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Refinement of an estimate for the Hurwitz zeta function in a neighbourhood of the line $\sigma = 1$

by

MIECZYSŁAW KULAS (Poznań)

The well-known estimate of the order of the Hurwitz zeta function

$$\zeta(s,\alpha) - \alpha^{-s} \ll t^{c(1-\sigma)^{3/2}} \log^{2/3} t$$

is proved with the constant $c = 18.4974$ for $1/2 \le \sigma \le 1$, $t \ge t_0 > 0$.

The improvement of the constant c is a consequence of some technical modifications in the method of estimating exponential sums sketched by Heath-Brown ([11], p. 136).

I. Introduction. In 1967 H. E. Richert [9] proved for the Hurwitz zeta function (defined in the half plane $\mathrm{Re}(s) > 1$ by $\zeta(s,\alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s}$, $0 < \alpha \le 1$) that

$$|\zeta(s,\alpha) - \alpha^{-s}| \le c_0 t^{c(1-\sigma)^{3/2}} \log^{2/3} t \tag{1}$$

for $1/2 \le \sigma \le 1$, $t \ge 2$, where c_0 is an absolute positive constant and $c = 100$. This leads to the same bound for the Riemann zeta function,

$$|\zeta(\sigma+it)| \le c_1 t^{100(1-\sigma)^{3/2}} \log^{2/3} t \tag{2}$$

for $1/2 \le \sigma \le 1$, $t \ge 2$ and a positive constant c_1.

More generally, one can deduce from (1) that if $L(s,\chi)$ denotes the Dirichlet L-function associated with the Dirichlet character $\chi \pmod k$, $k \ge 1$, then

$$|L(\sigma+it,\chi)| \le c_2 k^{1-\sigma} t^{100(1-\sigma)^{3/2}} \log^{2/3} t + k^{1-\sigma} \log k \tag{3}$$

for the same range as (2) and a positive constant c_2.

The bounds of the type (2) or (3) have existed for a long time in the literature and have various applications (zero-free regions, a problem of Dirichlet divisors in number fields, the order of the Dedekind zeta function of a quadratic field, and so on).

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Several authors have reduced the constant $c = 100$ in (2) or (3). For example, $c = 86$ [2], $c = 39$ [12] (also compare [4], Chapter 6 with $c = 122$, $c = 2^{15}$ [10]).

In 1988 using Tyrina's version (see [13]) of Vinogradov's mean value theorem, Panteleeva [6] proved that $c = 21$ in (3) but it seems that this result is incorrect (since Tyrina's result has a factor n^{4k^2} in the "constant" and it is not clear how to make it an absolute constant). Note that in [7] (1994) Panteleeva postulated $c = 21.57$.

Heath-Brown ([11], p. 135) pointed out that "the best result up-to-date appears to be one in which 100 is replaced by 18.8" (Heath-Brown, unpublished).

In this paper we will show that Richert's result (1) can be sharpened for a given range of σ and sufficiently large $t > 0$. We shall prove the following

THEOREM. *If $s = \sigma + it$ and $0 < \alpha \le 1$, then there exists an absolute positive constant c_0 such that*

$$|\zeta(s,\alpha) - \alpha^{-s}| \le c_0 t^{c(1-\sigma)^{3/2}} \log^{2/3} t$$

for $1/2 \le \sigma \le 1$, $t \ge t_0 > 0$ and $c = 18.4974$.

The improvement of the constant c is a consequence of some technical modifications in the method of estimating exponential sums sketched by Heath-Brown ([11], p. 136).

Perhaps, the latest developments in the theory of I. M. Vinogradov's mean value theorem (due to T. Wooley and others) could be used to obtain an even better value of c. Of course the up-to-date constant $c = 18.4974$ is still large, particularly in view of the fact that according to the Riemann hypothesis it should tend to zero.

II. Lemmas. In the proof of the Theorem we use some lemmas, presented below for convenience. We suppose that $s = \sigma + it$ and $0 < \alpha \le 1$. All constants occurring in the Vinogradov symbol $\ll$ are absolute.

