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p-adic logarithmic forms and group varieties II

by

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1. Introduction. The present paper is a continuation of the studies in Yu [10], where we have brought the p-adic theory of linear forms in logarithms more in line with the complex theory as in Baker and Wüstholz [1]. The purpose here is to refine upon our results in [10].

Let $\alpha_1, \ldots, \alpha_n$ $(n \geq 1)$ be non-zero algebraic numbers and K be a number field containing $\alpha_1, \ldots, \alpha_n$ with $d = [K : \mathbb{Q}]$. Denote by $\mathfrak{p}$ a prime ideal of the ring $\mathcal{O}_K$ of integers in K, lying above the prime number p, by $e_\mathfrak{p}$ the ramification index of $\mathfrak{p}$, and by $f_\mathfrak{p}$ the residue class degree of $\mathfrak{p}$. For $\alpha \in K$, $\alpha \neq 0$, write $\mathrm{ord}_\mathfrak{p}\, \alpha$ for the exponent to which $\mathfrak{p}$ divides the principal fractional ideal generated by α in K; define $\mathrm{ord}_\mathfrak{p}\, 0 = \infty$. We shall estimate $\mathrm{ord}_\mathfrak{p} \Xi$, where

$$\Xi = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1 \tag{1.1}$$

with $b_1, \ldots, b_n$ being rational integers and $\Xi \neq 0$. Let

$$h_j = \max(h_0(\alpha_j), \log p) \quad (1 \leq j \leq n),$$

where $h_0(\alpha)$ for algebraic α is defined by the formula below (1.5), and let B be given by (1.8). Then as a consequence of Theorem 1, we have

$$\mathrm{ord}_\mathfrak{p}\, \Xi < 19(20\sqrt{n+1}\, d)^{2(n+1)} e_\mathfrak{p}^{n-1} \cdot \frac{p^{f_\mathfrak{p}}}{(f_\mathfrak{p} \log p)^2} \cdot \log(e^5 nd)\, h_1 \cdots h_n \log B.$$

From now on we shall keep the notation introduced in the third paragraph of §1 in [10]. (For the self-containedness, we repeat part of it here.) We assume that K satisfies the following condition:

$$\begin{cases} \zeta_3 \in K & \text{if } p = 2, \\ \text{either } p^{f_\mathfrak{p}} \equiv 3 \pmod 4 \text{ or } \zeta_4 \in K & \text{if } p > 2, \end{cases} \tag{1.2}$$

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where $\zeta_m = e^{2\pi i/m}$ $(m = 1, 2, \ldots)$. Set

$$q = \begin{cases} 2 & \text{if } p > 2, \\ 3 & \text{if } p = 2. \end{cases} \tag{1.3}$$

Let $\mathbb{N}$ be the set of non-negative rational integers and set

$$u = \max\{k \in \mathbb{N} \mid \zeta_{q^k} \in K\}, \qquad \alpha_0 = \zeta_{q^u}. \tag{1.4}$$

Define

$$h'(\alpha_j) = \max\left(h_0(\alpha_j), \frac{f_{\mathfrak{p}} \log p}{d}\right) \qquad (1 \le j \le n), \tag{1.5}$$

where $h_0(\alpha)$ denotes the absolute logarithmic Weil height of an algebraic number α, i.e.,

$$h_0(\alpha) = \delta^{-1}\Big(\log a_0 + \sum_{i=1}^{\delta} \log \max(1, |\alpha^{(i)}|)\Big),$$

where the minimal polynomial for α is

$$a_0 x^{\delta} + a_1 x^{\delta-1} + \ldots + a_{\delta} = a_0(x - \alpha^{(1)}) \cdots (x - \alpha^{(\delta)}), \qquad a_0 > 0.$$

Let $\kappa \in \mathbb{N}$ and ϑ be defined by

$$\begin{aligned} &\phi(p^{\kappa}) \le 2e_{\mathfrak{p}} < \phi(p^{\kappa+1}), \\ &\vartheta = \begin{cases} (p-2)/(p-1) & \text{if } p \ge 5 \text{ with } e_{\mathfrak{p}} = 1, \\ p^{\kappa}/(2e_{\mathfrak{p}}) & \text{otherwise,} \end{cases} \end{aligned} \tag{1.6}$$

where ϕ is Euler's ϕ-function. Denote by

$$\Phi(\mathfrak{p}) = p^{f_{\mathfrak{p}}} - 1 \tag{1.7}$$

the Euler function of the prime ideal $\mathfrak{p}$. Let B be a real number satisfying

$$B \ge \max(|b_1|, \ldots, |b_n|, 3). \tag{1.8}$$

Let $\omega_q(n)$ $(n = 1, 2, \ldots)$ be the two sequences (for $q = 2, 3$) of positive rational numbers, defined in [10], §5, which we shall recall in §2 for self-containedness. We note that $\omega_q(n) \le n!/2^{n-1}$ for $n = 1, 2, \ldots$, and a lower bound for ϑ is given in §2, (2.8).

Theorem 1. *Suppose that*

$$\operatorname{ord}_{\mathfrak{p}} \alpha_j = 0 \qquad (1 \le j \le n). \tag{1.9}$$

If $\Xi = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1 \ne 0$, *then*

$$\operatorname{ord}_{\mathfrak{p}} \Xi < C(n, d, \mathfrak{p}) h'(\alpha_1) \cdots h'(\alpha_n) \log B,$$

where

$$(1.10)\quad C(n,d,\mathfrak{p}) = \frac{c}{e_{\mathfrak{p}}\vartheta}\max(p^{-f_{\mathfrak{p}}e_{\mathfrak{p}}\vartheta}, q^{-n})(aep^{\kappa})^n \cdot \frac{(n+1)^{n+2}}{(n-1)!}\cdot \omega_q(n) \times \frac{\Phi(\mathfrak{p})}{q^u}\cdot\frac{d^{n+2}}{(f_{\mathfrak{p}}\log p)^3}\cdot \max(f_{\mathfrak{p}}\log p, \log(e^4(n+1)d)),$$

with

$$a = 16,\ c = 712 \quad \text{if } p > 2,$$
$$a = 32,\ c = 58 \quad \text{if } p = 2.$$

Furthermore if $\alpha_1,\ldots,\alpha_n$ *satisfy*

$$(1.11)\qquad [K(\alpha_0^{1/q},\alpha_1^{1/q},\ldots,\alpha_n^{1/q}) : K] = q^{n+1},$$

then $C(n,d,\mathfrak{p})$ *can be replaced by* $C(n,d,\mathfrak{p})/\omega_q(n)$.

Let

$$(1.12)\qquad C^*(n,d,\mathfrak{p}) = C(n,d,\mathfrak{p})/(n+1),$$

where $C(n,d,\mathfrak{p})$ is given by (1.10).

THEOREM 2. *Suppose that* (1.9) *holds and*

$$(1.13)\qquad \operatorname{ord}_p b_n = \min_{1\le j\le n}\operatorname{ord}_p b_j.$$

Let B, B_n, Ψ *be such that*

$$(1.14)\qquad \max_{1\le j\le n}|b_j| \le B,\quad |b_n| \le B_n \le B,\quad \Psi = p^{f_{\mathfrak{p}}}(8n^3 d\log(5d))^n.$$

If $\Xi = \alpha_1^{b_1}\cdots\alpha_n^{b_n} - 1 \ne 0$, *then for all real* δ *with*

$$0 < \delta \le d^{n-1}h'(\alpha_1)\cdots h'(\alpha_{n-1})f_{\mathfrak{p}}\log p$$

we have

$$\operatorname{ord}_{\mathfrak{p}}\Xi < C^*(n,d,\mathfrak{p})d^{-n}\max(d^n h'(\alpha_1)\cdots h'(\alpha_n)\widetilde{h}, \delta B/B_n),$$

where

$$\widetilde{h} = 3\log(\delta^{-1}\Psi(d^{n-1}h'(\alpha_1)\cdots h'(\alpha_{n-1}))^2 B_n).$$

If $\alpha_1,\ldots,\alpha_n$ *satisfy condition* (1.11), *then* $C^*(n,d,\mathfrak{p})$ *can be replaced by* $C^*(n,d,\mathfrak{p})/\omega_q(n)$, *and* Ψ *in* (1.14) *can be replaced by* $\Psi = \max(p^{f_{\mathfrak{p}}}, (5n)^{2n}d)$.

It is straightforward to deduce, from Theorems 1 and 2, precise versions in terms of $K_0 = \mathbb{Q}(\alpha_1,\ldots,\alpha_n)$ and $\operatorname{ord}_{\mathfrak{p}_0}$, and without assuming $\operatorname{ord}_{\mathfrak{p}_0}\alpha_j = 0$ $(1 \le j \le n)$, where $\mathfrak{p}_0$ is a prime ideal of the ring of integers in K_0, also versions for $\alpha_1,\ldots,\alpha_n$ being rational (see Yu [9], III, §4).

For $n \ge 2$, Theorems 1 and 2 indicate that we can replace the term n^{n-1} that occurs in the expressions for $C(n,d,\mathfrak{p})$ and $C^*(n,d,\mathfrak{p})$ in [10], Theorems 1 and 2, by $c_{\mathfrak{p}}^{n-1}$ where $c_{\mathfrak{p}} = 2e\vartheta e_{\mathfrak{p}} f_{\mathfrak{p}}\log p$ with ϑ defined by (1.6). Plainly this gives an improvement on our results in [10] when $n > c_{\mathfrak{p}}$ and this is significant in applications. Indeed, the present paper and [10] have led to an improvement on Stewart and Yu [7] to the effect that 2/3 in the Theorem of [7] can be replaced by 1/3. (See our subsequent joint paper, *On the abc conjecture II*.) The refinement was stimulated by a lecture given by Matveev in Oberwolfach in 1996 (see [5]) in which he indicated that he could eliminate a term $n!$ from certain linear form estimates in the complex case. The crucial new idea in the present paper came out from discussions with M. Waldschmidt during his short visit to my university at the end of May 1996. This is to apply the pigeon-hole principle to the set of integral points $(\lambda_1, \ldots, \lambda_r)$, where $\lambda = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_r) \in \Lambda^{\clubsuit}$ with $\Lambda^{\clubsuit}$ defined by §10, (10.14), thereby constructing improved auxiliary rational functions (see §10). This idea is then incorporated into the whole structure of [10]. Thus, though stimulated by Matveev's lecture, our work in the p-adic case involves a different approach and it is in substance quite independent.

Throughout the remaining part of this paper, [10] will be quoted frequently; for convenience, we shall refer to formulae, theorems and so on in [10] by adjoining a ♣, e.g., (8.9)♣, Theorem 7.1♣, §6♣.

I am indebted to discussions with M. Waldschmidt mentioned above.

The search for constants c_0, c_1, c_3, c_4, c_5 in §9 was carried out by my wife Dehua Liu, who designed a program, using PARI GP 1.39, for the search and verification. I would like to express my warm gratitude to her for her great support.

2. Preliminaries. Let ς_1 and ς_2 be constants given by the following list:

$$\begin{aligned} &\text{(I)} && \text{if } p = 3 \text{ or } p = 5 \text{ with } e_{\mathfrak{p}} \geq 2, \text{ then } \varsigma_1 = 1.2133,\ \varsigma_2 = 1.0202; \\ &\text{(II)} && \text{if } p \geq 5 \text{ with } e_{\mathfrak{p}} = 1, \text{ then } \varsigma_1 = 1.3128,\ \varsigma_2 = 1.0415; \\ &\text{(III)} && \text{if } p \geq 7 \text{ with } e_{\mathfrak{p}} \geq 2, \text{ then } \varsigma_1 = 1.3701,\ \varsigma_2 = 1.0568; \\ &\text{(IV)} && \text{if } p = 2, \text{ then } \varsigma_1 = 1.176,\ \varsigma_2 = 1.014. \end{aligned} \tag{2.1}$$

Let $r \in \mathbb{Z}$ be such that

$$r/(f_{\mathfrak{p}} \log p) > e_{\mathfrak{p}} \vartheta + 0.6, \tag{2.2}$$

where ϑ is given by (1.6). Set

$$\varrho = (f_{\mathfrak{p}} \log p)/r, \qquad \chi(x) = e^{-\varrho x}(x + e_{\mathfrak{p}} \vartheta), \qquad x_0 = 1/\varrho - e_{\mathfrak{p}} \vartheta. \tag{2.3}$$

Thus $x_0 > 0.6$ and $x_0 \notin \mathbb{Q}$, since $\log p \notin \mathbb{Q}$. Further we let $\mathfrak{m}$ and $\mathfrak{N}$ be defined by

$$\mathfrak{m} = \begin{cases} 1 & \text{if } 0.6 < x_0 < 1, \\ [x_0] & \text{if } x_0 > 1 \text{ and } \{x_0\} \leq 1/\varrho - 1/(e^{\varrho} - 1), \\ [x_0] + 1 & \text{if } x_0 > 1 \text{ and } \{x_0\} > 1/\varrho - 1/(e^{\varrho} - 1), \end{cases} \tag{2.4}$$

$$e_{\mathfrak{p}} \mathfrak{N} = (1 + (2n)^{-1} \cdot 10^{-100})^{-1} (\mathfrak{m} + e_{\mathfrak{p}} \vartheta). \tag{2.5}$$

REMARK (The roles of $\mathfrak{m}$ and $\mathfrak{N}$). Here we introduce $\mathfrak{m}$ and $\mathfrak{N}$ under assumption (2.2), which will occur as (8.12) in §8. We shall define an equivalence relation on $\Lambda^{\clubsuit}$ (see §10, (10.14)) by (10.15), in which $\mathfrak{m}$ will play an important role. Note that $\mathfrak{N}$ will appear in §11, (11.24). In contrast, without assumption (2.2), we take $\mathfrak{m} = 0$ in (2.5); then $\mathfrak{N}$ becomes θ which is defined by (8.1)$^{\clubsuit}$, and (11.24) degenerates into (11.24)$^{\clubsuit}$, thus we return to the construction in [10].

LEMMA 2.1. *We have*

$$\mathfrak{m} + e_{\mathfrak{p}}\vartheta < \varsigma_1 r/(f_{\mathfrak{p}} \log p), \tag{2.6}$$

$$1/\chi(\mathfrak{m}) < \varsigma_2/\chi(x_0). \tag{2.7}$$

Proof. Note that we have, by (1.6),

$$\vartheta \geq \vartheta_0 \quad \text{with } e_{\mathfrak{p}}\vartheta_0 := 3/2,\ 3/4,\ 1/2,\ 2, \tag{2.8}$$

for cases (I), (II), (III), (IV) in (2.1). We first prove (2.6). If $0.6 < x_0 < 1$, then

$$\mathfrak{m} + e_{\mathfrak{p}}\vartheta < \frac{1}{\varrho}(1 + 0.4\varrho) < \frac{1}{\varrho}\left(1 + \frac{0.4}{e_{\mathfrak{p}}\vartheta + 0.6}\right) < \frac{\varsigma_1 r}{f_{\mathfrak{p}} \log p}.$$

If $x_0 > 1$ and $\{x_0\} > 1/\varrho - 1/(e^{\varrho} - 1)$, then, by (2.8) and (2.1),

$$\mathfrak{m} + e_{\mathfrak{p}}\vartheta < \frac{1}{\varrho} \cdot \frac{\varrho e^{\varrho}}{e^{\varrho} - 1} < \frac{1}{\varrho}\left(\frac{x e^x}{e^x - 1}\right)_{x = 1/(e_{\mathfrak{p}}\vartheta + 1)} \leq \frac{\varsigma_1 r}{f_{\mathfrak{p}} \log p}.$$

(2.6) is trivially true for the remaining case of (2.4).

We now show (2.7). If $0.6 < x_0 < 1$, then by (2.8) and (2.1),

$$\chi(\mathfrak{m})/\chi(x_0) = e^{-\varrho(1-x_0)}(1 + \varrho(1 - x_0)) \geq (e^{-x}(1 + x))_{x = 0.4/(e_{\mathfrak{p}}\vartheta + 0.6)} > 1/\varsigma_2.$$

If $x_0 > 1$ and $\{x_0\} \leq 1/\varrho - 1/(e^{\varrho} - 1)$, then by (2.8) and (2.1),

$$\chi(\mathfrak{m})/\chi(x_0) = e^{\varrho\{x_0\}}(1 - \varrho\{x_0\}) \geq (e^x(1 - x))_{x = x_1} > 1/\varsigma_2,$$

where $x_1 = (1 - y/(e^y - 1))_{y = 1/(e_{\mathfrak{p}}\vartheta + 1)}$.

If $x_0 > 1$ and $\{x_0\} > 1/\varrho - 1/(e^{\varrho} - 1)$, then by (2.8) and (2.1),

$$\chi(\mathfrak{m})/\chi(x_0) = e^{-\varrho(1-\{x_0\})}(1 + \varrho(1 - \{x_0\})) \geq (e^{-x}(1 + x))_{x = x_2} > 1/\varsigma_2,$$

where $x_2 = 1/(e_{\mathfrak{p}}\vartheta + 1) - x_1$. The proof of (2.7) and the lemma is thus complete.

Recall $\omega_2(n)$ and $\omega_3(n)$ defined in §5$^{\clubsuit}$. That is, for $q = 2, 3$ we define $\omega_q(1) = \omega_q(2) = 1$ and for $n > 2$,

$$\omega_2(n) = 4^{s-n} \cdot (s + n + 1)!/(2s + 1)!, \tag{5.1$^{\clubsuit}$}$$

$$\omega_3(n) = 6^{t-n} \cdot (2t + n + 1)!/(3t + 1)!, \tag{5.2$^{\clubsuit}$}$$

where

$$s = [1/4 + \sqrt{n + 17/16}] \tag{5.3$^{\clubsuit}$}$$

and t is the unique rational integer such that

(5.4)$^\clubsuit$ $\quad g(t) := 9t^3 - 8t^2 - (8n+5)t - 2n(n+1) \le 0 \quad \text{and} \quad g(t+1) > 0.$

Hence $t = [x_n]$, where x_n is the unique real zero of $g(x)$, which can be determined explicitly by Cardano's formula.

Lemma 2.2. *We have*, *for* $n = 2, 3, \ldots$,

$$\frac{\omega_2(n)}{\omega_2(n-1)} \ge \frac{n+2}{4}, \tag{2.9}$$

$$\frac{\omega_3(n)}{\omega_3(n-1)} \ge \frac{n+4}{6}. \tag{2.10}$$

Proof. It is easy to verify that (2.9) holds for $2 \le n \le 6$. Write $s = s(n)$ for s defined by (5.3)$^\clubsuit$. Suppose now $n \ge 7$. Then $s(n) \ge 3$ and $s(n) - s(n-1) \in \{0, 1\}$. If $s(n) - s(n-1) = 0$, then by (5.1)$^\clubsuit$,

$$\frac{\omega_2(n)}{\omega_2(n-1)} = \frac{s+n+1}{4} > \frac{n+2}{4}.$$

If $s(n) - s(n-1) = 1$, then by (5.1)$^\clubsuit$ and the discussion in [10] following (5.20)$^\clubsuit$, we have

$$\frac{\omega_2(n)}{\omega_2(n-1)} = \frac{(s+n+1)(s+n)}{(2s+1)2s} \ge \frac{s+n}{4} > \frac{n+2}{4}.$$

This completes the proof of (2.9).

