbing. We hope that he will long continue to enjoy a life of both fruitful work and enjoyable recreation.

The contributions which Mordell has made to mathematics can best be considered under five headings:

I. Diophantine equations;
II. Theta functions and modular functions;
III. The geometry of numbers;
IV. Congruences and exponential sums;
V. Other work.

It is interesting to note that, unlike some other mathematicians whose work on different subjects falls into distinct periods of time, Mordell has maintained a continuous interest in each of his four principal fields; and each of them is represented in recent, as well as in earlier, papers.

I. Diophantine equations

Although there is a vast literature on the subject of Diophantine (or 'indeterminate') equations, as one may see from Dickson's *History of the theory of numbers*, it is only in the present century that results of any real generality have been discovered, for equations of higher degree than the second. Mordell has been a great pioneer in these discoveries.

One may perhaps best classify the many types of Diophantine equation with which Mordell's work is concerned by using geometrical language. The equations can be regarded as relating to one of the following three topics:

(a) the integer points on a cubic curve;
(b) the rational points on a cubic curve;
(c) the integer points and rational points on a cubic surface.

In each case there may, however, be a variety of different forms of equation, giving rise to different arithmetical problems.

Mordell's first substantial paper [2] dealt with the integer solutions of the equation

$$y^2-k=x^3, \tag{1}$$

and so falls under (a). The account given there of the three possible methods of treatment can still be read with interest; the methods discussed were (i) elementary, using congruences, (ii) algebraic, using ideal theory, and (iii) a method which related the equation to the theory of binary cubic forms. The last of these may perhaps contain the first trace of the point of view which led to the famous 'basis theorem'. It is interesting to note that at this time Mordell and others still thought that some equations of the type (1) might have infinitely many solutions in integers. Mordell himself [15] was one of those who later proved that this is never the case.

But even today we have no systematic way of determining whether an equation of the form (1) has a solution or not. Other papers on (a) are [27] and [29], and the subject was discussed again in the inaugural lecture [120] of 1946.

The papers which deal with the question (b) are not numerous but are of the highest importance. They include [28], in which the basis theorem was stated and proved. Following Poincaré, one can regard the rational points on a cubic (elliptic) curve, if there are any, as an additive group, the group operation being given by the rule that $A+B+C=0$ means that the points A, B, C lie on a straight line. The basis theorem asserts that the group is finitely generated, that is, that there is a finite set $A_1, \ldots, A_m$ of rational points on the curve such that every rational point on the curve is representable uniquely as

$$t_1A_1+\ldots+t_mA_m,$$

where $t_1, \ldots, t_m$ are integers. This makes it possible, in principle, to survey the totality of all the points. The starting point for the proof came from [3], but the essential idea consists of a very ingenious use of the 'principle of infinite descent'.

The basis theorem has been of fundamental importance for all later work on Diophantine equations. It was generalized by Weil in 1929 to apply to rational systems of points on any algebraic curve, and this generalization in turn played an important part in Siegel's complete determination of the plane curves which have on them infinitely many rational points. (For an account, and references, see Skolem's *Diophantische Gleichungen*, Springer, 1935.)

It may be of interest to mention that the subject of rational points on cubic curves, after receiving only intermittent attention for several years, has now come into the foreground of mathematical activity, thanks to the work of Selmer, Tate and Šafarevič, and to the very recent work of Cassels and of Birch and Swinnerton-Dyer.

The papers that fall under (c) are many and are not easily summarized. The main theorem on rational points on cubic surfaces is that of Segre, dating from 1943, that a cubic surface has on it either 0, 1, or infinitely many rational points. The stimulus to the discovery of this result came from discussions with Mordell during the war years in Manchester, and the starting point of this body of work was Mordell's discovery [101] that Ryley's parametric solution, found in 1825, for

$$x^3+y^3+z^3=n$$

can be extended to the equation $x^3+y^3+z^3+dxyz=n$, for arbitrary rational d and n. As regards *integer* points on cubic surfaces, Mordell has given many types of surface on which there are an infinity [102, 125, 135, 146, 147, 152], and some for which there are only finitely many [139].