LEMMA 1 (compare [9], p. 101). *For $0 < \sigma \le 1$, $t \ge t_1 > 0$,*

$$\sum_{n \le \exp(\log^{2/3} t)} (n+\alpha)^{-s} \ll t^{(1-\sigma)^{3/2}} \log^{2/3} t.$$

LEMMA 2 ([5], p. 124). *Let $\sigma > 0$, $t \ge t_2 > 0$, $M, N \in \mathbb{N}$, $N < M \le 2N$, $\exp(\log^{2/3} t) < N \le t^{1/1000}$. Then there exist positive constants γ, δ such that*

$$\Big|\sum_{N < n \le M} (n+\alpha)^{-s}\Big| \le \gamma N^{1-\sigma-\delta(\frac{\log N}{\log t})^2}$$

where $\gamma = 2.003$ and $\delta = (2309.525)^{-1}$.

This lemma plays the main role in the proof of our Theorem. The latest version of Vinogradov's mean value theorem joined with some technical modifications in the method of estimating exponential sums (Heath-Brown [11], p. 136) enables us to get a very good value of the constant δ. Numerical calculations show that we can get a very small improvement of δ if we decrease the exponent $1/1000$ of t, so in this light our choice of the range of N seems to be optimal.

COROLLARY 1. *For* $0 < \sigma \le 1$ *and* $t \ge t_3 > 0$,

$$S = \sum_{\exp(\log^{2/3} t) < n \le t^{1/1000}} (n+\alpha)^{-s} \ll t^{B(1-\sigma)^{3/2}} \log^{2/3} t,$$

where $B = 18.4974$.

Proof. Let

$$Q = [\exp(\log^{2/3} t)] + 1.$$

We see that

$$S = (Q+\alpha)^{-s} + \sum_{Q < n \le t^{1/1000}} (n+\alpha)^{-s}. \tag{1}$$

Let r be the largest integer such that $Q \cdot 2^r < T^{1/1000}$. Then

$$\begin{aligned}\sum_{Q < n \le t^{1/1000}} (n+\alpha)^{-s} \\ = \sum_{m=0}^{r-1} \sum_{Q\cdot 2^m < n \le Q\cdot 2^{m+1}} (n+\alpha)^{-s} + \sum_{Q\cdot 2^r < n \le t^{1/1000}} (n+\alpha)^{-s}.\end{aligned} \tag{2}$$

From Lemma 2 we get

$$\begin{aligned}\sum_{Q < n \le t^{1/1000}} (n+\alpha)^{-s} \\ \ll \sum_{m=0}^{r} (Q\cdot 2^m)^{1-\sigma-\delta(\frac{\log Q\cdot 2^m}{\log t})^2} \\ = \sum_{m=0}^{r} \exp\left\{(1-\sigma)\log(Q\cdot 2^m) - \delta\frac{\log^3 Q\cdot 2^m}{\log^2 t}\right\} = S_1.\end{aligned} \tag{3}$$

If $\delta = (2309.525)^{-1} = \delta_1 + \delta_2$, $\delta_1, \delta_2 > 0$, then

$$\begin{aligned}S_1 = \sum_{m=0}^{r} \exp\left\{(1-\sigma)\log(Q\cdot 2^m) - \delta_1\frac{\log^3 Q\cdot 2^m}{\log^2 t}\right\} \\ \times \exp\left\{-\delta_2\frac{\log^3 Q\cdot 2^m}{\log^2 t}\right\}.\end{aligned} \tag{4}$$

Considering the function $f(x)=(1-\sigma)x-\delta_1 x^3/\log^2 t$, $x>0$, we shall see that f has a maximum at the point $x_0=\sqrt{\frac{1-\sigma}{3\delta_1}}\log t$ and $f(x_0)=\frac{2}{3}(\sqrt{3\delta_1})^{-1}(1-\sigma)^{3/2}\log t$. This implies that

$$S_1 \le \Big(\sum_{m=0}^{r}\exp\Big\{-\delta_2\frac{\log^3 Q\cdot 2^m}{\log^2 t}\Big\}\Big)t^{(2/3)(\sqrt{3\delta_1})^{-1}(1-\sigma)^{3/2}}. \tag{5}$$