It remains to show (2.10). Write $t = t(n)$ for t defined by (5.4)$^\clubsuit$ and let

$$g(x, y) = 9x^3 - 8x^2 - (8y+5)x - 2y(y+1).$$

We now prove that

$$t(n) - t(n-1) \in \{0, 1\}, \qquad n = 5, 6, \ldots \tag{2.11}$$

By PARI GP 1.39, it is easy to verify that (2.11) holds for $5 \le n \le 31$. If $n \ge 32$, then $g(0.5n^{2/3} + 1, n) < g(0.6n^{2/3}, n) < 0$, whence by (5.4)$^\clubsuit$,

$$t = t(n) > 0.5n^{2/3} \qquad (n \ge 32). \tag{2.12}$$

In order to prove (2.11), it suffices to verify

$$g(t-1, n-1) \le 0 \qquad (n \ge 32). \tag{2.13}$$

Now by $n \ge 32$ and (2.12) we obtain

$$g(t, n) - g(t-1, n-1) = 27t^2 - 51t - 12n + 20 > 0.$$

This together with $g(t, n) \le 0$ (by (5.4)$^\clubsuit$) yields (2.13) and (2.11).

Now we are ready to prove (2.10). (2.10) is trivially true for $n = 2, 3, 4$. Note that $t = t(n) \ge 3$ $(n \ge 5)$. For $n \ge 5$ with $t(n) - t(n-1) = 0$, we get, by (5.2)$^\clubsuit$,

$$\frac{\omega_3(n)}{\omega_3(n-1)} = \frac{2t+n+1}{6} > \frac{n+4}{6};$$

and for $n \geq 5$ with $t(n) - t(n-1) = 1$, by (5.2)$^{\clubsuit}$ and the discussion in [10] following (5.20)$^{\clubsuit}$, we obtain

$$\frac{\omega_3(n)}{\omega_3(n-1)} = \frac{(2t+n+1)(2t+n)(2t+n-1)}{(3t+1)3t(3t-1)} \geq \frac{2t+n-1}{6} > \frac{n+4}{6}.$$

This completes the proof of (2.10) and the lemma.

We shall rely on §2$^{\clubsuit}$–§6$^{\clubsuit}$. For the convenience of exposition, we shall number the remaining sections as §7–§15, corresponding to §7$^{\clubsuit}$–§15$^{\clubsuit}$.

7. A central result. We now state a central result, which implies Theorems 1 and 2 of §1. We maintain the notation introduced in §1.

THEOREM 7.1. *Suppose that* (1.9) *and* (1.13) *hold*, $\alpha_1, \ldots, \alpha_n$ *are multiplicatively independent*, *and* $b_1, \ldots, b_n$ *are not all zero. Then*

$$\operatorname{ord}_{\mathfrak{p}} \Xi < C^*(n, d, \mathfrak{p}) h'(\alpha_1) \cdots h'(\alpha_n)(h^* + \log c^*), \tag{7.1}$$

where $C^*(n,d,\mathfrak{p})$ *is given by* (1.12), c^* *is given by* (5.24)$^{\clubsuit}$, *and*

$$h^* = \max\left\{ \log\left(\frac{f_{\mathfrak{p}} \log p}{2d} \max_{1\leq j<n} \left(\frac{|b_n|}{h'(\alpha_j)} + \frac{|b_j|}{h'(\alpha_n)} \right)\right), \right. \left. \log B^\circ, 6n\log(5n) + 1.2\log d, 2f_{\mathfrak{p}} \log p \right\} \tag{7.2}$$

with

$$B^\circ = \min_{1\leq j\leq n, b_j\neq 0} |b_j|. \tag{7.3}$$

Furthermore if $\alpha_1, \ldots, \alpha_n$ *satisfy* (1.11), *then* $C^*(n,d,\mathfrak{p})$ *and* $h^* + \log c^*$ *can be replaced by* $C^*(n,d,\mathfrak{p})/\omega_q(n)$ *and* h^*, *respectively.*

REMARK. If $n/(f_{\mathfrak{p}} \log p) < e_{\mathfrak{p}}\vartheta + 0.6$, then Theorem 7.1 is a consequence of Theorem 7.1$^{\clubsuit}$.

Proof (of Remark). We note, on recalling (1.10)$^{\clubsuit}$ and (1.10), that

$$\frac{a^{\clubsuit}}{a} = \begin{cases} (p-1)/(2(p-2)) & \text{if } p \geq 5 \text{ with } e_{\mathfrak{p}} = 1, \\ 1 & \text{otherwise.} \end{cases}$$

From (1.10)$^{\clubsuit}$, (1.12)$^{\clubsuit}$, (1.10), (1.12), (1.6) and (2.8) we get

$$\begin{aligned} \frac{C^*(n,d,\mathfrak{p})^{\clubsuit}}{C^*(n,d,\mathfrak{p})} &\leq \frac{c^{\clubsuit}}{c}\left(\frac{a^{\clubsuit}}{a}\right)^n \left(\frac{q}{ep^{\kappa}/(e_{\mathfrak{p}}\vartheta)}\right)^n \left(\frac{n}{f_{\mathfrak{p}}\log p} \cdot \frac{1}{e_{\mathfrak{p}}\vartheta}\right)^{n-1} \\ &< \frac{c^{\clubsuit}}{c}\left(\frac{a^{\clubsuit}}{a}\right)^n \left(\frac{q}{ep^{\kappa}/(e_{\mathfrak{p}}\vartheta)}\right)^n \left(1 + \frac{0.6}{e_{\mathfrak{p}}\vartheta}\right)^{n-1} < 1. \end{aligned} \tag{7.4}$$

Now by (7.4), Theorem 7.1 follows from Theorem 7.1$^{\clubsuit}$.

Thus we may assume

$$n/(f_{\mathfrak{p}} \log p) > e_{\mathfrak{p}}\vartheta + 0.6 \tag{7.5}$$

in §8–§13, where we shall prove Theorem 7.1.

8. Basic hypothesis. Let $b_1, \ldots, b_n$ be the rational integers in Theorem 7.1, satisfying (1.13), and set

$$L = b_1 z_1 + \ldots + b_n z_n. \tag{8.1}$$

Let ν be defined as in §5♣, $l_0, l_1, \ldots, l_n$ be defined by (5.7)♣.

Our *basic hypothesis* is that *there exists a set of linear forms* $L_0, L_1, \ldots, L_r$ *in* $z_0, z_1, \ldots, z_n$ *with rational integer coefficients having the following properties*:

(i) $L_0 = q^\nu z_0$; $L_0, L_1, \ldots, L_r$ *are linearly independent, and*

$$L = B_0 L_0 + B_1 L_1 + \ldots + B_r L_r \tag{8.2}$$

for some rationals $B_0, B_1, \ldots, B_r$, *with* $B_r \neq 0$.

(ii) *On writing*

$$l'_i = q^{-\nu} L_i(l_0, l_1, \ldots, l_n) \quad (1 \le i \le r),$$

the numbers $\alpha'_i = e^{l'_i}$ $(1 \le i \le r)$ *are in* K, *and satisfy* $\operatorname{ord}_{\mathfrak{p}} \alpha'_i = 0$ $(1 \le i \le r)$ *and*

$$[K(\alpha_0^{1/q}, {\alpha'_1}^{1/q}, \ldots, {\alpha'_r}^{1/q}) : K] = q^{r+1}. \tag{8.3}$$

(iii) *We have*

$$h'(\alpha'_i) \le \sigma_i \quad (1 \le i \le r), \tag{8.4}$$

$$\sum_{j=1}^{n} |\partial L_i / \partial z_j| h'(\alpha_j) \le q^\nu \sigma_i \quad (1 \le i \le r) \tag{8.5}$$

for some positive real numbers $\sigma_1, \ldots, \sigma_r$ *satisfying*

$$\sigma_1 \cdots \sigma_r \le \psi(r) h'(\alpha_1) \cdots h'(\alpha_n), \tag{8.6}$$

where

$$\psi(r) = \left(\frac{a}{q} e^2 p^\kappa d\right)^{n-r} \omega_q(n) \min\left(\frac{c^*}{q^\nu}, 1\right) \tag{8.7}$$

with a *given by* (1.10) *and* c^* *given by* (5.24)♣. *Furthermore, if* $\nu = 0$ *then* $\psi(r)$ *in* (8.6) *is replaced by* $\psi(r)/\omega_q(n)$.

The construction of §5♣ establishes the existence of linear forms as above for $r = n$. *We now take* r *as the least integer for which such a set of linear forms exists. Hence* $B_i \neq 0$ $(1 \le i \le r)$ *and we may assume*

$$(8.8) \qquad \sigma_r \geq \sigma_i \quad (1 \leq i \leq r)$$

in our basic hypothesis.

REMARK. The difference between the basic hypotheses in §8$^♣$ and here is only at (8.9)$^♣$ and (8.7).

LEMMA 8.1. *If* $r = 1$, *then Theorem* 7.1 *holds.*

Proof. Similar to the proof of Lemma 8.2$^♣$.

By Lemma 8.1, *we may assume* $r \geq 2$ *in our basic hypothesis.*

LEMMA 8.2. *If* r *in the basic hypothesis satisfies*

$$(8.9) \qquad r \geq 2 \quad \text{and} \quad r/(f_{\mathfrak{p}} \log p) < e_{\mathfrak{p}}\vartheta + 0.6,$$

then Theorem 7.1 *holds.*

Proof. We assert that, under (8.9), if the basic hypothesis in §8$^♣$ is replaced with that of this section, then all arguments in §9$^♣$–§13$^♣$ remain valid. Hence Proposition 9.1$^♣$, with r given in the basic hypothesis of this section and with the choice of parameters and constants of §9$^♣$, holds. To see this, we note that (8.9) and (2.8) yield

$$(8.10) \qquad \left(\frac{r}{f_{\mathfrak{p}} \log p} \cdot \frac{1}{e_{\mathfrak{p}}\theta}\right)^{r-\rho} < \left(\frac{e_{\mathfrak{p}}\vartheta + 0.6}{e_{\mathfrak{p}}\theta}\right)^{r-\rho} < e^{r-\rho},$$

where θ is given by (8.1)$^♣$ and ρ is the integer with $1 \leq \rho < r$ appearing in the text of [10] above (13.19)$^♣$. Further by the inequalities in the text of [10] above (13.9)$^♣$,

$$(8.11) \qquad \eta^{(r+1)I}/\eta^{\rho} < \eta^{(r+1)(I-1)} < 10^{-21}.$$

On observing (8.10) and (8.11), it is readily seen, similarly to proving that (13.24)$^♣$ implies (13.23)$^♣$ in [10], that in order to prove (13.23)$^♣$, it suffices to show

$$\frac{e^{\rho}(\rho!)^3 \rho^{\rho} r^{\rho} p^{f_{\mathfrak{p}}}}{\frac{1}{r} \cdot \frac{2}{3}(q-1)\frac{e_{\mathfrak{p}}}{d} - \frac{10^{-21}}{g_2}} \leq (q\eta^{r+1})^{I\rho},$$

which is a consequence of (13.24)$^♣$ (it is still true; see §13$^♣$).

Now we deduce Theorem 7.1 for $\nu > 0$. On replacing (8.8)$^♣$, (8.9)$^♣$ by (8.6), (8.7), and on observing $(n+1)^{n+1}/(n-1)! \geq e^{n-r}(r+1)^{r+1}/(r-1)!$, and by (8.9) and (2.8), we have

$$\frac{e_{\mathfrak{p}} U^♣}{(q^{-n}/\max(p^{-f_{\mathfrak{p}} e_{\mathfrak{p}}\vartheta}, q^{-n})) \times \text{the right side of (7.1)}} \leq \frac{c^♣}{c}\left(\frac{q}{2e}\right)^r \left(\frac{r}{f_{\mathfrak{p}} \log p} \cdot \frac{1}{e_{\mathfrak{p}}\vartheta}\right)^{r-1} \leq \frac{c^♣}{c}\left(\frac{q}{2e}\right)^r \left(\frac{e_{\mathfrak{p}}\vartheta + 0.6}{e_{\mathfrak{p}}\vartheta}\right)^{r-1} \leq 1.$$

The verification in the case $\nu = 0$, i.e., when $\alpha_1, \ldots, \alpha_n$ satisfy (1.11), is similar. This completes the proof of Lemma 8.2.

By Lemma 8.2, *we may and shall assume*, *in* §9–§13 *below*, *that* r *in the basic hypothesis satisfies*

$$r/(f_{\mathfrak{p}} \log p) > e_{\mathfrak{p}}\vartheta + 0.6. \tag{8.12}$$

Finally, note that we shall use $(8.13)^{\clubsuit}$–$(8.16)^{\clubsuit}$ in §9–§13.

9. Choices of parameters and Proposition 9.1. Recall that κ, ϑ are defined by (1.6), and ς_1, ς_2 are given by (2.1). We define $g_0, \ldots, g_{12}, \epsilon_1, \epsilon_2, f_6$ by the following formulae:

$$\begin{aligned}
&g_0 = 6r\log(5r) + 1.2\log d,\\
&g_1 = \log(e^4(r+1)d), \quad g_2 = 2c_3q(r+1)d,\\
&g_3 = 2c_0c_4(\varsigma_2c_2qp^{\kappa})^r(p^{f_{\mathfrak{p}}}-1)p^{0.6f_{\mathfrak{p}}}(f_{\mathfrak{p}}\log p)^{r-1}\frac{(r+1)^{r+1}}{r!}(q-1)\frac{g_1}{\vartheta},\\
&g_4 = 2c_0c_4(\varsigma_2q)^r(c_2p^{\kappa})^{r-1}p^{0.6f_{\mathfrak{p}}}(f_{\mathfrak{p}}\log p)^{r-1}\frac{(r+1)^r}{r!r}f_{\mathfrak{p}}\left(e_{\mathfrak{p}}+\frac{1}{\vartheta}\right)g_1,\\
&1+\epsilon_1 = (1+r/g_3)^r, \quad 1+\epsilon_2 = (1-1/g_4)^{-r},\\
&g_5 = g_4(p^{f_{\mathfrak{p}}}-1)(q-1)/(qf_{\mathfrak{p}}),\\
&g_6 = 2c_0c_3(\varsigma_2c_2qp^{\kappa})^r(p^{f_{\mathfrak{p}}}-1)p^{0.6f_{\mathfrak{p}}}(f_{\mathfrak{p}}\log p)^{r-1}\frac{(r+1)^{r+1}}{r!}(q-1)e_{\mathfrak{p}},\\
&g_7 = g_5c_1c_2rp^{\kappa}, \quad g_8 = g_7f_{\mathfrak{p}}\log p,\\
&g_9 = \frac{d}{g_2g_8}\left(\log\left(\frac{g_2}{d}\right) + (r+1)\log g_8\right),\\
&g_{10} = \frac{2}{g_0}\exp(-1+e^{-g_0-0.9})\frac{1}{c_2qp^{\kappa}}\cdot\frac{r-1}{r+1},\\
&g_{11} = \frac{\varsigma_1}{c_2c_5q}\cdot\frac{1}{p^{\kappa}f_{\mathfrak{p}}\log p}\cdot\left(q\left(1-\frac{c_5}{r+1}\right)^{r+1}\right)^{-\log g_5/\log q},\\
&g_{12} = \frac{1}{g_2}\left(\frac{d-1}{g_7} + \frac{d\log d}{2g_8}\right),\\
&f_6 = (1+10^{-100})(1+\epsilon_1)(1+\epsilon_2)(2+1/g_2)\varsigma_1c_0c_1c_3c_4q^2.
\end{aligned} \tag{9.1}$$

In (9.1), $c_0, \ldots, c_5$ are given in Table (9.2) below. The upper bounds for f_6 can be obtained from the above formulae by direct calculation. Blocks I, II, III, IV are for cases (I), (II), (III), (IV) of (2.1), respectively. The lower bounds $r \geq 3$ in I, II, III and $r \geq 4$ in IV are determined by (8.12), (2.8) and the fact that $f_{\mathfrak{p}} \geq 2$ when $p = 2$ (see [9], II, Appendix).

(9.2)

Case	r	c_0	c_1	$\varsigma_2 c_2$	c_3	c_4	c_5	$f_6 \leq$
I	$3 \leq r \leq 6$	2.7	0.6743	4	1.22	18.7	0.48	414
	$7 \leq r \leq 15$	2.6	0.655	4	0.986	20.72	0.512	344
	$r \geq 16$	2.5	0.65	4	0.856	23.53	0.54	321
II	$3 \leq r \leq 6$	2.8	0.6718	4	1.75	9.16	0.48	323
	$7 \leq r \leq 15$	2.7	0.652	4	1.418	10.15	0.512	270
	$r \geq 16$	2.6	0.6483	4	1.225	11.53	0.54	252
III	$3 \leq r \leq 6$	3	0.6546	4	1.93	16.98	0.47	712
	$7 \leq r \leq 15$	2.8	0.6425	4	1.612	19.01	0.504	608
	$r \geq 16$	2.7	0.6403	4	1.43	21.2	0.53	576
IV	$4 \leq r \leq 6$	2.7	0.7823	32/9	0.4082	3.107	0.7	58
	$7 \leq r \leq 15$	2.5	0.7646	32/9	0.3416	3.44	0.75	49
	$r \geq 16$	2.26	0.7458	32/9	0.31	3.835	0.8	43

Let

$$\eta = 1 - \frac{c_5}{r+1}. \tag{9.3}$$

It can be verified that $c_0, \ldots, c_5$ satisfy (9.4)–(9.7) and (9.9) below.

(9.4)

$$\text{(i)}\ 2c_5\eta^{r-1}\left(1 - \frac{1}{2g_2}\right) - \frac{\log q}{\log(q\eta)}\left(\frac{2}{c_2} + \left(1+\frac{1}{g_6}\right)\frac{\tau(p)}{c_4(\mathfrak{N}/\vartheta)g_1} \cdot \frac{1}{q}\right) > 0, \quad r \leq 15,$$

$$\text{(ii)}\ 2c_5e^{-c_5}\left(1-\frac{1}{2g_2}\right) - \frac{\log q}{\log(q\eta)}\left(\frac{2}{c_2} + \left(1+\frac{1}{g_6}\right)\frac{\tau(p)}{c_4(\mathfrak{N}/\vartheta)g_1} \cdot \frac{1}{q}\right) > 0, \quad r \geq 16,$$

where $\tau(p) = 1$ or $3/2$, according as $p > 2$ or $p = 2$,

$$\begin{aligned} 2c_5 q\left(1 - \frac{1}{2g_2}\right) &\geq c_1\left\{g_{12} + \left(1 + \frac{1}{2(c_0-1)}\right)g_9\right\} \\ &+ \left\{q + \frac{1}{2(c_0-1)}\left(1 + \frac{1}{2g_2+1}\right)\right\}\frac{2}{c_2} \\ &+ \left\{\frac{107}{103} \cdot \frac{1+10^{-100}}{e_{\mathfrak{p}}\vartheta_0+1}\left(1 - \frac{c_5}{r+1} + \frac{1}{c_0-1}\right) + \left(1 + \frac{1}{c_0-1}\right)g_{10}\right\}\frac{1}{c_3} \\ &+ \left(1 + \frac{1}{g_6}\right)\left\{1 + \frac{1}{c_0-1} + \left(\mathfrak{N} + \frac{1}{p-1}\right)\frac{1}{f_{\mathfrak{p}}}\right\}\frac{1}{c_4(\mathfrak{N}/\vartheta)}, \end{aligned} \tag{9.5}$$

where ϑ_0 and $\mathfrak{N}$ are defined by (2.8) and (2.5),

$$c_1 \geq c_5(\eta^r + g_{11})\left\{2 + \frac{1}{g_2} + \frac{1}{(r+1)q^{r+1}} \cdot \frac{1+10^{-100}}{e_{\mathfrak{p}}\vartheta_0 + 1} \cdot \frac{1}{c_3}\right\}, \tag{9.6}$$

(9.7)

$$\text{(i)}\quad \left(1+\frac{1}{g_6}\right)\frac{c_2}{2c_4(\mathfrak{N}/\vartheta)} \cdot \frac{\log q}{q} \cdot \frac{I}{\max\,(f_{\mathfrak{p}}\log p, g_1)} + \frac{1}{(q\eta^{r+1})^I} \leq 1 \quad \text{if } p > 2,$$

$$\text{(ii)}\quad \left(1+\frac{1}{g_6}\right)\frac{c_2}{2c_4(\mathfrak{N}/\vartheta)} \cdot \frac{\log q}{q^2} \cdot \frac{I}{\max(f_{\mathfrak{p}}\log p, g_1)} + \frac{1}{(q\eta^{r+1})^I} \leq 1 \quad \text{if } p = 2,$$

for $1 \leq I < I^*$, where

$$I^* = [5\max(f_{\mathfrak{p}}\log p, g_1)/\log(q\eta^{r+1})] + 1, \tag{9.8}$$

$$\eta^{-(r+1)} + g_2^{-1} \leq q. \tag{9.9}$$

Note that in verifying (9.4)–(9.7) and (9.9), we have used the fact that $\eta^{r+1} = (1 - c_5/(r+1))^{r+1}$ is increasing and $\eta^r = (1 - c_5/(r+1))^r$ is decreasing in each range of r considered in (9.2). For the details of the verification, see the paragraph below, which contains (9.33)–(9.35).