II. Theta functions and modular functions

Mordell is one of the few world experts on this important but somewhat technical subject. His contributions deal mainly with the applications of these functions to a variety of questions in the theory of numbers. Among them there are applications to

(a) *class-relation formulae* [6, 7, 12, 19, 21]. The class-number of binary quadratic forms of negative discriminant $-n$ satisfies various recurrence relations, and these are most easily deduced from the expression for the corresponding generating function in terms of the values of theta-functions $\theta(x, \omega)$ when $x = 0$. (There are also elementary methods of proof, but these are generally more in the nature of verifications of relations already discovered.) Mordell's investigations led him, rather unexpectedly, to the evaluation of the general definite integral mentioned below;

(b) *sums of squares* [13, 16]. The modular functions provide a powerful method for discovering exact formulae for the number of representations of an integer as the sum of a given number of squares. Mordell's work on this subject should be studied in conjunction with Hardy's, in which a different point of view is adopted;

(c) *particular arithmetical functions* [14, 25]. In the former of these papers, Mordell gave a simple and elegant proof of the multiplicative property of Ramanujan's function $\tau(n)$, defined by

$$\sum_{1}^{\infty} \tau(n) x^n = x\Big\{\prod_{m=1}^{\infty} (1-x^m)\Big\}^{24}.$$

This property had been conjectured empirically by Ramanujan. There remain, however, other conjectures of Ramanujan about $\tau(n)$ which still await proof. For further developments related to Mordell's work, see Hecke, Math. Werke, 644-671.

To this general subject there belong also the papers [5, 18, 68] concerning a type of definite integral, namely

$$\int_{-\infty}^{\infty} \frac{e^{ax^2+bx}}{e^{cx}+d}\,dx$$

particular cases of which had been studied by Kronecker and many later mathematicians. This integral is related to Gauss's sum and to the approximate functional equation of the theta-functions.

Other papers which fall into this general division are [34, 45, 50, 57].

III. The geometry of numbers

Mordell's interest in this subject—the creation of the genius of Minkowski—showed itself first in his simple proof [37] in 1928 of Minkowski's theorem on the product of two non-homogeneous linear forms. The interest has been an abiding one, but perhaps the most creative period was the decade 1940-50. The present writer well recalls the time, early in 1940, when Mordell told him that he was working on a most interesting problem in the geometry of numbers, which would throw new light on recent results concerning the product of three homogeneous linear forms. Mordell said then that he would not reveal what the problem was, lest it should be solved at once by one of the rest of us, then at Manchester. (I think his confidence in our powers, and lack of confidence in his own, was misplaced.) The problem related to the minimum of a binary cubic form, and the result which he duly proved is as follows. Let $f(x, y)$ be a binary cubic form, with real coefficients, of discriminant $D \neq 0$. Then there exist integers x, y (not both 0) such that

$$|f(x, y)| \leqslant \begin{cases} (D/49)^{1/4} & \text{if} \quad D > 0, \\ (-D/23)^{1/4} & \text{if} \quad D < 0. \end{cases}$$

These inequalities are best possible, the 'critical cases' being

$$f(x, y) = x^3+x^2y-2xy^2-y^3 \quad \text{or} \quad x^3-xy^2-y^3$$

as the case may be. These are theorems of classical elegance and simplicity. The first proofs [99, 103, 104, 107] depended on geometrical reasoning of some complexity, but simpler proofs were given later, both by Mordell and by others.

This result and others led Mordell to the considerable advances in the geometry of numbers represented by papers [116] and [117]. Here he developed methods for solving the fundamental problem of the subject for two dimensional non-convex regions of some generality. Other notable papers are [119, 123, 136]; and there is also an earlier group on the analytical approach [42, 46, 49, 51]. Mordell's work gave a great stimulus to that of others, and this is especially true of Professor Mahler and myself.

The subject of the geometry of numbers is one in which there is much less activity now than there was ten or twenty years ago. But the problems of the subject are of general interest and are of a fundamental character, and there can be little doubt that there will be a revival sometime.