Now

$$\begin{aligned}\sum_{m=0}^{r}\exp\Big\{-\delta_2\frac{\log^3 Q\cdot 2^m}{\log^2 t}\Big\} &\le \exp\Big\{-\delta_2\frac{\log^3 Q}{\log^2 t}\Big\}+\int_0^r\exp\Big\{-\delta_2\frac{\log^3 Q\cdot 2^x}{\log^2 t}\Big\}\,dx\\ &< e^{-\delta_2}+\int_0^\infty\exp\Big\{-\delta_2\frac{\log^3 Q\cdot 2^x}{\log^2 t}\Big\}\,dx\\ &\le e^{-\delta_2}+1.5(\log 2)^{-1}\Big(\frac{\log^2 t}{\delta_2}\Big)^{1/3}.\end{aligned} \tag{6}$$

Choose the parameters δ_1,δ_2 in the way that $\delta_1=(2309.526)^{-1}$. This gives

$$\tfrac{2}{3}(\sqrt{3\delta_1})^{-1}=18.497351\ldots<18.4974=B \tag{7}$$

and $\delta_2=\delta-\delta_1>10^{-3}\cdot(2309.526)^{-2}$.

From (4)–(7) we see that

$$S_1 \ll t^{B(1-\sigma)^{3/2}}\log^{2/3} t \tag{8}$$

and from (1)–(3) and (8) we have

$$S \ll Q^{-\sigma}+t^{B(1-\sigma)^{3/2}}\log^{2/3} t \ll t^{B(1-\sigma)^{3/2}}\log^{2/3} t.$$

LEMMA 3 ([9], Hilfssatz 4). *For* $1-1/2^{r+1}\le\sigma$, $4\le r\le\log\log t$ *and* $t\ge t_4>0$,

$$\sum_{t^{1/(r-1)}<n\le t^{1/2}}(n+\alpha)^{-s}=O(1).$$

From Lemma 3 we get immediately:

COROLLARY 2. *For* $1-2^{-1002}\le\sigma$ *and* $t\ge t_5>0$,

$$\sum_{t^{1/1000}<n\le t^{1/2}}(n+\alpha)^{-s}=O(1).$$

According to Vinogradov's theorem in the form given in [14] (Th. 1b, p. 114, compare an example after that theorem) one can easily achieve the following:

LEMMA 4. *Let* $\sigma > 0$, $t \ge t_6 > 0$, $k, P_0, P \in \mathbb{N}$, $k \ge 11$, $0 < P_0 \le P$ *and* $t^{1/k} \le P \le t^{1/(k-1)}$. *Then*

$$\sum_{P \le n \le P+P_0-1} (n+\alpha)^{-s} \ll (8k)^{(k \log 120k)/2} P^{1-\sigma-\varrho},$$

where $\varrho = (3k^2 \log 120k)^{-1}$.

As an application of this lemma one can obtain

COROLLARY 3. *For* $0 < \sigma < 1 - 2^{-1002}$ *and* $t \ge t_7 > 0$,

$$\sum_{t^{1/1000} < n \le t^{1/10}} (n+\alpha)^{-s} \ll t^{18.1(1-\sigma)^{3/2}} \log^{2/3} t.$$

Proof. Consider a sum $\sum_{t^{1/k} < n \le t^{1/(k-1)}} (n+\alpha)^{-s}$, $11 \le k \le 1000$. If $Q = [t^{1/k}] + 1$ and r denotes the largest integer such that $Q \cdot 2^r < t^{1/(k-1)}$, then

$$\sum_{t^{1/k} < n \le t^{1/(k-1)}} (n+\alpha)^{-s} = \sum_{m=0}^{r-1} \sum_{Q\cdot 2^m \le n \le Q \cdot 2^{m+1}-1} (n+\alpha)^{-s} + \sum_{Q\cdot 2^r \le n \le t^{1/(k-1)}} (n+\alpha)^{-s}.$$

With the help of Lemma 4 we see that

$$\sum_{t^{1/k} < n \le t^{1/(k-1)}} (n+\alpha)^{-s} \ll \sum_{m=0}^{r} (Q \cdot 2^m)^{1-\sigma-\varrho}, \qquad \varrho = (3k^2 \log 120k)^{-1},$$

because the dependence on k can be incorporated in the order constant ($k \le 1000$).