Let

$$h = \max\left\{\log\left(\frac{f_{\mathfrak{p}}\log p}{2d}\max_{1\leq j<n}\left(\frac{|b_n|}{h'(\alpha_j)} + \frac{|b_j|}{h'(\alpha_n)}\right)\right),\ \log B^\circ, g_0, 2f_{\mathfrak{p}}\log p\right\}, \tag{9.10}$$

where B° is given by (7.3), and

$$G_0 = (p^{f_{\mathfrak{p}}} - 1)/q^u, \tag{9.11}$$

which is a positive integer by Hasse ([4], p. 220) and (1.3), (1.4). Set

$$S = \frac{c_3 q(r+1)d(h + \nu\log q)}{f_{\mathfrak{p}}\log p}, \tag{9.12}$$

where ν is defined at the end of the paragraph above (5.23)$^\clubsuit$ in [10],

$$D = (1+10^{-100})(1+\epsilon_1)(1+\epsilon_2)\left(2+\frac{1}{g_2}\right)c_0c_1c_4\frac{\mathfrak{N}}{\vartheta}(\varsigma_2 c_2 q p^\kappa)^r \times \frac{(r+1)^r}{r!}G_0 p^{f_{\mathfrak{p}}x_0} d^{r+1}\sigma_1\cdots\sigma_r\max\,(f_{\mathfrak{p}}\log p, g_1), \tag{9.13}$$

where x_0 is defined by (2.3),

$$T = \frac{q(r+1)D}{c_1\mathfrak{N}e_{\mathfrak{p}}f_{\mathfrak{p}}\log p}, \tag{9.14}$$

$$\widetilde{D}_{-1} = h + \nu \log q - 1, \qquad D_{-1} = [\widetilde{D}_{-1}], \tag{9.15}$$

$$\widetilde{D}_0 = \frac{SDd^{-1}}{c_1 c_4 (\mathfrak{N}/\vartheta)(D_{-1}+1)\max(f_{\mathfrak{p}}\log p, g_1)}, \qquad D_0 = [\widetilde{D}_0], \tag{9.16}$$

$$\widetilde{D}_i = \frac{D}{c_1 c_2 r p^{\kappa} d\sigma_i}, \quad D_i = [\widetilde{D}_i], \quad 1 \le i \le r. \tag{9.17}$$

It is readily seen that

$$(9.j) := (9.j)^{\clubsuit} \qquad \text{for } 18 \le j \le 31 \text{ with } j \ne 23, 25, 26$$

hold. The following three inequalities are also true:

$$\frac{(D_{-1}+1)(D_0+1)}{G_0 p^{f_{\mathfrak{p}}\mathfrak{m}}} \prod_{i=1}^{r} (D_i + 1 - G_0) \ge c_0(2S+1)\binom{[T]+r}{r}, \tag{9.23}$$

$$T(\widetilde{D}_{-1}+1) \le \frac{1}{c_1 c_3 e_{\mathfrak{p}} \mathfrak{N}} \cdot \frac{SD}{d} < \frac{1+10^{-100}}{e_{\mathfrak{p}}\vartheta_0 + 1} \cdot \frac{1}{c_1 c_3} \cdot \frac{SD}{d} \tag{9.25}$$

(the second inequality in (9.25) follows from (2.5), (2.4) and (2.8)),

$$\begin{aligned} &\widetilde{D}_0 \ge g_6, \\ &(D_{-1}+1)(D_0+1)\max(f_{\mathfrak{p}}\log p, g_1) \le \left(1+\frac{1}{g_6}\right)\frac{1}{c_1 c_4 (\mathfrak{N}/\vartheta)} \cdot \frac{SD}{d}. \end{aligned} \tag{9.26}$$

P r o o f o f (9.23). By (9.19)–(9.21), it suffices to show that $D \ge D'$, where

$$\begin{aligned} D' = (1+\epsilon_1)(1+\epsilon_2)\left(2+\frac{1}{g_2}\right) c_0 c_1 c_4 \frac{\mathfrak{N}}{\vartheta}\left(c_2 q \frac{p^{\kappa}}{e_{\mathfrak{p}}\mathfrak{N}}\right)^r G_0 p^{f_{\mathfrak{p}}\mathfrak{m}} \\ \times \frac{r^r (r+1)^r}{r!} \cdot \frac{d^{r+1}\sigma_1 \cdots \sigma_r}{(f_{\mathfrak{p}}\log p)^r} \max(f_{\mathfrak{p}}\log p, g_1). \end{aligned}$$

Now by (2.5), (2.3) and (2.7), we have

$$\begin{aligned} \frac{p^{f_{\mathfrak{p}}\mathfrak{m}}}{(e_{\mathfrak{p}}\mathfrak{N})^r} &= \left(1+\frac{1}{2n}\cdot 10^{-100}\right)^r \frac{p^{f_{\mathfrak{p}}\mathfrak{m}}}{(\mathfrak{m}+e_{\mathfrak{p}}\vartheta)^r} < (1+10^{-100})\frac{1}{(\chi(\mathfrak{m}))^r} \\ &< (1+10^{-100})\varsigma_2^r \frac{1}{(\chi(x_0))^r} = (1+10^{-100})\varsigma_2^r p^{f_{\mathfrak{p}} x_0} \frac{(f_{\mathfrak{p}}\log p)^r}{r^r}, \end{aligned}$$

which implies $D \ge D'$. (We omit the proofs of (9.25) and (9.26) here.)

Set

$$U = \frac{q^{r+1}}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p} SD. \tag{9.32}$$

PROPOSITION 9.1. *Under the hypotheses of Theorem* 7.1, *we have*

$$\operatorname{ord}_p \Xi < U.$$

Note that Proposition 9.1 implies Theorem 7.1. We verify this for the case $\nu > 0$. By (9.12), (9.13), (8.6), (9.2), (8.7), (8.13)$^{\clubsuit}$, (7.2), (9.10), (2.3), (2.5), (2.6), (1.3), and the inequality $(n+1)^{n+1}/(n-1)! \ge e^{n-r}(r+1)^{r+1}/(r-1)!$, we have

$$\frac{e_{\mathfrak{p}}U}{(p^{-f_{\mathfrak{p}}e_{\mathfrak{p}}\vartheta}/\max(p^{-f_{\mathfrak{p}}e_{\mathfrak{p}}\vartheta}, q^{-n})) \times \text{the right side of (7.1)}} \le \frac{f_6}{cq^{n-r}} \le 1.$$

The verification in the case $\nu = 0$, i.e., when $\alpha_1, \ldots, \alpha_n$ satisfy (1.11), is similar.

In the following §10–§13, we shall prove Proposition 9.1.

Now we indicate how we verify (9.4)–(9.7) and (9.9). We divide the verification into four cases, which are (I), (II), (III), (IV) of (2.1). We have

$$\text{(9.33)}\quad \begin{aligned} &\text{(I) } q = 2,\ d \ge 1,\ \vartheta \le 3/2,\ f_{\mathfrak{p}} \ge 1,\ e_{\mathfrak{p}} \ge 1,\ p^{\kappa} \ge 3,\\ &\quad \mathfrak{N}/\vartheta > (1+10^{-100})^{-1};\\ &\text{(II) } q = 2,\ d \ge 1,\ \vartheta \le 1,\ f_{\mathfrak{p}} \ge 1,\ e_{\mathfrak{p}} = 1,\ p^{\kappa} = 1,\\ &\quad \mathfrak{N}/\vartheta > 2(1+10^{-100})^{-1};\\ &\text{(III) } q = 2,\ d \ge 2,\ \vartheta \le 7/6,\ f_{\mathfrak{p}} \ge 1,\ e_{\mathfrak{p}} \ge 2,\ p^{\kappa} \ge 1,\\ &\quad \mathfrak{N}/\vartheta > (1+10^{-100})^{-1};\\ &\text{(IV) } q = 3,\ d \ge 2,\ \vartheta \le 2,\ f_{\mathfrak{p}} \ge 2,\ e_{\mathfrak{p}} \ge 1,\ p^{\kappa} \ge 4,\\ &\quad \mathfrak{N}/\vartheta > (1+10^{-100})^{-1}. \end{aligned}$$

We can prove (9.4)–(9.7) and (9.9) for $r \le 15$ by direct computation, using (9.1)–(9.3) and (9.33). It remains to verify them for $r \ge 16$. By direct computation, we see that (9.4)(ii) is true for $r = 16$, whence it holds for $r \ge 16$, since its left side is an increasing function of r. In case (I), LHS$-$RHS of (9.5) is increasing in d, and this difference at $d = 1$ is increasing in r with $r \ge 16$; further this difference at $d = 1$, $r = 16$ is positive by direct computation, whence (9.5) for case (I) with $r \ge 16$ follows. In cases (II), (III) and (IV) we have

$$\text{(9.34)}\quad \begin{aligned} &\left\{\frac{107}{103}\cdot\frac{1}{e_{\mathfrak{p}}\vartheta_0+1}\cdot\frac{c_5}{r+1} - \left(1+\frac{1}{c_0-1}\right)g_{10}\right\}\frac{1}{c_3} - \frac{c_5 q}{g_2}\\ &- c_1\left\{g_{12} + \left(1+\frac{1}{2(c_0-1)}\right)g_9\right\} - \frac{1}{(c_0-1)(2g_2+1)c_2}\\ &- \frac{1}{g_6}\left\{1+\frac{1}{c_0-1}+\left(\mathfrak{N}+\frac{1}{p-1}\right)\frac{1}{f_{\mathfrak{p}}}\right\}\frac{1}{c_4(\mathfrak{N}/\vartheta)} > 0 \quad \text{for } r \ge 16, \end{aligned}$$

since its left side is positive for $r = 16$, decreasing in r and tends to 0 as $r \to \infty$; also it is readily verified, using (9.1), (9.3) and (9.33), that for $r \ge 16$,

$$(9.35)\qquad 2c_5q \ge \left\{q+\frac{1}{2(c_0-1)}\right\}\frac{2}{c_2}+\frac{107}{103}\cdot\frac{1+10^{-100}}{e_{\mathfrak p}\vartheta_0+1}\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_3} + \left\{1+\frac{1}{c_0-1}+\left(\mathfrak{N}+\frac{1}{p-1}\right)\frac{1}{f_{\mathfrak p}}\right\}\frac{1}{c_4(\mathfrak{N}/\vartheta)}.$$

Now (9.5) for cases (II), (III), (IV) with $r\ge 16$ follows from (9.34) and (9.35). Finally (9.6), (9.7) and (9.9) for $r=16$ can be verified by direct computation, using (9.1)–(9.3) and (9.33), whence they hold for $r\ge 16$ by monotonicity in r. Our computation is carried out on a SUN SPARCstation 10 with PARI GP 1.39.

In order to prove Lemma 11.2 of §11 in the sequel, we show the following inequality. Let $I\in\mathbb{Z}$ satisfy $0\le I<I^*$ with I^* given by (9.8) and $\delta_I=0$ or 1 according to $I=0$ or $I>0$. Then for $k=0,\dots,r-1$ when $I=0$ and for $k=1,\dots,r-1$ when $I>0$ we have

$$(9.36)\qquad 2c_5q^{k+1}\eta^k\left(1-\frac{1}{2g_2}\right) \ge c_1\left\{g_{12}+\left(1+\frac{1}{2(c_0-1)}\right)g_9\right\} + \left\{\frac{q^{k+1}}{(q\eta^{r+1})^I}+\frac{1}{2(c_0-1)}\left(1+\frac{1}{2g_2+1}\right)\right\}\frac{2}{c_2} + \left\{\frac{107}{103}\cdot\frac{1+10^{-100}}{e_{\mathfrak p}\vartheta_0+1}\left(\eta^{k+1}+\frac{1}{c_0-1}\right)+\left(1+\frac{1}{c_0-1}\right)g_{10}\right\}\frac{1}{c_3} + \left(1+\frac{1}{g_6}\right)\left\{1+\frac{1}{c_0-1}+\frac{[k+\delta_I(I+1/(q-1))]\log q}{\max(f_{\mathfrak p}\log p,g_1)} + \left(\mathfrak{N}+\frac{1}{p-1}\right)\frac{1}{f_{\mathfrak p}}\right\}\frac{1}{c_4(\mathfrak{N}/\vartheta)}.$$

Proof of (9.36). By (9.7), we see that the right side of (9.36) is bounded above by $\mathfrak{R}(k)$ which is obtained from the right side of (9.36) by replacing $q^{k+1}/(q\eta^{r+1})^I$ with q^{k+1}, replacing $k+\delta_I(I+1/(q-1))$ with $\tau(p)k$, and replacing $\max(f_{\mathfrak p}\log p,\,g_1)$ with g_1. Write $\mathfrak{L}(k)$ for the left side of (9.36). Now (9.4) implies $(\mathfrak{L}(x)-\mathfrak{R}(x))'>0$ for $0\le x\le r-1$ and (9.5) implies $\mathfrak{L}(0)-\mathfrak{R}(0)\ge 0$. Hence $\mathfrak{L}(k)\ge\mathfrak{R}(k)$ for $k=0,\dots,r-1$, which yields (9.36).

10. The auxiliary rational functions. Let

$$(10.1)\qquad G=p^{f_{\mathfrak p}}-1,\quad G_0=G/q^u,\quad G_1=G/q^{\mu}\quad\text{with } \mu=\operatorname{ord}_q G.$$

Denote by ζ a fixed Gth primitive root of 1 in $K_{\mathfrak p}$ such that

$$(10.2)\qquad \zeta^{G_0}=\zeta_{q^u}\ (=\alpha_0),$$

by ξ a fixed (qG)th root of 1 in $\mathbb{C}_p$, and by $\alpha_0^{1/q}$ a qth root of α_0 in $\mathbb{C}_p$, satisfying

$$\xi^q = \zeta \quad \text{and} \quad \xi^{G_0} = \alpha_0^{1/q}. \tag{10.3}$$

By (1.9), there exist $\widetilde{a}_1, \ldots, \widetilde{a}_n \in \mathbb{N}$ such that $\alpha_j \zeta^{\widetilde{a}_j} \equiv 1 \pmod{\mathfrak{p}}$ $(1 \le j \le n)$. Now [9], III, Lemma 1.1 yields

$$\operatorname{ord}_p(\alpha_j^{p^\kappa} \zeta^{a_j} - 1) \ge \vartheta + \frac{1}{p-1}, \quad 1 \le j \le n, \tag{10.4}$$

where $a_j = p^\kappa \widetilde{a}_j$, and ϑ is given by (1.6). Note also, by (10.2),

$$\alpha_0^{p^\kappa} \zeta^{a_0} = 1, \quad \text{where} \quad a_0 = p^\kappa (G - G_0). \tag{10.5}$$

Thus the p-adic logarithms of $\alpha_j^{p^\kappa} \zeta^{a_j}$ satisfy

$$\log(\alpha_0^{p^\kappa} \zeta^{a_0}) = 0, \quad \operatorname{ord}_p \log(\alpha_j^{p^\kappa} \zeta^{a_j}) \ge \vartheta + \frac{1}{p-1}, \quad 1 \le j \le n. \tag{10.6}$$

We shall freely use the fundamental properties of the p-adic exponential and logarithmic functions (see, for example, [8], §1.1).

Recall $L_i(z_0, \ldots, z_n)$ and α'_i $(1 \le i \le r)$ specified in the basic hypothesis in §8. It is proved (as (10.7)$^\clubsuit$) that there exist $a'_1, \ldots, a'_r \in \mathbb{N}$ such that for $1 \le i \le r$,

$$\exp\left(\frac{1}{q^\nu} L_i(0, \log(\alpha_1^{p^\kappa} \zeta^{a_1}), \ldots, \log(\alpha_n^{p^\kappa} \zeta^{a_n}))\right) = \alpha_i'^{p^\kappa} \zeta^{a'_i}. \tag{10.7}$$

By (10.6) and (10.7) we have

$$\operatorname{ord}_p(\alpha_i'^{p^\kappa} \zeta^{a'_i} - 1) \ge \vartheta + \frac{1}{p-1}, \quad 1 \le i \le r; \tag{10.8}$$

or equivalently,

$$\alpha_i'^{p^\kappa} \zeta^{a'_i} \equiv 1 \pmod{\mathfrak{p}^{\mathfrak{m}_0}}, \quad 1 \le i \le r, \tag{10.9}$$

where $\mathfrak{m}_0$ is the least integer $\ge e_\mathfrak{p}(\vartheta + 1/(p-1))$. Hence

$$\prod_{i=1}^r (\alpha_i'^{p^\kappa} \zeta^{a'_i})^{\lambda_i} \equiv 1 \pmod{\mathfrak{p}^{\mathfrak{m}_0}} \quad \text{for all } (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r. \tag{10.10}$$

We define $(\alpha_i'^{p^\kappa} \zeta^{a'_i})^{1/q}$ by the p-adic exponential and logarithmic functions:

$$(\alpha_i'^{p^\kappa} \zeta^{a'_i})^{1/q} = \exp\left(\frac{1}{q} \log(\alpha_i'^{p^\kappa} \zeta^{a'_i})\right), \quad 1 \le i \le r; \tag{10.11}$$

and we fix a choice of qth roots of $\alpha'_1, \ldots, \alpha'_r$ in $\mathbb{C}_p$, denoted by $\alpha_1'^{1/q}, \ldots$ $\ldots, \alpha_r'^{1/q}$, such that

$$(\alpha_i'^{p^\kappa} \zeta^{a'_i})^{1/q} = (\alpha_i'^{1/q})^{p^\kappa} \xi^{a'_i}, \quad 1 \le i \le r, \tag{10.12}$$

where ξ is given by (10.3).