IV. Congruences and exponential sums

When the present writer was a research student, he was given by Littlewood the problem of estimating the quadratic character-sum

$$\sum_x \left(\frac{(x+a)(x+b)(x+c)}{p}\right),$$

where p is a prime and a, b, c are mutually incongruent $(\bmod\, p)$, and the summation is over a complete set of residues $(\bmod\, p)$. In his first paper (J. London Math. Soc. 6 (1931), 49–54) it was proved that the sum is $O(p^{3/4})$. Mordell became interested in the problem, and by an ingenious new approach he improved the result to $O(p^{2/3})$ in [64] and [65]. More important than this, he connected the problem with that of estimating exponential sums of the form

$$\sum_x e^{2\pi i f(x)/p},$$

where $f(x)$ is a polynomial $(\bmod\, p)$, and he also directed attention to the more general question of approximating to the number of solutions of a congruence $f(x, y) \equiv 0 \,(\bmod\, p)$. In a paper [63] which had considerable influence on other mathematicians, he obtained the estimate $O(p^{1-1/n})$ for the above exponential sum, where n is the degree of $f(x)$ and where the implied constant depends only on n.

A notable advance in the congruence (or character-sum) problem was made by Hasse in 1934. He proved that if $f(x)$ is a cubic polynomial then

$$\left|\sum_x \left(\frac{f(x)}{p}\right)\right| < 2p^{1/2}.$$

This had, in fact, been conjectured by Artin in his memoir of 1924 on quadratic function-fields; it is substantially equivalent to the analogue of the Riemann hypothesis for a particular type of congruence zeta-function.

All these results have been superseded by the work of A. Weil, in which the analogue of the Riemann hypothesis was proved for the congruence zeta-function corresponding to any 'curve' $f(x, y) \equiv 0 \,(\bmod\, p)$, provided the curve is absolutely irreducible. Weil's work is based on the theories of algebraic geometry, and there seems to be a clearly defined limit to what can be proved by the direct methods used by Mordell and others in the early 1930's.

Mordell's interest in 'mod p' problems has been an enduring one, as in shown by several of his recent papers [158, 159, 190, 197, 200, 202, 203].

V. Other work

In a group of papers of about 1930 [56, 60, 61, 78], Mordell considered the problem of representing a binary quadratic form

$$ax^2 + 2hxy + by^2,$$

where $a \geq 0$ and $ab - h^2 > 0$, as a sum of squares of linear forms:

$$(a_1 x + \beta_1 y)^2 + \ldots + (a_n x + \beta_n y)^2,$$

where the a_j and β_j are integers. He showed that such a representation is always possible with $n = 5$, and not always with $n < 5$. The analogous result for rational a_j, β_j had been proved earlier by Landau.

The problem of solving two simultaneous homogeneous quadratic equations

$$Q_1(x_1, \ldots, x_n) = 0, \qquad Q_2(x_1, \ldots, x_n) = 0$$

in integers was attacked in [163]. It was proved, subject to certain reasonable conditions, that there is always a solution if $n \geq 13$. A more precise result, with 13 replaced by 11, has since been proved by Swinnerton-Dyer (in couse of publication in this volume).

In quite a different field of mathematics, Mordell is known for his simple and elegant proof [36] of Hadamard's gap theorem: if

$$f(m+1) > \lambda f(m)$$

for all m, where $\lambda > 1$, then the power series

$$\sum a_n z^{f(n)}$$

has its circle of convergence as a natural boundary.

Finally, I mention some of Mordell's conjectures and problems which have aroused widespread interest:

(i) There can only be a finite number of rational points on any curve of genus greater than 1.

(ii) Are there infinitely many integral solutions of $x^3 + y^3 + z^3 = 3$? The only known solutions are (1, 1, 1) and (4, 4, −5).

(iii) The general conjecture, that any homogeneous cubic equation in 4 unknowns has a non-trivial solution if and only if the corresponding congruence has a non-trivial solution for every modulus, has been disproved by Swinnerton-Dyer (Mathematika 9 (1962), 54–56). But some classes of equations are known for which the result is true, and it may be that it is true subject to some mild restrictions.

(iv) The series $\sum_{-\infty}^{\infty} a_n(1-xe^{2\pi i a n})^{-1}$, where $|x| < 1$ and a are real, obviously converges if $\sum |a_n|$ converges. Can one give a less stringent condition? See [194].

(v) A non-homogeneous cubic congruence $f(x, y, z) \equiv 0 \pmod{p}$ has $p^2 + O(p)$ solutions, apart from certain exceptions. (This has been substantially proved by Davenport and Lewis, using a result of Dwork; see Quart. J. of Math. (2), 14 (1963), 154-159.)

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