Clearly $2^{-1002} - \varrho < 1 - \sigma - \varrho < 1 - \varrho$ and $r \ll \log t$. If $1 - \sigma - \varrho \le 0$ then

$$\sum_{t^{1/k} < n \le t^{1/(k-1)}} (n+\alpha)^{-s} \ll \log t \ll t^{18(1-\sigma)^{3/2}} \log t \ll t^{18.1(1-\sigma)^{3/2}} \log^{2/3} t. \tag{1}$$

Otherwise

$$\sum_{t^{1/k} < n \le t^{1/(k-1)}} (n+\alpha)^{-s} \ll t^{(1/(k-1))(1-\sigma-\varrho)} \log t = t^{18(1-\sigma)^{3/2} - f(1-\sigma)} \log t, \tag{2}$$

on defining $f(x) = 18x^{3/2} - (x-\varrho)(k-1)^{-1}$, $x > 0$.

Simple calculations show that f has a global minimum on $(0, \infty)$ at the point $x_0 = (27(k-1))^{-2}$ and $f(x_0) = -(3^7(k-1)^3)^{-1} + \varrho(k-1)^{-1}$.

It follows that

$$\sum_{t^{1/k}<n\le t^{1/(k-1)}} (n+\alpha)^{-s} \ll t^{18(1-\sigma)^{3/2}-f(x_0)}\log t. \tag{3}$$

However, for $11\le k\le 1000$ we can find that

$$\begin{aligned} f(x_0) &= -(3^7(k-1)^3)^{-1}+(3k^2(k-1)\log 120k)^{-1}\\ &= \frac{1}{3(k-1)^3}\bigg\{\bigg(\frac{k-1}{k}\bigg)^2\frac{1}{\log 120k}-\frac{1}{3^6}\bigg\}\\ &\ge \frac{1}{3(k-1)^3}\bigg\{\bigg(\frac{10}{11}\bigg)^2\frac{1}{\log 120\cdot 10^3}-\frac{1}{3^6}\bigg\} = \frac{1}{3(k-1)^3}\lambda_1. \end{aligned}$$

Numerical calculations show that $\lambda_1>0$ whence

$$\sum_{t^{1/k}<n\le t^{1/(k-1)}} (n+\alpha)^{-s} \ll t^{18(1-\sigma)^{3/2}}\log t \ll t^{18.1(1-\sigma)^{3/2}}\log^{2/3} t. \tag{4}$$

From (1) and (2)–(4) we can easily conclude that

$$\sum_{t^{1/1000}<n\le t^{1/10}} (n+\alpha)^{-s} = \sum_{k=11}^{1000}\sum_{t^{1/k}<n\le t^{1/(k-1)}} (n+\alpha)^{-s} \ll t^{18.1(1-\sigma)^{3/2}}\log^{2/3} t$$

for $0<\sigma<1-2^{-1002}$ and sufficiently large $t>0$.

LEMMA 5 ([1], Lemma 6, p. 15). *Let* $\sigma\ge 0$, $t\ge t_8>0$, $k,K,M,N\in\mathbb{N}$, $k\ge 2$, $K=2^k$, $N\le M\le 2N$ *and* $1\le N\le t^{2/3}$. *Then*

$$\sum_{N\le n\le M} (n+\alpha)^{-s} \ll N^{1-\sigma-k/(K-2)}t^{1/(K-2)}.$$

COROLLARY 4. *For* $0<\sigma<1<2^{-1002}$, $t\ge t_9>0$ *and* $4\le k\le 17$,

$$\sum_{t^{1/(k-1)}<n\le t^{1/2}} (n+\alpha)^{-s} \ll t^{18.1(1-\sigma)^{3/2}}\log^{2/3} t.$$

Proof. Taking $k\ge 4$, $Q=[t^{1/(k-1)}]+1$, and defining r as the largest integer such that $Q\cdot 2^r\le t^{1/2}$ we have

$$\begin{aligned} &\sum_{t^{1/(k-1)}<n\le t^{1/2}} (n+\alpha)^{-s}\\ &\qquad= \sum_{m=0}^{r-1}\sum_{Q\cdot 2^m\le n\le Q\cdot 2^{m+1}} (n+\alpha)^{-s} + \sum_{Q\cdot 2^r\le n\le t^{1/2}} (n+\alpha)^{-s}. \end{aligned}$$