We shall use the notation introduced in [1], §12:

$$\Delta(z;k) = (z+1)\cdots(z+k)/k! \quad \text{for } k \in \mathbb{Z}_{>0} \quad \text{and} \quad \Delta(z;0) = 1,$$

$$\Pi(z_1,\dots,z_{r-1};t_1,\dots,t_{r-1}) = \prod_{i=1}^{r-1} \Delta(z_i;t_i) \quad (t_1,\dots,t_{r-1} \in \mathbb{N}),$$

$$\Theta(z;k,l,m) = \frac{1}{m!}\left(\frac{d}{dz}\right)^m (\Delta(z;k))^l \quad (l,m \in \mathbb{N}).$$

Recalling (10.1) and writing $\lambda = (\lambda_{-1},\dots,\lambda_r)$, we define

$$\mathcal{B} = \{b \in \mathbb{N} \mid b < q^{\mu-u},\ \gcd(a'_1,\dots,a'_r,G_0) \mid bG_1\}, \tag{10.13}$$

$$\Lambda^{\clubsuit} = \Big\{\lambda \in \mathbb{N}^{r+2} \,\Big|\, \lambda_i \le D_i,\ -1 \le i \le r,\ \sum_{i=1}^{r} a'_i\lambda_i \equiv 0 \pmod{G_1}\Big\}, \tag{10.14}$$

where $D_{-1},\dots,D_r$ are given by (9.15)–(9.17). We define an equivalence relation on $\Lambda^{\clubsuit}$: λ and λ' in $\Lambda^{\clubsuit}$ are said to be *equivalent* if $\lambda_i = \lambda'_i$ $(i = -1, 0)$ and

$$\prod_{i=1}^{r} (\alpha_i'^{p^\kappa}\zeta^{a'_i})^{\lambda_i} \equiv \prod_{i=1}^{r} (\alpha_i'^{p^\kappa}\zeta^{a'_i})^{\lambda'_i} \pmod{\mathfrak{p}^{\mathfrak{m}_0+\mathfrak{m}}}, \tag{10.15}$$

where $\mathfrak{m}_0$ is given in (10.9) and $\mathfrak{m}$ is given by (2.4). From (10.10) and the fact that $\mathfrak{m}_0 \ge 1$, we see that $\Lambda^{\clubsuit}$ decomposes into at most $N(\mathfrak{p})^{\mathfrak{m}} = p^{f_{\mathfrak{p}}\mathfrak{m}}$ equivalence classes, whence there exists an equivalence class, denoted by Λ, such that its cardinality satisfies

$$\#\Lambda \ge \frac{\#\Lambda^{\clubsuit}}{p^{f_{\mathfrak{p}}\mathfrak{m}}}. \tag{10.16}$$

We have $\Lambda = \bigcup_{b\in\mathcal{B}} \Lambda_b$, where

$$\Lambda_b = \Big\{\lambda \in \Lambda \,\Big|\, \sum_{i=1}^{r} a'_i\lambda_i \equiv bG_1 \pmod{G_0}\Big\}. \tag{10.17}$$

From now on we fix $(\lambda_1^{(0)},\dots,\lambda_r^{(0)})$ and $b^{(0)} \in \mathcal{B}$ such that

$$(\lambda_{-1},\lambda_0,\lambda_1^{(0)},\dots,\lambda_r^{(0)}) \in \Lambda_{b^{(0)}} \quad \text{for } 0 \le \lambda_{-1} \le D_{-1},\ 0 \le \lambda_0 \le D_0. \tag{10.18}$$

We shall construct a rational function $P = P(Y_0,\dots,Y_r)$ of the form

$$P = \sum_{\lambda\in\Lambda} \rho(\lambda)(\Delta(Y_0+\lambda_{-1};D_{-1}+1))^{\lambda_0+1} Y_1^{\lambda_1-\lambda_1^{(0)}} \cdots Y_r^{\lambda_r-\lambda_r^{(0)}} \tag{10.19}$$

with coefficients $\rho(\lambda) = \rho(\lambda_{-1},\dots,\lambda_r)$ in $\mathcal{O}_K$. We write $P = \sum_{b\in\mathcal{B}} P_b$, where P_b is given by the right side of (10.19) with Λ replaced by Λ_b.

Put $\partial_0^* = \partial/\partial Y_0$ and recall (8.14)♣:

$$\partial_j^* = (1/B_r)\sum_{i=1}^{r-1}(b_n\partial L_i/\partial z_j - b_j\partial L_i/\partial z_n)\partial_i \quad (1 \le j < n).$$

Then we have

$$\partial_j^* Y_1^{\lambda_1-\lambda_1^{(0)}}\cdots Y_r^{\lambda_r-\lambda_r^{(0)}} = \gamma_j Y_1^{\lambda_1-\lambda_1^{(0)}}\cdots Y_r^{\lambda_r-\lambda_r^{(0)}} \quad (1 \le j < n),$$

where

$$\gamma_j = \sum_{i=1}^{r}(b_n\partial L_i/\partial z_j - b_j\partial L_i/\partial z_n)(\lambda_i - \lambda_i^{(0)}). \tag{10.20}$$

For any $t = (t_0, \ldots, t_{r-1}) \in \mathbb{N}^r$, we write $|t| = t_0 + \ldots + t_{r-1}$ and put

$$\Pi(t) = \Pi(\gamma_1, \ldots, \gamma_{r-1}; t_1, \ldots, t_{r-1}),$$
$$\Theta(Y_0; t) = (v(D_{-1}+1))^{t_0}\Theta(Y_0 + \lambda_{-1}; D_{-1}+1, \lambda_0 + 1, t_0),$$

where $v(k) = \mathrm{lcm}(1, 2, \ldots, k)$ for $k \in \mathbb{Z}_{>0}$. We introduce further rational functions $Q(t) = Q(Y_0, \ldots, Y_r; t)$ by

$$Q(t) = \sum_{\lambda\in\Lambda}\rho(\lambda)\Pi(t)\Theta(Y_0;t)Y_1^{\lambda_1-\lambda_1^{(0)}}\cdots Y_r^{\lambda_r-\lambda_r^{(0)}}, \tag{10.21}$$

and write $Q(t) = \sum_{b\in\mathcal{B}} Q_b(t)$, where $Q_b(t)$ is given by the right side of (10.21) with Λ replaced by Λ_b.

We shall use the notation of heights introduced in [1], §2. Now we apply [1], Lemma 1, which is a consequence of Bombieri and Vaaler [2], Theorem 9, to prove the following lemma, where

$\rho = (\rho(\lambda) : \lambda \in \Lambda) \in \mathbb{P}^N$ with $N = \#\Lambda$ (= the number of elements of Λ).

LEMMA 10.1. *There exist $\rho(\lambda) \in \mathcal{O}_K$, $\lambda \in \Lambda$, not all zero, with*

$$h_0(\rho) \le \frac{SD}{d}\biggl\{g_{12} + \frac{1}{c_0-1}\biggl[\frac{1}{2}g_9 + \biggl(1 + \frac{1}{2g_2+1}\biggr)\frac{1}{c_1c_2} + \biggl(\frac{107}{103}\cdot\frac{1}{e_{\mathfrak{p}}\mathfrak{N}} + g_{10}\biggr)\frac{1}{c_1c_3} + \biggl(1 + \frac{1}{g_6}\biggr)\frac{1}{c_1c_4(\mathfrak{N}/\vartheta)}\biggr]\biggr\}, \tag{10.22}$$

such that for all $b \in \mathcal{B}$ we have

$$Q_b(s, (\alpha_1'^{p^\kappa}\zeta^{a_1'})^s, \ldots, (\alpha_r'^{p^\kappa}\zeta^{a_r'})^s; t) = 0 \quad \text{for } s \in \mathbb{Z} \text{ with } |s| \le S \text{ and } t \in \mathbb{N}^r \text{ with } |t| \le T. \tag{10.23}$$

REMARK. In the sequel s always denotes a rational integer and t always denotes an r-tuple $(t_0, \ldots, t_{r-1}) \in \mathbb{N}^r$. The expressions $s \in \mathbb{Z}$ and $t \in \mathbb{N}^r$ will be omitted.

Proof (of Lemma 10.1). For each $\lambda \in \Lambda_b$, by (10.14), (10.17) and (10.18), there exists $w(\lambda) \in \mathbb{Z}$ such that

$$\sum_{i=1}^{r} a_i'(\lambda_i - \lambda_i^{(0)}) = (b - b^{(0)})G_1 + w(\lambda)G_0,$$

whence, by (10.2),

$$\prod_{i=1}^{r} (\alpha_i'^{p^\kappa} \zeta^{a_i'})^{s(\lambda_i - \lambda_i^{(0)})} = \zeta^{s(b-b^{(0)})G_1} \alpha_0^{sw(\lambda)} \prod_{i=1}^{r} \alpha_i'^{p^\kappa s(\lambda_i - \lambda_i^{(0)})}.$$

Thus it suffices to construct $\rho(\lambda) \in \mathcal{O}_K$, $\lambda \in \Lambda$, not all zero, such that

$$(10.24) \qquad \sum_{\lambda \in \Lambda_b} \rho(\lambda) \Pi(t) \Theta(s;t) \alpha_0^{sw(\lambda)} \prod_{i=1}^{r} \alpha_i'^{p^\kappa s(\lambda_i - \lambda_i^{(0)} + D_i)} = 0$$
$$\text{for } b \in \mathcal{B},\ |s| \le S,\ |t| \le T,$$

which is a system of

$$M \le q^{\mu - u}(2S+1)\binom{[T]+r}{r}$$

homogeneous linear equations in N unknowns $\rho(\lambda)$, $\lambda \in \Lambda$, with coefficients in $E = \mathbb{Q}(\alpha_0, \alpha_1', \ldots, \alpha_r') \subseteq K$. Evidently (10.22)♣ remains true. Now our proof follows closely that of Lemma 10.1♣, and we indicate here only modifications. From (10.16) and (9.23), we get

$$N \ge \frac{(D_{-1}+1)(D_0+1)G_1^{r-1}}{p^{f_\mathfrak{p}\mathfrak{m}}} \prod_{i=1}^{r} \left[\frac{D_i+1}{G_1}\right] \ge c_0 M.$$

By (9.28), (10.23)♣ still holds. Also (10.24)♣ remains true, since $|\lambda_i - \lambda_i^{(0)}| \le D_i$ $(1 \le i \le r)$ for $\lambda \in \Lambda$. (10.25)♣ should be modified to: for $\lambda \in \Lambda$, $|s| \le S$, $|t| \le T$,

$$(10.25) \qquad \log|\Theta(s;t)| \le \frac{107}{103} t_0 (D_{-1}+1) + \left(1 + \frac{1}{g_6}\right) \frac{1}{c_1 c_4 (\mathfrak{N}/\vartheta)} \cdot \frac{SD}{d},$$

because (9.26)♣ is changed to (9.26).

On combining (10.24)♣ with (10.25), and by (9.25) and $g_0 > 48$, we obtain for $\lambda \in \Lambda$, $|s| \le S$, $|t| \le T$,

$$(10.26) \qquad \log|\Pi(t)\Theta(s;t)|$$
$$\le \left\{ \left(\frac{107}{103} \cdot \frac{1}{e_\mathfrak{p}\mathfrak{N}} + g_{10} \right) \frac{1}{c_1 c_3} + \left(1 + \frac{1}{g_6}\right) \frac{1}{c_1 c_4 (\mathfrak{N}/\vartheta)} \right\} \frac{SD}{d}.$$

Now let $|\ |_v$ be an absolute value on E normalized as in [1], §2. On noting that $\Pi(t)\Theta(s;t) \in \mathbb{Z}$, we have for $v \mid \infty$, and for $\lambda \in \Lambda$, $|s| \le S$, $|t| \le T$,

$$\log \Big| \Pi(t)\Theta(s;t)\alpha_0^{sw(\lambda)} \prod_{i=1}^r {\alpha_i'}^{p^\kappa s(\lambda_i - \lambda_i^{(0)} + D_i)}\Big|_v \le \log|\Pi(t)\Theta(s;t)| + \sum_{i=1}^r 2D_i \log\max(1, |{\alpha_i'}^{p^\kappa s}|_v);$$

for $v \nmid \infty$, and for $\lambda \in \Lambda$, $|s| \le S$, $|t| \le T$,

$$\log \Big| \Pi(t)\Theta(s;t)\alpha_0^{sw(\lambda)} \prod_{i=1}^r {\alpha_i'}^{p^\kappa s(\lambda_i - \lambda_i^{(0)} + D_i)}\Big|_v \le \sum_{i=1}^r 2D_i \log\max(1, |{\alpha_i'}^{p^\kappa s}|_v).$$

Thus, by the product formula, by (10.26), (8.4), (9.24), and on noting $[S]([S]+1) \le S(2S+1) \cdot \frac{1}{2}(1 + 1/(2g_2+1))$ (by (9.19)), we have

$$\tag{10.27} \begin{aligned} &\frac{1}{N-M} \times \sum_{\substack{b\in\mathcal{B}\\ |s|\le S, |t|\le T}} \frac{1}{d'} \sum_v \log \max_{\lambda\in\Lambda_b} \Big| \Pi(t)\Theta(s;t)\alpha_0^{sw(\lambda)} \prod_{i=1}^r {\alpha_i'}^{p^\kappa s(\lambda_i - \lambda_i^{(0)} + D_i)}\Big|_v \\ &\le \frac{1}{c_0 - 1}\bigg\{\Big(1 + \frac{1}{2g_2+1}\Big)\frac{1}{c_1c_2} + \Big(\frac{107}{103}\cdot\frac{1}{e_{\mathfrak{p}}\mathfrak{N}} + g_{10}\Big)\frac{1}{c_1c_3} \\ &\qquad + \Big(1 + \frac{1}{g_6}\Big)\frac{1}{c_1c_4(\mathfrak{N}/\vartheta)}\bigg\}\frac{SD}{d}. \end{aligned}$$

Now by [1], Lemma 1, Lemma 10.1 follows from (10.22)$^\clubsuit$, (10.23)$^\clubsuit$ and (10.27).

11. Double inductive procedure. For $\varepsilon^{(I)} \in \mathbb{N}$, $D_i^{(I)} \in \mathbb{N}$ $(-1 \le i \le r)$ and $\rho^{(I)}(\lambda) = \rho^{(I)}(\lambda_{-1}, \ldots, \lambda_r) \in \mathcal{O}_K$, which will be constructed in the following main inductive argument, we set

$$\tag{11.1} \mathcal{B}^{(I)} = \{b \in \mathbb{N} \mid b < q^{\mu-u},\ \gcd(a_1', \ldots, a_r', G_0) \mid (\varepsilon^{(I)} + bG_1)\},$$

and let $\Lambda^{(I)}$ be a subset of $\mathbb{Z}^{r+2}$ having the following three properties:

(11.2) (i) $0 < \#\Lambda^{(I)} \le \prod_{i=-1}^r (D_i + 1)$,

(ii) $\lambda = (\lambda_{-1}, \ldots, \lambda_r) \in \Lambda^{(I)}$ implies that λ_i runs over $0, 1, \ldots, D_i^{(I)}$ $(i = -1, 0)$ and

$$\sum_{i=1}^r a_i'\lambda_i \equiv \varepsilon^{(I)} \pmod{G_1},$$

(iii) if $\lambda, \lambda' \in \Lambda^{(I)}$ then (10.15) holds and $|\lambda_i - \lambda_i'| \le D_i^{(I)}$ $(1 \le i \le r)$.

For $b \in \mathcal{B}^{(I)}$, set

$$\Lambda_b^{(I)} = \Big\{ \lambda \in \Lambda^{(I)} \Bigm| \sum_{i=1}^{r} a_i' \lambda_i \equiv \varepsilon^{(I)} + bG_1 \pmod{G_0} \Big\}. \tag{11.3}$$

Fix $(\lambda_1^{(I)}, \ldots, \lambda_r^{(I)})$ and $b^{(I)} \in \mathcal{B}^{(I)}$ such that

$$(\lambda_{-1}, \lambda_0, \lambda_1^{(I)}, \ldots, \lambda_r^{(I)}) \in \Lambda_{b^{(I)}}^{(I)} \quad \text{for } 0 \le \lambda_{-1} \le D_{-1}^{(I)},\ 0 \le \lambda_0 \le D_0^{(I)}. \tag{11.4}$$

Put

$$\gamma_j^{(I)} = \sum_{i=1}^{r} (b_n \partial L_i/\partial z_j - b_j \partial L_i/\partial z_n)(\lambda_i - \lambda_i^{(I)}) \quad (1 \le j \le n), \tag{11.5}$$

$$\Pi^{(I)}(t) = \Pi(\gamma_1^{(I)}, \ldots, \gamma_{r-1}^{(I)}; t_1, \ldots, t_{r-1}). \tag{11.6}$$

Define $Q^{(I)}(t) = Q^{(I)}(Y_0, \ldots, Y_r; t)$ by

$$Q^{(I)}(t) = \sum_{\lambda \in \Lambda^{(I)}} \rho^{(I)}(\lambda) \Pi^{(I)}(t) \Theta(q^{-I} Y_0; t) Y_1^{\lambda_1 - \lambda_1^{(I)}} \cdots Y_r^{\lambda_r - \lambda_r^{(I)}}, \tag{11.7}$$

and write $Q^{(I)}(t) = \sum_{b \in \mathcal{B}^{(I)}} Q_b^{(I)}(t)$, where $Q_b^{(I)}(t)$ is given by the right side of (11.7) with $\Lambda^{(I)}$ replaced by $\Lambda_b^{(I)}$.

We now define the linear forms

$$M_i = L_i - \frac{1}{b_n} \cdot \frac{\partial L_i}{\partial z_n} L \quad (1 \le i \le r), \tag{11.8}$$

where $L = b_1 z_1 + \ldots + b_n z_n$. Then

$$b_n M_i = b_n (\partial L_i/\partial z_0) z_0 + \sum_{j=1}^{n-1} (b_n \partial L_i/\partial z_j - b_j \partial L_i/\partial z_n) z_j \quad (1 \le i \le r).$$

For $z_0, z_1, \ldots, z_n$ with $\mathrm{ord}_p\, z_0 \ge 0$ and $\mathrm{ord}_p\, z_j > 1/(p-1)$ $(1 \le j \le n)$, we define the p-adic functions

$$\varphi^{(I)}(z_0, \ldots, z_n; t) = Q^{(I)}(z_0, e^{L_1(0, z_1, \ldots, z_n)}, \ldots, e^{L_r(0, z_1, \ldots, z_n)}; t), \tag{11.9}$$

$$f^{(I)}(z_0, \ldots, z_{n-1}; t) = Q^{(I)}(z_0, e^{M_1(0, z_1, \ldots, z_{n-1})}, \ldots, e^{M_r(0, z_1, \ldots, z_{n-1})}; t). \tag{11.10}$$

Let ν be defined as in §5♣ (see the paragraph above (5.23)♣). We put for $z \in \mathbb{Z}_p$

$$\varphi^{(I)}(z; t) = \varphi^{(I)}(z, zq^{-\nu} \log(\alpha_1^{p^\kappa} \zeta^{a_1}), \ldots, zq^{-\nu} \log(\alpha_n^{p^\kappa} \zeta^{a_n}); t), \tag{11.11}$$

$$f^{(I)}(z; t) = f^{(I)}(z, zq^{-\nu} \log(\alpha_1^{p^\kappa} \zeta^{a_1}), \ldots, zq^{-\nu} \log(\alpha_{n-1}^{p^\kappa} \zeta^{a_{n-1}}); t). \tag{11.12}$$

For $b \in \mathcal{B}^{(I)}$, let $\varphi_b^{(I)}(z_0, \ldots, z_n; t)$ and $f_b^{(I)}(z_0, \ldots, z_{n-1}; t)$ be given by the right side of (11.9) and (11.10) with $Q^{(I)}$ replaced by $Q_b^{(I)}$; let $\varphi_b^{(I)}(z;t)$ and $f_b^{(I)}(z;t)$ be given by the right side of (11.11) and (11.12), with $\varphi^{(I)}$ and $f^{(I)}$ replaced by $\varphi_b^{(I)}$ and $f_b^{(I)}$, respectively. Note that by (10.7), we have

$$\varphi^{(I)}(z;t) = Q^{(I)}(z, ({\alpha'_1}^{p^\kappa}\zeta^{a'_1})^z, \ldots, ({\alpha'_r}^{p^\kappa}\zeta^{a'_r})^z; t) \quad \text{for any } z \in \mathbb{Z}_p, \tag{11.13}$$

$$\varphi_b^{(I)}(z;t) = Q_b^{(I)}(z, ({\alpha'_1}^{p^\kappa}\zeta^{a'_1})^z, \ldots, ({\alpha'_r}^{p^\kappa}\zeta^{a'_r})^z; t) \quad \text{for any } z \in \mathbb{Z}_p. \tag{11.14}$$

Recall η, I^*, S, T given by (9.3), (9.8), (9.12), (9.14). Let

$$S^{(I)} = \eta^{-(r+1)I}S, \qquad T^{(I)} = \eta^{(r+1)I}T. \tag{11.15}$$

We write $\rho^{(I)} = (\rho^{(I)}(\lambda) : \lambda \in \Lambda^{(I)})$.