From Lemma 5 we get

$$\sum_{t^{1/(k-1)}<n\le t^{1/2}} (n+\alpha)^{-s} \ll \sum_{m=0}^{r} (Q\cdot 2^m)^{1-\sigma-k/(K-2)}t^{1/(K-2)}.$$

Clearly

$$2^{-1002}-\frac{k}{K-2}<1-\sigma-\frac{k}{K-2}<1-\frac{k}{K-2},\qquad r\ll\log t.$$

If $1-\sigma-k/(K-2)\le 0$, then

$$\sum_{t^{1/(k-1)}<n\le t^{1/2}}(n+\alpha)^{-s}\le t^{\frac{1}{k-1}(1-\sigma-\frac{k}{K-2})+\frac{1}{K-2}}\log t=t^{18(1-\sigma)^{3/2}-g(1-\sigma)}\log t$$

on defining

$$g(x)=18x^{3/2}-\frac{x}{k-1}+\frac{1}{(K-2)(k-1)},\qquad x>0.$$

Just like for the function f considered in the proof of Corollary 3 we find that g has a global minimum on $(0,\infty)$ at $x_0=(27(k-1))^{-2}$ but $g(x_0)=-(3^7(k-1)^3)^{-1}+((K-2)(k-1))^{-1}$. It follows that

$$\sum_{t^{1/(k-1)}<n\le t^{1/2}}(n+\alpha)^{-s}\ll t^{18(1-\sigma)^{3/2}-g(x_0)}\log t.$$

However, for $4\le k\le 19$, we can find that

$$\begin{aligned}g(x_0)&=-(3^7(k-1)^3)^{-1}+((2^k-2)(k-1))^{-1}\\&=\frac{1}{(k-1)^3}\left\{\frac{(k-1)^2}{2^k-2}-\frac{1}{3^7}\right\}\\&\ge\frac{1}{(k-1)^3}\left\{\frac{18^2}{2^{19}-2}-\frac{1}{3^7}\right\}=\frac{1}{(k-1)^3}\lambda_2,\end{aligned}$$

and $\lambda_2>0$.

This yields

$$\sum_{t^{1/(k-1)}<n\le t^{1/2}}(n+\alpha)^{-s}\ll t^{18(1-\sigma)^{3/2}}\log t\ll t^{18.1(1-\sigma)^{3/2}}\log^{2/3}t.$$

Otherwise $(0<1-\sigma-k/(K-2))$

$$\sum_{t^{1/(k-1)}<n\le t^{1/2}}(n+\alpha)^{-s}\ll t^{\frac{1}{2}(1-\sigma-\frac{k}{K-2})+\frac{1}{K-2}}\log t=t^{18(1-\sigma)^{3/2}-h(1-\sigma)}\log t$$

on defining $h(x)=18x^{3/2}-x/2+(k-2)/(2(K-2))$, $x>0$.

We find that the function h has a global minimum on $(0,\infty)$ at the point $x_0=54^{-2}$ and $h(x_0)=-(2^3\cdot 3^7)^{-1}+(k-2)/(2(K-2))$.

A not very difficult calculation shows that for $4\le k\le 17$,

$$h(x_0)=-\frac{1}{2^3\cdot 3^7}+\frac{k-2}{2(2^k-2)}\ge\frac{15}{2(2^{17}-2)}-\frac{1}{2^3\cdot 3^7}>0.$$

It follows that

$$\sum_{t^{1/(k-1)}<n\le t^{1/2}} (n+\alpha)^{-s} \ll t^{18(1-\sigma)^{3/2}-h(x_0)}\log t < t^{18(1-\sigma)^{3/2}}\log t$$
$$\ll t^{18.1(1-\sigma)^{3/2}}\log^{2/3} t.$$

III. Proof of the Theorem. According to van der Corput and Koksma's result ([1], p. 4) we have

$$\zeta(s,\alpha)-\alpha^{-s} \ll t^{\frac{1-\sigma}{\log(1/(1-\sigma))}\log 2}\frac{\log t}{\log\log t}$$

for $1/2 \le \sigma < 1$ and $t > e$.