THE MAIN INDUCTIVE ARGUMENT. *Suppose that Proposition* 9.1 *is false, that is,*

$$\operatorname{ord}_p \Xi \ge U \tag{11.16}$$

for some $\alpha_1, \ldots, \alpha_n \in K$ *and* $b_1, \ldots, b_n \in \mathbb{Z}$ *satisfying* (1.9) *and* (1.13) *with* $\alpha_1, \ldots, \alpha_n$ *multiplicatively independent and* $b_1, \ldots, b_n$ *not all zero. Then for every* $I \in \mathbb{Z}$ *with* $0 \le I \le \min([\log D_r/\log q] + 1, I^*)$ *there exist* $\varepsilon^{(I)} \in \mathbb{N}$, $D_i^{(I)} \in \mathbb{N}$ $(-1 \le i \le r)$, $\Lambda^{(I)} \subseteq \mathbb{Z}^{r+2}$ *with* $\gcd(a'_1, \ldots, a'_r, G_1) \mid \varepsilon^{(I)}$, $D_i^{(I)} = D_i$ $(i = -1, 0)$, $D_i^{(I)} \le q^{-I} D_i$ $(1 \le i \le r)$, $\Lambda^{(I)}$ *satisfying* (11.2), *and* $\rho^{(I)}(\lambda) = \rho^{(I)}(\lambda_{-1}, \ldots, \lambda_r) \in \mathcal{O}_K$ $(\lambda \in \Lambda^{(I)})$, *not all zero, with*

$$h_0(\rho^{(I)}) \le \frac{SD}{d}\biggl\{g_{12} + \frac{1}{c_0 - 1}\biggl[\frac{1}{2}g_9 + \biggl(1 + \frac{1}{2g_2+1}\biggr)\frac{1}{c_1c_2} + \biggl(\frac{107}{103}\cdot\frac{1}{e_{\mathfrak{p}}\mathfrak{N}} + g_{10}\biggr)\frac{1}{c_1c_3} + \biggl(1 + \frac{1}{g_6}\biggr)\frac{1}{c_1c_4(\mathfrak{N}/\vartheta)}\biggr]\biggr\}, \tag{11.17}$$

such that

$$\varphi_b^{(I)}(s;t) = 0 \quad \textit{for all } b \in \mathcal{B}^{(I)},\ |s| \le qS^{(I)},\ |t| \le \eta T^{(I)}. \tag{11.18}$$

In the remainder of this section, we always keep (11.16).

LEMMA 11.1. *Suppose* $b \in \mathcal{B}^{(I)}$ *and* $\rho^{(I)}(\lambda) \in \mathcal{O}_K$ $(\lambda \in \Lambda_b^{(I)})$ *are not all zero. Then for* $y \in \mathbb{Q} \cap \mathbb{Z}_p$, $|t| \le T$ *we have*

$$\operatorname{ord}_p(f_b^{(I)}(y;t) - \varphi_b^{(I)}(y;t)) \ge U - \operatorname{ord}_p b_n + \min_{\lambda \in \Lambda_b^{(I)}} \operatorname{ord}_p \rho^{(I)}(\lambda).$$

P r o o f. This is similar to the proof of Lemma 11.1♣. We omit the details.

Let $\varepsilon^{(0)} = 0$, $D_i^{(0)} = D_i$ $(-1 \le i \le r)$. Then $\mathcal{B}^{(0)} = \mathcal{B}$. We can choose $\Lambda^{(0)} = \Lambda$, and let $\rho^{(0)}(\lambda) = \rho(\lambda)$ $(\lambda \in \Lambda^{(0)})$, which are determined by Lemma 10.1. Thus $\Lambda_b^{(0)} = \Lambda_b$, $\gamma_j^{(0)} = \gamma_j$, $\Pi^{(0)}(t) = \Pi(t)$, $Q^{(0)}(t) = Q(t)$, $Q_b^{(0)}(t) = Q_b(t)$, and by Lemma 10.1, (11.14) and (11.15), we have

$$\varphi_b^{(0)}(s;t) = 0 \quad \text{for all } b \in \mathcal{B}^{(0)},\ |s| \le S^{(0)},\ |t| \le T^{(0)}. \tag{11.19}$$

Lemma 11.2. *Suppose $I = 0$ or I is a positive integer with*

$$I \le \min([\log D_r/\log q],\ I^* - 1) \tag{11.20}$$

for which the main inductive argument holds. Then for $J = 1, \ldots, r$, we have

$$\varphi_b^{(I)}(s;t) = 0 \quad \textit{for all } b \in \mathcal{B}^{(I)},\ |s| \le q^J S^{(I)},\ |t| \le \eta^J T^{(I)}. \tag{11.21}$$

Proof. We abbreviate $(\tau_0, \ldots, \tau_{r-1}) \in \mathbb{N}^r$ to τ, $(\mu_0, \ldots, \mu_{n-1}) \in \mathbb{N}^n$ to μ and write $|\tau| = \tau_0 + \ldots + \tau_{r-1}$, $|\mu| = \mu_0 + \ldots + \mu_{n-1}$. (There should be no confusion with the μ defined by (10.1).) Similarly to the proof of (11.19)$^\clubsuit$, it is readily verified that for every $m \in \mathbb{N}$ we have

$$\begin{aligned} \frac{1}{m!}\left(\frac{d}{dz}\right)^m f_b^{(I)}(z;t) &= \sum_{|\mu|=m} \binom{\tau_0}{\mu_0} \frac{q^{-I\mu_0}(b_n q^\nu)^{-(m-\mu_0)}}{(v(D_{-1}+1))^{\mu_0}} \\ &\times \prod_{j=1}^{n-1} \frac{(\log(\alpha_j^{p^\kappa}\zeta^{a_j}))^{\mu_j}}{\mu_j!} \sum_{\tau_1,\ldots,\tau_{r-1}} C(\mu,\tau) f_b^{(I)}(z;\tau), \end{aligned} \tag{11.22}$$

where $\tau = (\tau_0, \tau_1, \ldots, \tau_{r-1})$ with $\tau_0 = t_0 + \mu_0$, the second sum is over $\tau_1, \ldots, \tau_{r-1}$ with $|\tau| \le |t| + m$, and $C(\mu,\tau) \in \mathbb{Q} \cap \mathbb{Z}_p$.

Note that (11.21) holds for $J = 0$ when $I = 0$ by (11.19), and for $J = 1$ when $I > 0$ by (11.18). Assume (11.21) holds for $J = k$ with $0 \le k \le r$ when $I = 0$, and with $1 \le k \le r$ when $I > 0$. We shall prove (11.21) for $J = k+1$ with $k < r$ and include the case $k = r$ for later use.

Clearly, for any fixed $b \in \mathcal{B}^{(I)}$, we may assume $\rho^{(I)}(\lambda)$, $\lambda \in \Lambda_b^{(I)}$, are not all zero, and we write

$$\rho_b^{(I)} = (\rho^{(I)}(\lambda) : \lambda \in \Lambda_b^{(I)}).$$

Now we prove that for every $\lambda \in \Lambda^{(I)}$,

$$\prod_{i=1}^{r} \exp\{p^{-\mathfrak{N}} z q^{-\nu} M_i(0, \log(\alpha_1^{p^\kappa}\zeta^{a_1}), \ldots, \log(\alpha_{n-1}^{p^\kappa}\zeta^{a_{n-1}}))(\lambda_i - \lambda_i^{(I)})\} \tag{11.23}$$

is a p-adic normal function of z.

To see this, we note, by (11.8) and (10.7), that

$$\prod_{i=1}^{r}\exp\{q^{-\nu}M_i(0,\log(\alpha_1^{p^\kappa}\zeta^{a_1}),\ldots,\log(\alpha_{n-1}^{p^\kappa}\zeta^{a_{n-1}}))(\lambda_i-\lambda_i^{(I)})\}$$
$$= e^{\delta(\lambda)}\prod_{i=1}^{r}\exp\{q^{-\nu}L_i(0,\log(\alpha_1^{p^\kappa}\zeta^{a_1}),\ldots,\log(\alpha_n^{p^\kappa}\zeta^{a_n}))(\lambda_i-\lambda_i^{(I)})\}$$
$$= e^{\delta(\lambda)}\prod_{i=1}^{r}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{\lambda_i-\lambda_i^{(I)}},$$

where

$$\delta(\lambda)=-(q^\nu b_n)^{-1}L(\log(\alpha_1^{p^\kappa}\zeta^{a_1}),\ldots,\log(\alpha_n^{p^\kappa}\zeta^{a_n}))\sum_{i=1}^{r}(\lambda_i-\lambda_i^{(I)})\partial L_i/\partial z_n.$$

By (11.4) and (11.2)(iii), we have, for every $\lambda\in\Lambda^{(I)}$,

$$\prod_{i=1}^{r}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{\lambda_i-\lambda_i^{(I)}}\equiv 1 \pmod{\mathfrak{p}^{\mathfrak{m}_0+\mathfrak{m}}},$$

which implies, by (2.5) and the definition of $\mathfrak{m}_0$ given in (10.9),

$$\operatorname{ord}_p\Big(\prod_{i=1}^{r}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{\lambda_i-\lambda_i^{(I)}}-1\Big)>\mathfrak{N}+\frac{1}{p-1}.$$

Also, similarly to the proof of [8], Lemma 3.2, we have $a_1b_1+\ldots+a_nb_n\equiv 0$ (mod G), whence by (11.16) and recalling (7.3), (9.10) and (9.32), we get

$$\begin{aligned}\operatorname{ord}_p\delta(\lambda)&\geq\operatorname{ord}_p L(\log(\alpha_1^{p^\kappa}\zeta^{a_1}),\ldots,\log(\alpha_n^{p^\kappa}\zeta^{a_n}))-\operatorname{ord}_p b_n\\&\geq\operatorname{ord}_p(p^\kappa\log(\alpha_1^{b_1}\cdots\alpha_n^{b_n}))-(\log B^\circ)/\log p\\&\geq\operatorname{ord}_p\Xi-h/\log p\geq U-h/\log p>\mathfrak{N}+\frac{1}{p-1}.\end{aligned}$$

Thus (11.23) follows. Further

$$p^{(D_{-1}+1)(D_0+1)\mathfrak{N}}((D_{-1}+1)!)^{D_0+1}\Theta(q^{-I}p^{-\mathfrak{N}}z;t)$$

is a p-adic normal function of z. Hence for $|t|\leq\eta^{k+1}T^{(I)}$,

$$F_b^{(I)}(z;t):=p^{(D_{-1}+1)(D_0+1)(\mathfrak{N}+1/(p-1))-\Delta_b^{(I)}}f_b^{(I)}(p^{-\mathfrak{N}}z;t)\tag{11.24}$$

are p-adic normal functions of z, where

$$\Delta_b^{(I)}=\min_{\lambda\in\Lambda_b^{(I)}}\operatorname{ord}_p\rho^{(I)}(\lambda).$$

Obviously

$$\begin{aligned}&\frac{1}{m!}\Big(\frac{d}{dz}\Big)^m F_b^{(I)}(sp^{\mathfrak{N}};t)\\&\quad=p^{(D_{-1}+1)(D_0+1)(\mathfrak{N}+1/(p-1))-\Delta_b^{(I)}-m\mathfrak{N}}\frac{1}{m!}\Big(\frac{d}{dz}\Big)^m f_b^{(I)}(s;t).\end{aligned}\tag{11.25}$$

We now apply Lemma 2.1$^♣$ with $\theta = \mathfrak{N}$ to each function in (11.24), taking

$$R = [q^k S^{(I)}], \qquad M = \left[\frac{c_5}{r+1}\eta^k T^{(I)}\right]. \tag{11.26}$$

Similarly to §11$^♣$, we see, by (11.25), (11.22), (11.21) with $J = k$ and Lemma 11.1, that condition (2.3)$^♣$ with $\theta = \mathfrak{N}$ holds for each $F_b^{(I)}(z;t)$ with $|t| \le \eta^{k+1}T^{(I)}$ whenever

$$\begin{aligned} U + (D_{-1}+1)(D_0+1)\left(\mathfrak{N} + \frac{1}{p-1}\right) \\ \ge (M+1)(2R+1)\mathfrak{N} + (M+1)\frac{\max(h+\nu\log q, \log(2R+1))}{\log p}. \end{aligned} \tag{11.27}$$

Similarly to §11$^♣$, we see that $(11.j) := (11.j)^♣$ holds for $j = 28, 29$. Thus by (9.14) and (9.32), we have

$$\begin{aligned} \frac{2c_5}{c_1}q^{k-r}\eta^k\left(1 - \frac{1}{2g_2}\right)U < (M+1)(2R+1)\mathfrak{N} \\ \le \frac{2c_5}{c_1}q^{k-r}(\eta^k + g_{11})\left(1 + \frac{1}{2g_2}\right)U. \end{aligned} \tag{11.30}$$

From (11.29), (9.12), (9.10), (9.1) and (9.2), we see that

$$\eta^{(r+1)I}\log(2R+1) \le h + \nu\log q.$$

This together with (11.28), (9.12), (9.14) and (9.32) gives

$$\begin{aligned} (M+1)\frac{\max(h+\nu\log q, \log(2R+1))}{\log p} \\ \le (\eta^k + g_{11})\frac{1}{(r+1)q^{r+1}} \cdot \frac{1}{e_{\mathfrak{p}}\mathfrak{N}} \cdot \frac{c_5}{c_1c_3}U. \end{aligned} \tag{11.31}$$

It is readily seen that the sum of the (extreme) right sides of (11.30) and (11.31) is at most its value at $k = r$:

$$\frac{U}{c_1} \cdot c_5(\eta^r + g_{11})\left\{2 + \frac{1}{g_2} + \frac{1}{(r+1)q^{r+1}} \cdot \frac{1}{e_{\mathfrak{p}}\mathfrak{N}} \cdot \frac{1}{c_3}\right\}.$$

Thus (11.27) follows from (9.6), (2.5), (2.4) and (2.8). By Lemma 2.1$^♣$ with $\theta = \mathfrak{N}$ and (11.24), we get, for $s \in \mathbb{Z}$ and $|t| \le \eta^{k+1}T^{(I)}$,

$$\operatorname{ord}_p f_b^{(I)}\left(\frac{s}{q};t\right) \ge (M+1)(2R+1)\mathfrak{N} - (D_{-1}+1)(D_0+1)\left(\mathfrak{N} + \frac{1}{p-1}\right) + \Delta_b^{(I)}.$$

Further Lemma 11.1, (11.27) and (11.30) give

$$\begin{aligned} \operatorname{ord}_p \varphi_b^{(I)}\left(\frac{s}{q};t\right) + (D_{-1}+1)(D_0+1)\left(\mathfrak{N} + \frac{1}{p-1}\right) - \Delta_b^{(I)} \\ > \frac{U}{c_1} \cdot 2c_5q^{k-r}\eta^k\left(1 - \frac{1}{2g_2}\right) \quad \text{for } s \in \mathbb{Z} \text{ and } |t| \le \eta^{k+1}T^{(I)}. \end{aligned} \tag{11.32}$$

Now we assume $k < r$ and prove (11.21) for $J = k+1$. Suppose it were false, i.e., $\varphi_b^{(I)}(s;t) \neq 0$ for some s, t with $|s| \leq q^{k+1}S^{(I)}$, $|t| \leq \eta^{k+1}T^{(I)}$. We proceed to deduce a contradiction. In the remaining part of the proof we fix this set of s, t.

For $\lambda \in \Lambda_b^{(I)}$, by (11.3) and (11.4), there exists $w^{(I)}(\lambda) \in \mathbb{Z}$ such that

$$\sum_{i=1}^{r} a_i'(\lambda_i - \lambda_i^{(I)}) = (b - b^{(I)})G_1 + w^{(I)}(\lambda)G_0,$$

whence by (10.2), (10.3) and (10.12), we have

$$\text{(11.33)}\quad \text{(i)}\ \prod_{i=1}^{r}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{s(\lambda_i-\lambda_i^{(I)})} = \zeta^{s(b-b^{(I)})G_1}\alpha_0^{sw^{(I)}(\lambda)}\prod_{i=1}^{r}\alpha_i'^{p^\kappa s(\lambda_i-\lambda_i^{(I)})},$$

$$\text{(ii)}\ \prod_{i=1}^{r}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{\frac{s}{q}(\lambda_i-\lambda_i^{(I)})} = \xi^{s(b-b^{(I)})G_1}(\alpha_0^{1/q})^{sw^{(I)}(\lambda)}\prod_{i=1}^{r}(\alpha_i'^{1/q})^{p^\kappa s(\lambda_i-\lambda_i^{(I)})}.$$

Let

$$\delta_I = \begin{cases} 0 & \text{if } I = 0,\\ 1 & \text{if } I > 0.\end{cases}$$

Then, by [9], III, Lemma 1.3 (see the remark before (10.11)$^\clubsuit$ in [10]),

$$\text{(11.34)}\quad q^{\delta_I\{I(D_{-1}+1)(D_0+1)+(D_0+1)\operatorname{ord}_q((D_{-1}+1)!)\}}\Theta(q^{-I}s;t)\Pi^{(I)}(t) \in \mathbb{Z}.$$

By (11.33)(i) and $\operatorname{ord}_{\mathfrak{p}} \alpha_i' = 0$ $(1 \leq i \leq r)$ (see §8), we have

$$\operatorname{ord}_p \varphi_b^{(I)}(s;t) = \operatorname{ord}_p \varphi',$$

where

$$\varphi' = \sum_{\lambda\in\Lambda_b^{(I)}} \rho^{(I)}(\lambda)(q^{\delta_I(D_0+1)\{(D_{-1}+1)I+\operatorname{ord}_q((D_{-1}+1)!)\}}\Theta(q^{-I}s;t)\Pi^{(I)}(t)) \times \alpha_0^{sw^{(I)}(\lambda)}\prod_{i=1}^{r}\alpha_i'^{p^\kappa s(\lambda_i-\lambda_i^{(I)}+D_i^{(I)})}$$

is in K and non-zero. Now let $|\,|_v$ be an absolute value on K normalized as in [1], §2, and let $|\,|_{v_0}$ be the one corresponding to $\mathfrak{p}$, whence

$$\text{(11.35)}\quad \operatorname{ord}_p \varphi' = \frac{1}{e_{\mathfrak{p}}f_{\mathfrak{p}}\log p}(-\log|\varphi'|_{v_0}) = \frac{1}{e_{\mathfrak{p}}f_{\mathfrak{p}}\log p}\sum_{v\neq v_0}\log|\varphi'|_v,$$

by the product formula on K. Note (11.36) := (11.36)$^\clubsuit$ holds, by (11.2)(i) and (9.28). Also, $(11.j) := (11.j)^\clubsuit$ $(j = 37, 41)$ remain valid. Now by (9.26),

$$\text{(11.38)}\quad \delta_I(D_0+1)\{(D_{-1}+1)I + \operatorname{ord}_q((D_{-1}+1)!)\}\log q \leq \left(1+\frac{1}{g_6}\right)\frac{\delta_I(I+1/(q-1))\log q}{\max(f_{\mathfrak{p}}\log p, g_1)}\cdot\frac{1}{c_1c_4\mathfrak{N}/\vartheta}\cdot\frac{SD}{d}.$$

By (10.17)♣, [9], II, Lemma 1.6 and (11.15), $q\eta^{r+1} > 1$, (9.29), we have

$$\log|\Theta(q^{-I}s;t)| \le \frac{107}{103}t_0(D_{-1}+1) + (D_{-1}+1)(D_0+1) \times \max(f_{\mathfrak p}\log p,\, g_1)\left(1+\frac{k\log q}{\max(f_{\mathfrak p}\log p,\, g_1)}\right).$$

Further, it is readily seen that (10.24)♣ with $\Pi(t)$ and γ_j replaced by $\Pi^{(I)}(t)$ and $\gamma_j^{(I)}$ remains true for the fixed t and any $\lambda \in \Lambda_b^{(I)}$, since $|\lambda_i-\lambda_i^{(I)}| \le D_i^{(I)}$ $(1 \le i \le r)$ by (11.2)(iii). These and (9.25), (9.26), $g_0 > 48$ imply for all $\lambda \in \Lambda_b^{(I)}$

$$\begin{aligned}(11.39)\qquad \log|\Theta(q^{-I}s;t)\Pi^{(I)}(t)| &\le \Bigg\{\left(\frac{107}{103}\cdot\frac{1}{e_{\mathfrak p}\mathfrak{N}}\eta^{k+1}+g_{10}\right)\frac{1}{c_1c_3} \\ &\quad+\left(1+\frac{k\log q}{\max(f_{\mathfrak p}\log p,\, g_1)}\right)\left(1+\frac{1}{g_6}\right)\frac{1}{c_1c_4(\mathfrak{N}/\vartheta)}\Bigg\}\frac{SD}{d}.\end{aligned}$$

Observe that for $\lambda \in \Lambda_b^{(I)}$,

$$(11.40)\qquad \log\Big|\alpha_0^{sw^{(I)}(\lambda)}\prod_{i=1}^{r}\alpha_i'^{\,p^\kappa s(\lambda_i-\lambda_i^{(I)}+D_i^{(I)})}\Big|_v \le p^\kappa\sum_{i=1}^{r}2D_i^{(I)}\log\max(1,|\alpha_i'^{\,s}|_v).$$

Now, using (11.34)–(11.41), (11.17), (9.24), (9.26), (9.32), we obtain

$$(11.42)\qquad \operatorname{ord}_p\varphi_b^{(I)}(s;t)+(D_{-1}+1)(D_0+1)\left(\mathfrak{N}+\frac{1}{p-1}\right)-\Delta_b^{(I)} \le \frac{U}{c_1q^{r+1}}\times\text{the right side of (9.36)}.$$

Hence by (9.36), (11.42) contradicts (11.32). This contradiction proves (11.21) for $J=k+1$. Thus the induction on J is complete and Lemma 11.2 follows at once.