This estimate is better than ours for $1/2 \le \sigma \le 1-1/2^{16}$ and sufficiently large t. Indeed, for $1/2 \le \sigma \le 1-1/2^{16}$ we have $1/2^{16} \le 1-\sigma \le 1/2$ and

$$(1-\sigma)\log\frac{1}{1-\sigma}\log 2 \le 16(1-\sigma)^{3/2}.$$

It follows that for sufficiently large $t > t_{10} > 0$,

$$\zeta(s,\alpha)-\alpha^{-s} \ll t^{16(1-\sigma)^{3/2}}\frac{\log t}{\log\log t} \ll t^{16.1(1-\sigma)^{3/2}}\log^{2/3} t.$$

Clearly it suffices to consider the range $1-1/2^{16} < \sigma \le 1$.

For $1-2^{-5} \le \sigma \le 1$ and $t \ge t_{11} > 0$ (compare [9], p. 101, and [8], pp. 270–271) we have

$$|\zeta(s,\alpha)-\alpha^{-s}| \le \Big|\sum_{n\le t^{1/2}}(n+\alpha)^{-s}\Big| + \mathrm{const}$$

and

$$\sum_{n\le t^{1/2}}(n+\alpha)^{-s} = \Big\{\sum_{n\le N_1} + \sum_{N_1<n\le N_2} + \sum_{N_2<n\le N_3} + \sum_{N_3<n\le t^{1/2}}\Big\}(n+\alpha)^{-s}$$

where $N_1 = \exp(\log^{2/3} t)$, $N_2 = t^{1/1000}$, $N_3 = t^{1/10}$.

For the first sum we use Lemma 1. Corollary 1 gives an estimate for the second sum. To estimate the third and fourth term one can use Corollary 3, Corollary 4 with $k = 11$ and Corollary 2. In this way our Theorem is proved.

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References

[1] J. G. van der Corput et J. F. Koksma, *Sur l'ordre de grandeur de la fonction $\zeta(s)$ de Riemann dans la bande critique*, Ann. Fac. Sci. Univ. Toulouse (3) 22 (1930), 1–39.

[2] W. J. Ellison et M. Mendès-France, *Les nombres premiers*, Hermann, Paris, 1975.

[3] D. R. Heath-Brown, private correspondence, 1992.

[4] A. Ivić, *The Riemann Zeta Function*, Wiley, 1985.

[5] M. Kulas, *Some effective estimation in the theory of the Hurwitz-zeta function*, Funct. Approx. Comment. Math. 23 (1994), 123–134.

[6] E. I. Panteleeva, *On a problem of Dirichlet divisors in number fields*, Mat. Zametki 44 (1988), 494–505 (in Russian).

[7] —, *On mean values of some arithmetical functions*, ibid. 55 (1994), no. 2, 109–117 (in Russian).

[8] K. Prachar, *Primzahlverteilung*, Springer, Berlin, 1957.

[9] H. E. Richert, *Zur Abschätzung der Riemannschen Zetafunktion in der Nähe der Vertikalen* $\sigma = 1$, Math. Ann. 169 (1967), 97–101.

[10] W. Staś, *Über das Verhalten der Riemannschen ζ-Funktion und einiger verwandter Funktionen, in der Nähe der Geraden* $\sigma = 1$, Acta Arith. 7 (1962), 217–224.

[11] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Clarendon Press, Oxford, 1986.

[12] P. Turán, *On some recent results in the analytical theory of numbers*, in: Proc. Sympos. Pure Math. 20, Amer. Math. Soc., 1971, 339–347.

[13] O. V. Tyrina, *A new estimate for a trigonometric integral of I. M. Vinogradov*, Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987), 363–378 (in Russian).

[14] I. M. Vinogradov, *General theorems on the upper bound of the modulus of a trigonometric sum*, ibid. 15 (1951), 109–130 (in Russian).

Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Matejki 48/49, 60-769 Poznań, Poland
E-mail: kulas@math.amu.edu.pl

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