LEMMA 11.3. *For every I as in Lemma* 11.2 *we have*

$$(11.43)\qquad \varphi_b^{(I)}(s/q;t)=0 \quad \textit{for all } b\in\mathcal{B}^{(I)},\ |s|\le q([S^{(I+1)}]+1),\ |t|\le T^{(I+1)}.$$

Proof. Note that $T^{(I+1)}=\eta^{r+1}T^{(I)}$ and by (9.9) we have (11.44) := (11.44)♣, whence (11.43) for s with $q\mid s$ follows from Lemma 11.2 with $J=1$. Now we consider s with $(s,q)=1$. For any fixed $b\in\mathcal{B}^{(I)}$, we may assume that $\rho^{(I)}(\lambda)$, $\lambda\in\Lambda_b^{(I)}$, are not all zero. Now, by (11.32) with $k=r$ we have, for $s \in \mathbb{Z}$, $|t| \le T^{(I+1)}$,

$$\operatorname{ord}_p \varphi_b^{(I)}\left(\frac{s}{q};t\right) + (D_{-1}+1)(D_0+1)\left(\mathfrak{N} + \frac{1}{p-1}\right) - \Delta_b^{(I)} > \frac{U}{c_1} \cdot 2c_5\eta^r\left(1 - \frac{1}{2g_2}\right). \tag{11.45}$$

Let $K' = K(\alpha_0^{1/q}, \alpha_1'^{1/q}, \ldots, \alpha_r'^{1/q})$. By consecutively applying Fröhlich and Taylor [3], III.2, (2.28)(c) $r+1$ times, we see that

$$\mathfrak{p}\mathcal{O}_{K'} = \mathfrak{P}_1\mathfrak{P}_2 \cdots \mathfrak{P}_{q^{r_1}}$$

for some r_1 with $0 \le r_1 \le r+1$, where $\mathfrak{P}_j$ are distinct prime ideals of $\mathcal{O}_{K'}$ with ramification index and residue class degree (over $\mathbb{Q}$)

$$e_{\mathfrak{P}_j} = e_{\mathfrak{p}}, \quad f_{\mathfrak{P}_j} = q^{r+1-r_1} f_{\mathfrak{p}}, \quad j = 1, \ldots, q^{r_1}.$$

Denote by $|\,|_{v'}$ an absolute value on K' normalized as in [1], §2, and let $|\,|_{v'_j}$ be the one corresponding to $\mathfrak{P}_j$, and $K'_{\mathfrak{P}_j}$ be the completion of K' with respect to $|\,|_{v'_j}$. The embedding of $K_{\mathfrak{p}}$ into $\mathbb{C}_p$ (see §1) can be extended to an embedding of $K'_{\mathfrak{P}_j}$ into $\mathbb{C}_p$ and we define, for $\beta \in K'_{\mathfrak{P}_j}$ with $\beta \ne 0$,

$$\operatorname{ord}_p^{(j)} \beta := \frac{1}{e_{\mathfrak{P}_j} f_{\mathfrak{P}_j} \log p}(-\log|\beta|_{v'_j}) = \frac{1}{q^{r+1-r_1} e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}(-\log|\beta|_{v'_j}).$$

On noting that $\varphi_b^{(I)}(s/q;t) \in K_{\mathfrak{p}}$ $(\subset K'_{\mathfrak{P}_j})$, we have

$$\operatorname{ord}_p^{(j)} \varphi_b^{(I)}(s/q;t) = \operatorname{ord}_p \varphi_b^{(I)}(s/q;t) \quad (j = 1, \ldots, q^{r_1}),$$

whence (11.45) yields

$$\sum_{j=1}^{q^{r_1}} \operatorname{ord}_p^{(j)} \varphi_b^{(I)}\left(\frac{s}{q};t\right) + q^{r_1}(D_{-1}+1)(D_0+1)\left(\mathfrak{N} + \frac{1}{p-1}\right) - q^{r_1}\Delta_b^{(I)} > \frac{U}{c_1} \cdot 2c_5 q^{r_1}\eta^r\left(1 - \frac{1}{2g_2}\right). \tag{11.45$^\spadesuit$}$$

Suppose that (11.43) were false, that is,

$$\varphi_b^{(I)}(s/q;t) \ne 0$$

for some s, t with $(s,q) = 1$, $|s| \le q([S^{(I+1)}]+1)$, $|t| \le T^{(I+1)}$. We proceed to deduce a contradiction. In the sequel we fix this set of s, t.

Now by [9], III, Lemma 1.3, we have, for $\lambda \in \Lambda_b^{(I)}$,

$$q^{(D_0+1)\{(D_{-1}+1)(I+1)+\operatorname{ord}_q((D_{-1}+1)!)\}} \Theta(q^{-(I+1)}s;t)\Pi^{(I)}(t) \in \mathbb{Z}. \tag{11.46}$$

Hence, by (11.33)(ii) and $\operatorname{ord}_{\mathfrak{p}} \alpha'_i = 0$ $(1 \le i \le r)$ (see §8), we have, for $j = 1, \ldots, q^{r_1}$,

$$\operatorname{ord}_p^{(j)} \varphi_b^{(I)}(s/q;t) = \operatorname{ord}_p^{(j)} \varphi'', \tag{11.47}$$

where

$$(11.48)\qquad \varphi'' = \sum_{\lambda\in\Lambda_b^{(I)}} \rho^{(I)}(\lambda)(q^{(D_0+1)\{(D_{-1}+1)(I+1)+\mathrm{ord}_q((D_{-1}+1)!)\}} \times \Theta(q^{-(I+1)}s;t)\Pi^{(I)}(t))(\alpha_0^{1/q})^{sw^{(I)}(\lambda)} \times \prod_{i=1}^{r}(\alpha_i'^{1/q})^{p^\kappa s(\lambda_i-\lambda_i^{(I)}+D_i^{(I)})} \neq 0$$

is in $K' = K(\alpha_0^{1/q}, \alpha_1'^{1/q}, \ldots, \alpha_r'^{1/q})$. Then, by the product formula on K',

$$(11.49)\qquad \sum_{j=1}^{q^{r_1}} \mathrm{ord}_p^{(j)}\,\varphi'' = \frac{1}{q^{r+1-r_1}e_{\mathfrak p}f_{\mathfrak p}\log p}\Big(-\sum_{j=1}^{q^{r_1}}\log|\varphi''|_{v_j'}\Big) = \frac{1}{q^{r+1-r_1}e_{\mathfrak p}f_{\mathfrak p}\log p}{\sum}'\log|\varphi''|_{v'},$$

where $\sum'$ signifies the summation over all $v' \neq v_1', \ldots, v_{q^{r_1}}'$. Note that (11.50) := (11.50)♣ holds. By (9.26), (11.51) := (11.51)♣ with its right side divided by $(\mathfrak{N}/\vartheta)$ is valid. Similarly to the proof of (11.52)♣ and (11.39), we can verify that for all $\lambda\in\Lambda_b^{(I)}$,

$$(11.52)\qquad \log|\Theta(q^{-(I+1)}s;t)\Pi^{(I)}(t)| \le \left(\frac{107}{103}\cdot\frac{1}{e_{\mathfrak p}\mathfrak{N}}\eta^{r+1}+g_{10}\right)\cdot\frac{1}{c_1c_3}\cdot\frac{SD}{d} + \left(1+\frac{1}{g_6}\right)\cdot\frac{1}{c_1c_4(\mathfrak{N}/\vartheta)}\cdot\frac{SD}{d}.$$

Note that

$$(11.53)\qquad \log\Big|\prod_{i=1}^{r}(\alpha_i'^{1/q})^{p^\kappa s(\lambda_i-\lambda_i^{(I)}+D_i^{(I)})}\Big|_{v'} \le p^\kappa\sum_{i=1}^{r}2D_i^{(I)}\cdot\log\max(1,|(\alpha_i'^{1/q})^s|_{v'}).$$

Also (11.54) := (11.54)♣ is valid. Observe that by (8.3) we have $[K':\mathbb{Q}] = q^{r+1}d$. Utilizing (11.36), (11.46)–(11.54), (11.17), (9.24), (9.26), (9.32), we see that

$$(11.55)\qquad \sum_{j=1}^{q^{r_1}}\mathrm{ord}_p^{(j)}\,\varphi_b^{(I)}\left(\frac{s}{q};t\right)+q^{r_1}(D_{-1}+1)(D_0+1)\left(\mathfrak{N}+\frac{1}{p-1}\right)-q^{r_1}\Delta_b^{(I)} \le \frac{U}{q^{r+1-r_1}c_1}\left\{c_1\left[g_{12}+\left(1+\frac{1}{2(c_0-1)}\right)g_9\right] + \left[\frac{q}{(q\eta^{r+1})^I}+\frac{1}{2(c_0-1)}\left(1+\frac{1}{2g_2+1}\right)\right]\frac{2}{c_2}\right.$$

$$+\left[\frac{107}{103}\cdot\frac{1}{e_{\mathfrak{p}}\mathfrak{N}}\left(\eta^{r+1}+\frac{1}{c_0-1}\right)+\left(1+\frac{1}{c_0-1}\right)g_{10}\right]\frac{1}{c_3}+\left(1+\frac{1}{g_6}\right)$$
$$\times\left[1+\frac{1}{c_0-1}+\frac{(I+q/(q-1))\log q}{\max(f_{\mathfrak{p}}\log p,g_1)}+\left(\mathfrak{N}+\frac{1}{p-1}\right)\frac{1}{f_{\mathfrak{p}}}\right]\frac{1}{c_4(\mathfrak{N}/\vartheta)}\Bigg\}.$$

Now we prove

$$\frac{q^{r+1-r_1}c_1}{U}\times\text{the right side of (11.55)} \le (q\eta)^r\times\text{the right side of (9.5).} \tag{11.56}$$

If $p>2$, then by (11.20) and (9.7)(i), the left side of (11.56) as a function of I attains its maximum at $I=0$, whence (11.56) follows from the inequality (which is a consequence of (9.1)–(9.3))

$$(q\eta)^r-1\ge 2^re^{-c_5}-1>\frac{2\log 2}{\log(e^4\cdot 4)}\ge\frac{q\log q}{(q-1)g_1}.$$

If $p=2$, then by (11.20), (9.7)(ii), (9.8), the value of the left side of (11.56) increases when $q/(q\eta^{r+1})^I$ is replaced by q and $I\log q/\max(f_{\mathfrak{p}}\log p,\,g_1)$ is replaced by $5(1-1/q)\log q/\log(q\eta^{r+1})$, whence (11.56) follows from

$$((q\eta)^r-1)\frac{2q}{c_2}\ge\left(1+\frac{1}{g_6}\right)\frac{1}{c_4(\mathfrak{N}/\vartheta)}\left\{\frac{q\log q}{(q-1)g_1}+\left(1-\frac{1}{q}\right)\frac{5\log q}{\log(q\eta^{r+1})}\right\},$$

which can be verified by direct computation, using (9.1)–(9.3) and (9.33). Thus (11.56) is proved. By (11.56) and (9.5), (11.55) contradicts (11.45)$^\spadesuit$. This contradiction proves (11.43), and the proof of Lemma 11.3 is thus complete.

In order to prove Lemma 11.4, we need the following observation (recalling that q is given by (1.3)):

(11.57) *If $\beta\in K_{\mathfrak{p}}$ satisfies $\beta\equiv 1\pmod{\mathfrak{p}}$ and $\beta^q\equiv 1\pmod{\mathfrak{p}^m}$ for a positive integer m, then $\beta\equiv 1\pmod{\mathfrak{p}^m}$.*

Proof. We prove (11.57) for the case $p=2$. Then $\mathfrak{p}\,|\,2$, $q=3$ and $\zeta_3\in K_{\mathfrak{p}}$ by (1.2). Thus $(\beta-1)(\beta-\zeta_3)(\beta-\zeta_3^2)\equiv 0\pmod{\mathfrak{p}^m}$. We assert that $\beta-\zeta_3\not\equiv 0\pmod{\mathfrak{p}}$, for otherwise we would have $\zeta_3-1\equiv 0\pmod{\mathfrak{p}}$, whence $2\,|\,N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(\zeta_3-1)$ $(=3)$, which is absurd. Similarly, $\beta-\zeta_3^2\not\equiv 0\pmod{\mathfrak{p}}$. Hence $\beta\equiv 1\pmod{\mathfrak{p}^m}$. We omit the proof for the case $p>2$.

LEMMA 11.4. *For every I as in Lemma 11.2, there exist $\varepsilon^{(I+1)}\in\mathbb{N}$, $D_i^{(I+1)}\in\mathbb{N}$ $(-1\le i\le r)$, $\Lambda^{(I+1)}$ with*

$$\gcd(a_1',\dots,a_r',G_1)\,|\,\varepsilon^{(I+1)}, \tag{11.58}$$

$$D_i^{(I+1)}=D_i\quad(i=-1,0),\qquad D_i^{(I+1)}\le q^{-(I+1)}D_i\quad(1\le i\le r), \tag{11.59}$$

(11.60) $\Lambda^{(I+1)}$ *having properties* (11.2) *with I replaced by $I+1$,* *and* $\rho^{(I+1)}(\lambda) = \rho^{(I+1)}(\lambda_{-1}, \ldots, \lambda_r) \in \mathcal{O}_K$ $(\lambda \in \Lambda^{(I+1)})$, *not all zero*, *satisfying* (11.17) *with* I *replaced by* $I+1$, *such that*

$$(11.61) \quad \varphi_b^{(I+1)}(s;t) = 0 \quad \text{for all } b \in \mathcal{B}^{(I+1)},\ |s| \le q([S^{(I+1)}]+1),\ |t| \le \eta T^{(I+1)},$$

where $\mathcal{B}^{(I+1)}$, $\Lambda_b^{(I+1)}$, *and* $\varphi_b^{(I+1)}(z;t)$ *are defined by* (11.1), (11.3) *and* (11.14) *with* I *replaced by* $I+1$.

Proof. From Lemma 11.3, we have

$$(11.62) \quad \varphi^{(I)}(s/q;t) = \sum_{b\in\mathcal{B}^{(I)}} \varphi_b^{(I)}(s/q;t) = 0 \quad \text{for } |s| \le q([S^{(I+1)}]+1),\ |t| \le T^{(I+1)}\ .$$

By (1.2), (8.3), (10.3) and an argument similar to that in the proof of [9], III, Lemma 2.5, it is readily seen that

$$(11.63) \qquad [K(\xi^{G_1})({\alpha'_1}^{1/q}, \ldots, {\alpha'_r}^{1/q}) : K(\xi^{G_1})] = q^r.$$

Recalling (11.4), for $(\lambda_1^*, \ldots, \lambda_r^*) \in \mathbb{Z}^r$ with $0 \le \lambda_i^* < q$ $(1 \le i \le r)$ we define

$$\Lambda^{(I)}(\lambda_1^*, \ldots, \lambda_r^*) = \{\iota = (\iota_{-1}, \ldots, \iota_r) \in \Lambda^{(I)} \mid \iota_i - \lambda_i^{(I)} \equiv \lambda_i^* \pmod q,\ 1 \le i \le r\},$$

$$\Gamma^{(I)}(\lambda_1^*, \ldots, \lambda_r^*) = \{\lambda = (\lambda_{-1}, \ldots, \lambda_r) \mid \lambda_i = \iota_i\ (i=-1,0),\ \lambda_i = (\iota_i - \lambda_i^{(I)} - \lambda_i^*)/q\ (1 \le i \le r) \text{ with } \iota \in \Lambda^{(I)}(\lambda_1^*, \ldots, \lambda_r^*)\}.$$

By the hypothesis that the main inductive argument holds for I, there exists an r-tuple $(\lambda_1^*, \ldots, \lambda_r^*)$ such that $\Lambda^{(I)}(\lambda_1^*, \ldots, \lambda_r^*) \ne \emptyset$ and $\rho^{(I)}(\iota)$, $\iota \in \Lambda^{(I)}(\lambda_1^*, \ldots, \lambda_r^*)$, are not all zero. Set

$$\Lambda^{(I+1)} = \Gamma^{(I)}(\lambda_1^*, \ldots, \lambda_r^*),$$
$$\rho^{(I+1)}(\lambda) = \rho^{(I)}(\iota) \quad \text{for } \lambda \in \Lambda^{(I+1)} \text{ corresponding to } \iota \in \Lambda^{(I)}(\lambda_1^*, \ldots, \lambda_r^*).$$

Thus $h_0(\rho^{(I+1)})$ $(\le h_0(\rho^{(I)}))$ satisfies (11.17), and $0 < \#\Lambda^{(I+1)} \le \#\Lambda^{(I)} \le \prod_{i=-1}^r (D_i+1)$. Further

$$\sum_{i=1}^r a'_i(q\lambda_i + \lambda_i^*) = \sum_{i=1}^r a'_i(\iota_i - \lambda_i^{(I)}) \equiv 0 \pmod{G_1},$$

whence

$$\sum_{i=1}^r a'_i\lambda_i \equiv \varepsilon^{(I+1)} \pmod{G_1},$$

where $\varepsilon^{(I+1)} \in \mathbb{N}$ is a solution to the congruence $qx \equiv -\sum_{i=1}^r a'_i\lambda_i^* \pmod{G_1}$, which is soluble uniquely mod G_1 by (10.1). Thus $\Lambda^{(I+1)}$ satisfies (11.2)(ii) with I replaced by $I+1$ and (11.58) is valid. Finally if λ and λ' in $\Lambda^{(I+1)}$ correspond to ι and ι' in $\Lambda^{(I)}(\lambda_1^*,\ldots,\lambda_r^*)$, then

$$\iota_i-\iota_i'=q(\lambda_i-\lambda_i')\quad (1\le i\le r).$$

So, by (11.2), $|\lambda_i-\lambda_i'|\le D_i^{(I+1)}:=[q^{-1}D_i^{(I)}]\le q^{-(I+1)}D_i$ $(1\le i\le r)$. We choose $D_i^{(I+1)}=D_i$ $(i=-1,0)$. Also by (11.2)(iii)

$$\Big(\prod_{i=1}^{r}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{\lambda_i-\lambda_i'}\Big)^q\equiv\prod_{i=1}^{r}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{\iota_i-\iota_i'}\equiv 1 \pmod{\mathfrak{p}^{\mathfrak{m}_0+\mathfrak{m}}},$$

whence by (11.57),

$$\prod_{i=1}^{r}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{\lambda_i}\equiv\prod_{i=1}^{r}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{\lambda_i'}\equiv 1 \pmod{\mathfrak{p}^{\mathfrak{m}_0+\mathfrak{m}}}.$$

Thus we have proved that $\Lambda^{(I+1)}$ satisfies (11.2) with I replaced by $I+1$. We define $\mathcal{B}^{(I+1)}$ and $\Lambda_b^{(I+1)}$ by (11.1) and (11.3) with I replaced by $I+1$.

Now we fix $(\lambda_1^{(I+1)},\ldots,\lambda_r^{(I+1)})$ and $b^{(I+1)}\in\mathcal{B}^{(I+1)}$ such that (11.4) with I replaced by $I+1$ holds. Note that for $\iota\in\Lambda^{(I)}(\lambda_1^*,\ldots,\lambda_r^*)$ and $1\le j\le r-1$,

$$\gamma_j^{(I)}:=\sum_{i=1}^{r}(b_n\partial L_i/\partial z_j-b_j\partial L_i/\partial z_n)(\iota_i-\lambda_i^{(I)})=q\gamma_j^{(I+1)}+\gamma_j^*,$$

where $\gamma_j^{(I+1)}$ is given by the right side of (10.20) with $\lambda_i^{(0)}$ replaced by $\lambda_i^{(I+1)}$ $(1\le i\le r)$ and γ_j^* is given by the right side of (10.20) with $\lambda_i-\lambda_i^{(0)}$ replaced by $q\lambda_i^{(I+1)}+\lambda_i^*$ $(1\le i\le r)$. Thus $\gamma_j^*\in\mathbb{Z}$ $(1\le j\le r-1)$. By (11.62), (11.63) and arguing similarly to the proof of [9], II, Lemma 2.5, we obtain

$$\sum_{\lambda\in\Lambda^{(I+1)}}\rho^{(I+1)}(\lambda)\Theta(q^{-(I+1)}s;t)\times\prod_{j=1}^{r-1}\Delta(q\gamma_j^{(I+1)}+\gamma_j^*;t_j)\prod_{i=1}^{r}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{\lambda_i s}=0 \tag{11.64}$$

for $|s|\le q([S^{(I+1)}]+1)$ with $(s,q)=1$, $|t|\le T^{(I+1)}$.

On multiplying both sides of (11.64) by $\prod_{i=1}^{r}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{-\lambda_i^{(I+1)}s}$ and defining $\varphi^{(I+1)}(z;t)$ by (11.13) with I replaced by $I+1$, and utilizing the argument of [8], p. 160, which is based on [8], Lemma 2.6, we get

$$\varphi^{(I+1)}(s;t)=0,\qquad |s|\le q([S^{(I+1)}]+1)\text{ with }(s,q)=1,\ |t|\le T^{(I+1)}. \tag{11.65}$$

Note that ζ^{G_1sb}, $b=0,1,\ldots,q^{\mu-u}-1$, are linearly independent over K. (See the proof of [9], III, Lemma 2.1. Here we have used $(s,q)=1$.) By this fact and (11.33)(i) with I replaced by $I+1$, (11.65) implies

$$\varphi_b^{(I+1)}(s;t) = 0 \tag{11.66}$$

for all $b \in \mathcal{B}^{(I+1)}$, $|s| \le q([S^{(I+1)}]+1)$ with $(s,q)=1$, $|t| \le T^{(I+1)}$.

It remains to verify (11.61) for s with $q \mid s$. In order to prove (11.61) with any fixed $b \in \mathcal{B}^{(I+1)}$, we may assume $\rho^{(I+1)}(\lambda)$, $\lambda \in \Lambda_b^{(I+1)}$, are not all zero, and set

$$\Delta_b^{(I+1)} = \min_{\lambda \in \Lambda_b^{(I+1)}} \operatorname{ord}_p \rho^{(I+1)}(\lambda).$$

We now apply Lemma 2.2$^{\clubsuit}$ to each function in (11.24) with I replaced by $I+1$ and with $|t| \le \eta T^{(I+1)}$, taking

$$R = q([S^{(I+1)}]+1), \qquad M = \left[\frac{c_5}{r+1} T^{(I+1)}\right]. \tag{11.67}$$

Similarly to the deduction of (11.27), by utilizing (11.25), (11.22), Lemma 11.1 (with I replaced by $I+1$) and (11.66), we see that condition (2.6)$^{\clubsuit}$ with $\vartheta = \mathfrak{N}$ holds for each $F_b^{(I+1)}(z;t)$ with $|t| \le \eta T^{(I+1)}$ whenever

$$\begin{aligned} U + (D_{-1}+1)(D_0+1)\left(\mathfrak{N} + \frac{1}{p-1}\right) \\ \ge 2\left(1 - \frac{1}{q}\right) R(M+1)\mathfrak{N} + (2M+2)\frac{\max(h+\nu\log q, \log(2R))}{\log p}. \end{aligned} \tag{11.68}$$

We now verify (11.68). Note that (11.69):=(11.63)$^{\clubsuit}$ and (11.70):=(11.64)$^{\clubsuit}$ hold, whence $\eta^{(r+1)I}\log(2R) \le h + \nu \log q$ (by (9.12), (9.10), (9.1), (9.2)) and

$$\begin{aligned} (2M+2)\frac{\max(h+\nu\log q, \log(2R))}{\log p} \\ \le (\eta^{r+1} + g_{11})\frac{1}{(r+1)q^{r+1}} \cdot \frac{1}{e_{\mathfrak{p}}\mathfrak{N}} \cdot \frac{2c_5}{c_1 c_3} U. \end{aligned} \tag{11.71}$$

By (11.69), (11.70), (9.14) and (9.32) we have

$$\frac{2c_5}{c_1} \cdot \frac{q-1}{q^r} U < 2\left(1 - \frac{1}{q}\right) R(M+1)\mathfrak{N} \le \frac{2c_5}{c_1} \cdot \frac{q-1}{q^{r-1}} \cdot (\eta^{r+1} + g_{11}) U. \tag{11.72}$$

From (9.2), (2.5), (2.4) and (2.8) we see that

$$\frac{2(q-1)}{q^{r-1}} + \frac{1}{(r+1)q^{r+1} e_{\mathfrak{p}} \mathfrak{N} c_3} \le 2.$$

Hence

$$\frac{c_1}{U} \times \text{the sum of the (extreme) right sides of (11.71) and (11.72)} \le \text{the right side of (9.6).}$$

Thus (11.68) follows from (9.6), whence (2.6)$^\clubsuit$ with $\vartheta = \mathfrak{N}$ holds for each $F_b^{(I+1)}(z;t)$ with $|t| \le \eta T^{(I+1)}$. By applying Lemma 2.2$^\clubsuit$ with $\theta = \mathfrak{N}$ to $F_b^{(I+1)}(z;t)$, and utilizing (11.24), Lemma 11.1 with I replaced by $I+1$, (11.68) and (11.72), we obtain

$$(11.73)\qquad \operatorname{ord}_p \varphi_b^{(I+1)}(s;t) + (D_{-1}+1)(D_0+1)\left(\mathfrak{N} + \frac{1}{p-1}\right) - \Delta_b^{(I+1)} > \frac{2c_5}{c_1}\cdot\frac{q-1}{q^r}U$$

for $|s| \le q([S^{(I+1)}]+1)$ with $q \mid s$, $|t| \le \eta T^{(I+1)}$.

We now assume $\varphi_b^{(I+1)}(s;t) \ne 0$ for some s, t in the range stated in (11.73) and proceed to deduce a contradiction. In the sequel we fix this set of s, t.

For $\lambda \in \Lambda_b^{(I+1)}$, by [9], III, Lemma 1.3 and the fact that $q \mid s$, we see that (recalling $\delta_I = 0$ if $I = 0$ and $\delta_I = 1$ if $I > 0$)

$$(11.74)\qquad q^{\delta_I(D_0+1)\{(D_{-1}+1)I + \operatorname{ord}_q((D_{-1}+1)!)\}}\Theta(q^{-(I+1)}s;t)\Pi^{(I+1)}(t) \in \mathbb{Z}.$$

Now by (11.33)(i) with I replaced by $I+1$ and $\operatorname{ord}_{\mathfrak{p}} \alpha_i' = 0$ $(1 \le i \le r)$ (see §8),

$$(11.75)\qquad \operatorname{ord}_p \varphi_b^{(I+1)}(s;t) = \operatorname{ord}_p \varphi''',$$

where

$$(11.76)\qquad \varphi''' = \sum_{\lambda \in \Lambda_b^{(I+1)}} \rho^{(I+1)}(\lambda)(q^{\delta_I(D_0+1)\{(D_{-1}+1)I + \operatorname{ord}_q((D_{-1}+1)!)\}} \times \Theta(q^{-(I+1)}s;t)\Pi^{(I+1)}(t))\alpha_0^{sw^{(I+1)}(\lambda)} \prod_{i=1}^{r} \alpha_i'^{\,p^\kappa s(\lambda_i - \lambda_i^{(I+1)} + D_i^{(I+1)})}$$

is in K and non-zero. Let $|\,|_v$ and $|\,|_{v_0}$ be as in the proof of Lemma 11.2. Thus

$$(11.77)\qquad \operatorname{ord}_p \varphi''' = \frac{1}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}(-\log|\varphi'''|_{v_0}) = \frac{1}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}\sum_{v \ne v_0} \log|\varphi'''|_v,$$

by the product formula on K. Note that (11.36)$^\clubsuit$ and (11.37)$^\clubsuit$ with I replaced by $I+1$ remain valid. On noting $|t| \le \eta T^{(I+1)} \le \eta^{r+2}T$ and (9.29), (10.24)$^\clubsuit$ with $\Pi(t)$ and γ_j replaced by $\Pi^{(I+1)}(t)$ and $\gamma_j^{(I+1)}$, we have

$$(11.78)\qquad \log|\Theta(q^{-(I+1)}s;t)\Pi^{(I+1)}(t)| \le \left(\frac{107}{103}\cdot\frac{1}{e_{\mathfrak{p}}\mathfrak{N}}\eta^{r+2} + g_{10}\right)\cdot\frac{1}{c_1c_3}\cdot\frac{SD}{d} + \left(1 + \frac{1}{g_6}\right)\cdot\frac{1}{c_1c_4(\mathfrak{N}/\vartheta)}\cdot\frac{SD}{d}.$$

Further, (11.40) with I replaced by $I+1$ holds for every $\lambda \in \Lambda_b^{(I+1)}$. Also, by (11.44) (= (11.44)$^{\clubsuit}$), for $1 \le i \le r$,

$$\sum_{v \ne v_0} \log\max(1, |\alpha_i'^{s}|_v) \le q^2 \eta^{-(r+1)I} S d\sigma_i. \tag{11.79}$$

Summing up, and using (11.38), we obtain

$$\begin{aligned} &\operatorname{ord}_p \varphi_b^{(I+1)}(s;t) + (D_{-1}+1)(D_0+1)\left(\mathfrak{N} + \frac{1}{p-1}\right) - \Delta_b^{(I+1)} \\ &\le \frac{U}{c_1 q^{r+1}} \left\{ c_1 \left[g_{12} + \left(1 + \frac{1}{2(c_0-1)}\right) g_9 \right] \right. \\ &\quad + \left[\frac{q}{(q\eta^{r+1})^I} + \frac{1}{2(c_0-1)}\left(1 + \frac{1}{2g_2+1}\right)\right] \frac{2}{c_2} \\ &\quad + \left[\frac{107}{103} \cdot \frac{1}{e_{\mathfrak{p}}\mathfrak{N}} \left(\eta^{r+2} + \frac{1}{c_0-1}\right) + \left(1 + \frac{1}{c_0-1}\right) g_{10} \right] \frac{1}{c_3} + \left(1 + \frac{1}{g_6}\right) \\ &\quad \left. \times \left[1 + \frac{1}{c_0-1} + \frac{\delta_I(I + 1/(q-1))\log q}{\max(f_{\mathfrak{p}}\log p,\, g_1)} + \left(\mathfrak{N} + \frac{1}{p-1}\right)\frac{1}{f_{\mathfrak{p}}} \right] \frac{1}{c_4(\mathfrak{N}/\vartheta)} \right\}. \end{aligned} \tag{11.80}$$

If $I = 0$, then

$$\frac{c_1 q^{r+1}}{U} \times \text{the right side of (11.80)} \le \text{the right side of (9.5)},$$

whence, by (9.5), (11.80) contradicts (11.73). If $I > 0$ and $p > 2$ (thus $q = 2$), then, by (11.20), (9.7)(i) and $\eta^{r+1} < e^{-c_5}$, we have

$$\begin{aligned} &\frac{c_1 q^{r+1}}{U} \times \text{the right side of (11.80)} - \text{the right side of (9.5)} \\ &\qquad \le \frac{107}{103} \cdot \frac{1}{e_{\mathfrak{p}}\vartheta_0 + 1} \cdot \frac{1}{c_3}(\eta^{r+2} - \eta) + \left(1 + \frac{1}{g_6}\right) \frac{1}{c_4(\mathfrak{N}/\vartheta)} \cdot \frac{\log q}{(q-1)g_1} \le 0. \end{aligned}$$

Now by (9.5), (11.80) contradicts (11.73). If $I > 0$ and $p = 2$ (thus $q = 3$), then

$$\frac{c_1 q^{r+1}}{U} \times \text{the right side of (11.80)} \le 2 \times \text{the right side of (9.5)}, \tag{11.81}$$

since by (11.20), (9.7)(ii) and (9.8), the value of the left side of (11.81) increases when $q/(q\eta^{r+1})^I$ is replaced by q and $I \log q/\max(f_{\mathfrak{p}}\log p,\, g_1)$ is replaced by $5(1 - 1/q)\log q/\log(q\eta^{r+1})$, whence (11.81) follows from

$$\begin{aligned} &\left(q + \frac{1}{2(c_0-1)}\right)\frac{2}{c_2} + \frac{107}{103} \cdot \frac{1}{e_{\mathfrak{p}}\vartheta_0 + 1}\left(\eta(2 - \eta^{r+1}) + \frac{1}{c_0-1}\right)\frac{1}{c_3} \\ &\qquad \ge \left(1 + \frac{1}{g_6}\right)\frac{1}{c_4(\mathfrak{N}/\vartheta)} \left\{ \frac{\log q}{(q-1)g_1} + \left(1 - \frac{1}{q}\right)\frac{5\log q}{\log(q\eta^{r+1})} \right\}, \end{aligned}$$

which can be verified by direct computation, using (9.1)–(9.3) and (9.33). By (11.81) and (9.5), (11.80) contradicts (11.73). The fact that (11.80) contradicts (11.73) for all I as in the hypothesis of the lemma proves

$$\varphi_b^{(I+1)}(s;t) = 0 \qquad \text{for } |s| \le q([S^{(I+1)}]+1) \text{ with } q \mid s,\ |t| \le \eta T^{(I+1)}.$$

Since $b \in \mathcal{B}^{(I+1)}$ is arbitrarily chosen, this and (11.66) establish Lemma 11.4.

By applying Lemma 11.2 to $I = 0$ and taking $J = 1$, and by applying Lemma 11.4 to $I = 0$, we see that the main inductive argument is true for $I = 0, 1$. Now the main inductive argument follows by induction on I, utilizing Lemma 11.4.

12. Simple reduction. We first deal with the case $I := [\log D_r/\log q] + 1 \le I^*$. Thus $D_r^{(I)} = 0$. We recall (11.2) and (11.4). On applying the main inductive argument and defining $\rho^{(I)}(\lambda_{-1}, \ldots, \lambda_r) = 0$ for $\lambda \notin \Lambda^{(I)}$ with $0 \le \lambda_i \le D_i^{(I)}$ $(i = -1, 0)$, $|\lambda_i - \lambda_i^{(I)}| \le D_i^{(I)}$ $(1 \le i \le r)$, we have

$$\sum_{\substack{0 \le \lambda_i \le D_i^{(I)},\, i=-1,0 \\ |\lambda_i - \lambda_i^{(I)}| \le D_i^{(I)},\, 1 \le i < r}} \rho^{(I)}(\lambda_{-1}, \ldots, \lambda_{r-1}, \lambda_r^{(I)}) \Theta(q^{-I}s;t) \times \Pi(\gamma_1^{(I)}, \ldots, \gamma_{r-1}^{(I)}; t_1, \ldots, t_{r-1}) \cdot \prod_{i=1}^{r-1} (\alpha_i'^{p^\kappa} \zeta^{a_i'})^{s(\lambda_i - \lambda_i^{(I)})} = 0 \tag{12.1}$$

for $|s| \le qS^{(I)}$, $|t| \le \eta T^{(I)}$, where

$$\gamma_j^{(I)} = \sum_{i=1}^{r-1} (b_n \partial L_i/\partial z_j - b_j \partial L_i/\partial z_n)(\lambda_i - \lambda_i^{(I)}) \qquad (1 \le j < r), \tag{12.2}$$

since $\lambda_r = \lambda_r^{(I)}$ because of $D_r^{(I)} = 0$. In virtue of (12.2) and (8.15)$^\clubsuit$, each of $\lambda_1 - \lambda_1^{(I)}, \ldots, \lambda_{r-1} - \lambda_{r-1}^{(I)}$ is a linear combination of $\gamma_1^{(I)}, \ldots, \gamma_{r-1}^{(I)}$. Thus $\prod_{i=1}^{r-1} \Delta(\lambda_i - \lambda_i^{(I)}; t_i)$ $(t_i \in \mathbb{N},\ 1 \le i < r)$ is a linear combination of ${\gamma_1^{(I)}}^{\tau_1} \cdots {\gamma_{r-1}^{(I)}}^{\tau_{r-1}}$, whence, by [8], Lemma 2.6, it is a linear combination of $\Pi(\gamma_1^{(I)}, \ldots, \gamma_{r-1}^{(I)}; \tau_1, \ldots, \tau_{r-1})$ with $(\tau_1, \ldots, \tau_{r-1}) \in \mathbb{N}^{r-1}$ and $\tau_1 + \ldots + \tau_{r-1} \le t_1 + \ldots + t_{r-1}$. So (12.1) gives

$$\sum_{\substack{0 \le \lambda_i \le D_i^{(I)},\, i=-1,0 \\ |\lambda_i - \lambda_i^{(I)}| \le D_i^{(I)},\, 1 \le i < r}} \rho^{(I)}(\lambda_{-1}, \ldots, \lambda_{r-1}, \lambda_r^{(I)}) \Theta(q^{-I}s;t) \times \prod_{i=1}^{r-1} \Delta(\lambda_i - \lambda_i^{(I)}; t_i) \prod_{i=1}^{r-1} (\alpha_i'^{p^\kappa} \zeta^{a_i'})^{s(\lambda_i - \lambda_i^{(I)})} = 0 \tag{12.3}$$

for $|s| \le qS^{(I)}$, $|t| \le \eta T^{(I)}$.

Note that $2D_1^{(I)} + \ldots + 2D_{r-1}^{(I)} \le \frac{1}{2}\eta T^{(I)}$ by (11.15), $D_i^{(I)} \le q^{-I} D_i$ $(1 \le i < r)$, (9.14), (9.17), (9.2), (9.3), (2.5), (2.6), (2.1), (9.33) and

$q\eta^{r+1} > 1$. Thus (12.3) holds for $|s| \le qS^{(I)}$ and t with

$$0 \le t_0 \le \tfrac{1}{2}\eta T^{(I)}, \quad 0 \le t_i \le 2D_i^{(I)} \quad (1 \le i < r).$$

It is readily seen, similarly to the proof of [8], Lemma 2.5, that for any $m \in \mathbb{N}$, the determinant of order $2m+1$

$$\det(\Delta(j;k))_{-m \le j \le m,\, 0 \le k \le 2m} \ne 0. \tag{12.4}$$

Thus, by an argument similar to [8], §3.5, we see that for any fixed $\lambda_1, \ldots, \lambda_{r-1}$ with $|\lambda_i - \lambda_i^{(I)}| \le D_i^{(I)}$ $(1 \le i < r)$, the polynomial (recalling $D_i^{(I)} = D_i$, $i = -1, 0$)

$$\sum_{\lambda_{-1}=0}^{D_{-1}} \sum_{\lambda_0=0}^{D_0} \rho^{(I)}(\lambda_{-1}, \ldots, \lambda_{r-1}, \lambda_r^{(I)})(\Delta(x+\lambda_{-1}; D_{-1}+1))^{\lambda_0+1}, \tag{12.5}$$

whose degree is at most $(D_{-1}+1)(D_0+1)$, has at least

$$M := (2[qS^{(I)}]+1)\big(\big[\tfrac{1}{2}\eta T^{(I)}\big]+1\big)$$

zeros. Now, by (9.12), $g_2 > 1$, $g_6 > 1$, $c_4 > 3$, $c_5 < 1$ (see (9.1), (9.2)), (9.3), (9.14), (9.19), (9.26) and (11.15), we have

$$M > \left(2q - \frac{1}{g_2}\right) \cdot \frac{1}{2}\eta S^{(I)} T^{(I)} \ge (D_{-1}+1)(D_0+1).$$

Thus the polynomial in (12.5) is identically zero. Further, $(\Delta(x+\lambda_{-1}; D_{-1}+1))^{\lambda_0+1}$ $(0 \le \lambda_i \le D_i,\ i = -1, 0)$ are linearly independent over $\mathbb{C}$ (see [1], §12). Hence

$$\rho^{(I)}(\lambda_{-1}, \ldots, \lambda_r) = 0, \quad 0 \le \lambda_i \le D_i^{(I)} \quad (i = -1, 0),$$
$$|\lambda_i - \lambda_i^{(I)}| \le D_i^{(I)} \quad (1 \le i \le r)$$

(recalling $D_r^{(I)} = 0$), contradicting the construction in the main inductive argument that $\rho^{(I)}(\lambda)$, $\lambda \in \Lambda^{(I)}$, are not all zero. This contradiction proves Proposition 9.1 in the case when $[\log D_r/\log q] + 1 \le I^*$.

13. Group variety reduction. It remains to prove Proposition 9.1 in the case

$$I^* < [\log D_r/\log q] + 1, \tag{13.1}$$

where I^* is given by (9.8). Take $I = I^*$ in the main inductive argument (see §11). There exists $b \in \mathcal{B}^{(I)}$ such that $\rho^{(I)}(\lambda)$, $\lambda \in \Lambda_b^{(I)}$, are not all zero. For every s with $q^u \mid s$ we have, by (11.33)(i), (1.4), and on multiplying (11.18) by $\prod_{i=1}^{r} \alpha_i'^{p^\kappa q^u s D_i^{(I)}}$,

$$\sum_{\lambda \in \Lambda_b^{(I)}} \rho^{(I)}(\lambda) \Pi^{(I)}(t) \Theta(q^{-I} q^u s; t) \prod_{i=1}^{r} \alpha_i'^{p^\kappa q^u s(\lambda_i - \lambda_i^{(I)} + D_i^{(I)})} = 0 \tag{13.2}$$

for $|s| \le q^{1-u} S^{(I)}$, $|t| \le \eta T^{(I)}$.

Recall that $\Pi^{(I)}(t)$ is given by (11.6). Now we take

$$\begin{aligned} &\mathcal{P}(Y_0, \ldots, Y_r) \\ &\quad = \sum_{\lambda \in \Lambda_b^{(I)}} \rho^{(I)}(\lambda) (\Delta(q^{-I} p^{-\kappa} Y_0; D_{-1} + 1))^{\lambda_0 + 1} \prod_{i=1}^{r} Y_i^{\lambda_i - \lambda_i^{(I)} + D_i^{(I)}}, \end{aligned} \tag{13.3}$$

$$N = p^\kappa q^u, \quad \mathcal{S} = q^{1-u} S^{(I)}, \quad \mathcal{T} = \eta T^{(I)}, \quad \theta_i = \alpha_i' \ (1 \le i \le r). \tag{13.4}$$

Recall that $\partial_0^* = \partial_0 = \partial/\partial Y_0$ and $\partial_1^*, \ldots, \partial_{r-1}^*$ are the differential operators specified in §8♣, and note that

$$\partial_j^* \prod_{i=1}^{r} Y_i^{\lambda_i - \lambda_i^{(I)} + D_i^{(I)}} = (\gamma_j^{(I)} + \gamma_j^\dagger) \prod_{i=1}^{r} Y_i^{\lambda_i - \lambda_i^{(I)} + D_i^{(I)}} \qquad (1 \le j < r),$$

with $\gamma_j^{(I)}$ given in (11.5), and $\gamma_j^\dagger \in \mathbb{Z}$ given by the right side of (11.5) with $\lambda_i - \lambda_i^{(I)}$ replaced by $D_i^{(I)}$. By [8], Lemma 2.6, we see from (13.2)–(13.4) that

$$\partial_0^{*t_0} \partial_1^{*t_1} \cdots \partial_{r-1}^{*t_{r-1}} \mathcal{P}(Ns, \theta_1^{Ns}, \ldots, \theta_r^{Ns}) = 0 \tag{13.5}$$

for $0 \le s \le \mathcal{S}$, $t_0 + \ldots + t_{r-1} \le \mathcal{T}$.

We note, as remarked in §8♣, that Proposition 6.1♣ holds with $\partial_1^*, \ldots, \partial_{r-1}^*$ in place of $\partial_1, \ldots, \partial_{r-1}$. Let

$$\mathcal{D}_0 = (D_{-1} + 1)(D_0 + 1), \qquad \mathcal{D}_i = 2q^{-I} \widetilde{D}_i \quad (1 \le i \le r), \tag{13.6}$$

$$\begin{gathered} \mathcal{S}_0 = \left[\frac{1}{4}\mathcal{S}\right], \quad \mathcal{S}_i = \left[\frac{1}{r} \cdot \frac{3}{4}\mathcal{S}\right] \quad (1 \le i \le r), \\ \mathcal{T}_i = \left[\frac{1}{r+1}\mathcal{T}\right] \quad (0 \le i \le r). \end{gathered} \tag{13.7}$$

Then $\mathcal{S}_0 \ge \mathcal{S}_1 = \ldots = \mathcal{S}_r$ since $r \ge 3$, $\mathcal{T}_0 = \ldots = \mathcal{T}_r$ and $\mathcal{S}_0 + \ldots + \mathcal{S}_r \le \mathcal{S}$, $\mathcal{T}_0 + \ldots + \mathcal{T}_r \le \mathcal{T}$.

For later convenience, we list the following inequalities derived from §9 and §2. We shall use them frequently in the remainder of this section.

$c_3/c_4 > 1/28, \quad c_5 \ge 0.47, \quad r \ge 3 \text{ if } p > 2, \quad r \ge 4 \text{ if } p = 2,$
$g_3/r > 10^7, \quad g_6 > 10^6, \quad (1+\epsilon_1)(1+\epsilon_2) < 1 + 10^{-5},$
$h \ge \max(2f_{\mathfrak{p}} \log p,\, g_0) \ge \max(f_{\mathfrak{p}} \log p,\, g_1), \quad \vartheta \le p/(p-1) \le 2,$
$e_{\mathfrak{p}}\vartheta \ge 1/2, \quad q\eta^{r+1} > 1, \quad \eta^{r+1} < e^{-c_5} \le e^{-0.47},$
$I = I^* \ge [5g_1/\log(q\eta^{r+1})] + 1 \ge 92, \quad \eta^{(r+1)I} < 10^{-18}.$

By (13.4), (13.7) and (9.20) we have, for $\rho \in \mathbb{Z}$ with $1 \le \rho \le r$,

$$\mathcal{T}_\rho + \rho \le \left(\frac{\eta^{(r+1)I}}{r+1} + \frac{\rho}{g_3}\right)T < 10^{-6}T. \tag{13.8}$$

Now (13.8), (13.6), (9.12), (9.14) and (9.16) yield

$$\mathcal{T}_\rho + \rho < 10^{-4}\mathcal{D}_0 \quad (1 \le \rho \le r), \tag{13.9}$$

whence (6.2)♣ follows. Further by (13.6), (9.17), (9.16), (9.12), (2.5), (2.6), we get

$$\mathcal{D}_i < \mathcal{D}_0 \quad (1 \le i \le r). \tag{13.10}$$

Now we verify (6.1)♣.

(i) $m = 0$. By (13.7), (13.4), (11.15), (9.18), (9.19), we have

$$\mathcal{S}_0(\mathcal{T}_0 + 1) > \left(\frac{(q-1)e_{\mathfrak{p}}}{4d} - \frac{10^{-18}}{g_2}\right)\frac{\eta}{r+1}ST. \tag{13.11}$$

From (13.11), (13.6), (9.2), (9.3), (9.26), (9.14), we obtain

$$\mathcal{S}_0(\mathcal{T}_0 + 1) > \mathcal{D}_0.$$

This and (13.10) establish (6.1)♣ with $m = 0$.

(ii) $1 \le m < r$. By (13.7), (13.4), (11.15), (9.18), (9.19), we have

$$\mathcal{S}_m\binom{\mathcal{T}_m + m + \delta_{m,r}}{m + \delta_{m,r}} > \left(\frac{1}{r}\cdot\frac{3}{4}(q-1)\frac{e_{\mathfrak{p}}}{d} - \frac{10^{-18}}{g_2}\right)\frac{\eta^{m+1+(r+1)Im}}{(r+1)^{m+1}(m+1)!}ST^{m+1}. \tag{13.12}$$

By (13.6), (13.10), (9.26) and (9.17) we get

$$(m+1)!\mathcal{D}_0^{m_0}\cdots\mathcal{D}_r^{m_r} \le \left(1 + \frac{1}{g_6}\right)\frac{1}{c_4(\mathfrak{N}/\vartheta)}\cdot\frac{1}{c_1^{m+1}(c_2p^\kappa)^m}\cdot\frac{(m+1)!2^m}{r^m q^{Im}}\cdot\frac{SD^{m+1}}{d(f_{\mathfrak{p}}\log p)^{m+1}}, \tag{13.13}$$

where $m_i \in \{0,1\}$ with $m_0 + \ldots + m_r = m + 1$. Now, by (13.12), (13.13), (9.14), (2.5), (2.6), (2.1) and

$$(q\eta^{r+1})^{Im} > (e^4(r+1)d)^{5m} \quad \text{(see (9.8), (9.1))},$$

we obtain (6.1)♣ for $1 \le m < r$.

(iii) $m = r$. We have, similarly to (13.12),

$$\mathcal{S}_r \binom{\mathcal{T}_r + r}{r} > \left(\frac{1}{r} \cdot \frac{3}{4}(q-1)\frac{e_{\mathfrak{p}}}{d} - \frac{10^{-18}}{g_2} \right) \frac{\eta^{r+(r+1)I(r-1)}}{(r+1)^r r!} ST^r. \tag{13.14}$$

By (13.6), (9.26), (9.17), we have

$$\begin{aligned} &(r+1)!\mathcal{D}_0\mathcal{D}_1\cdots\mathcal{D}_r \\ &\le \frac{(r+1)!2^r}{q^{Ir}r^r}\left(1+\frac{1}{g_6}\right)\frac{SD^{r+1}}{c_1^{r+1}(c_2p^{\kappa})^r c_4(\mathfrak{N}/\vartheta)d^{r+1}\sigma_1\cdots\sigma_r \max(f_{\mathfrak{p}}\log p, g_1)}. \end{aligned} \tag{13.15}$$

In virtue of (13.14), (13.15), (9.13), (9.14), (2.5), (2.6), (2.3), in order to prove (6.1)$^{\clubsuit}$ with $m = r$, it suffices to show

$$\begin{aligned} &\left(\frac{1}{r} \cdot \frac{3}{4}(q-1)\frac{e_{\mathfrak{p}}}{d} - \frac{10^{-18}}{g_2} \right)(q\eta^{r+1})^{Ir}\eta^{r-(r+1)I} \\ &\qquad \ge (1+10^{-100})\left(1+\frac{1}{g_6}\right)(1+\epsilon_1)(1+\epsilon_2)\left(2+\frac{1}{g_2}\right) \\ &\qquad\quad \times c_0(r+1)!(r+1)^r(2e\varsigma_1\varsigma_2)^r \cdot \frac{p^{f_{\mathfrak{p}}}-1}{q^u}. \end{aligned} \tag{13.16}$$

Now, by $I = I^*$, (9.8) and (9.1),

$$(q\eta^{r+1})^I > \exp(5\max(f_{\mathfrak{p}}\log p, g_1)) \ge p^{f_{\mathfrak{p}}}(e^4(r+1)d)^4.$$

This implies (13.16) at once. Hence (6.1)$^{\clubsuit}$ with $m = r$ is valid.

Note that in (13.5), $\mathcal{P}(Y_0,\ldots,Y_r) \ne 0$ and $\theta_i = \alpha_i'$ $(1 \le i \le r)$ are multiplicatively independent, since l_0' $(= l_0), l_1', \ldots, l_r'$ are linearly independent over $\mathbb{Q}$ by (5.8)$^{\clubsuit}$ and §8(i), (ii). Having verified (6.1)$^{\clubsuit}$ and (6.2)$^{\clubsuit}$, we can apply Proposition 6.1$^{\clubsuit}$ with $a_i = \sigma_i$ $(1 \le i \le r)$. Thus there exists $\rho \in \mathbb{Z}$ with $1 \le \rho < r$ and there is a set of primitive linear forms $\mathcal{L}_1, \ldots, \mathcal{L}_\rho$ in $Z_1, \ldots, Z_r$ with coefficients in $\mathbb{Z}$ such that $B_1Z_1 + \ldots + B_rZ_r$ is in the module generated by $\mathcal{L}_1, \ldots, \mathcal{L}_\rho$ over $\mathbb{Q}$ and, on defining

$$\mathcal{R}_i = \sum_{j=1}^{r} |\partial\mathcal{L}_i/\partial Z_j|\sigma_j \qquad (1 \le i \le \rho), \tag{13.17}$$

we have at least one of (6.3)$^{\clubsuit}$ and (6.4)$^{\clubsuit}$, whence (6.4)$^{\clubsuit}$ holds always, since (6.3)$^{\clubsuit}$ implies (6.4)$^{\clubsuit}$ by (13.4), (13.6), (13.7) and (13.9). We shall prove shortly that (6.4)$^{\clubsuit}$ implies

$$\mathcal{R}_1\cdots\mathcal{R}_\rho \le \psi(\rho)h'(\alpha_1)\cdots h'(\alpha_n), \tag{13.18}$$

where $\psi(\rho)$ is given by (8.7) with r replaced by ρ; thus (13.18) together with the same analysis as in §13$^{\clubsuit}$ shows that the basic hypothesis in §8 holds with ρ in place of r. By the minimal choice of r, we have a contradiction and this establishes Proposition 9.1.

Now, by (6.4)♣, (13.4), (13.6), (13.7), (9.13), (9.14), (9.17), (9.18), (9.19), (9.26), (8.6), (8.7), (2.5), (2.6), (2.3), (2.1) and $e^r \ge r^r/r!$, in order to prove (13.18), it suffices to show

$$\frac{(\varsigma_1\varsigma_2 e^2)^\rho(\rho+1)(\rho!)^3\rho^\rho r^\rho p^{f_{\mathfrak p}}}{\frac{1}{r}\cdot\frac{3}{4}(q-1)\frac{e_{\mathfrak p}}{d}-\frac{10^{-18}}{g_2}} \le (q\eta^{r+1})^{I\rho}. \tag{13.19}$$

Note that $(\rho+1)(\rho!)^3\rho^\rho r^\rho \le (r+1)^{5\rho-2}$ and that by $I = I^*$ and (9.8) we have

$$(q\eta^{r+1})^{I\rho} > p^{f_{\mathfrak p}}(e^4(r+1)d)^{5\rho-1}.$$

Hence (13.19) follows. The proof of Proposition 9.1 is thus complete, whence Theorem 7.1 is established.

14. Proof of Theorem 1

LEMMA 14.1. *It suffices to prove Theorem* 1 *on a further assumption that* $\alpha_1, \ldots, \alpha_n$ *are multiplicatively independent.*

Proof. Note that from (1.10), (1.12), Lemma 2.2 and $(1+1/n)^{n+1} > e$, we have for $n = 2, 3, \ldots,$

$$\frac{C(n,d,\mathfrak p)}{C(n-1,d,\mathfrak p)} > \frac{C^*(n,d,\mathfrak p)}{C^*(n-1,d,\mathfrak p)} > \frac{16}{9}e^2(n+2)d. \tag{14.1}$$

Using (14.1), the proof is the same as that of Lemma 14.1♣.

Proof of Theorem 1. By [9], II, Lemma 1.4, it is readily seen that Theorem 1 is true for $n = 1$. Thus we may assume that $n \ge 2$ and $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent by Lemma 14.1. Note that

$$\frac{d}{f_{\mathfrak p}\log p}\log 2 < 0.001C(n,d,\mathfrak p)h'(\alpha_1)\cdots h'(\alpha_n)\log B.$$

Hence we may assume, by (1.6), (1.10), (14.2)♣, (14.3)♣ and Stirling's formula, that

$$\frac{B}{\log B} > 30(8e^2)^n n^{3/2}(p^{f_{\mathfrak p}}-1)d > e^{13},$$

whence we obtain

$$B > 260(8e^2)^n n^{3/2} p^{f_{\mathfrak p}} d. \tag{14.2}$$

We may further assume, without loss of generality, that (1.13) is satisfied, since the main inequality in Theorem 1 is symmetric in $\alpha_1, \ldots, \alpha_n$. On appealing to Theorem 7.1 and observing that (14.2) implies

$$(n+1)\log B \ge \log\max\{\widetilde{c}B, \widetilde{c}(5n)^{6n}d^{1.2}, \widetilde{c}p^{2f_{\mathfrak p}}\} \ge h^* + \log c^*,$$

where h^* is given by (7.2) and

$$\widetilde{c} = \left(\frac{3}{4}\log^3(5d)\cdot f_{\mathfrak{p}}\log p\right)^n \cdot n! \ge c^* \tag{14.3}$$

by (5.21)♣ and (5.24)♣, Theorem 1 follows at once.

15. Proof of Theorem 2

LEMMA 15.1. *It suffices to prove Theorem* 2 *on a further assumption that* $\alpha_1, \ldots, \alpha_n$ *are multiplicatively independent.*

Proof. Using (14.1), the proof is the same as that of Lemma 15.1♣.

Proof of Theorem 2. It is readily seen, by Theorem 1 with $n = 1$, that Theorem 2 is true for $n = 1$. Thus we may assume that $n \ge 2$ and $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent by Lemma 15.1. On using (14.1), the proof is the same as that of Theorem 2♣.

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