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On a question regarding visibility of lattice points, II

by

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1. Introduction. For $d \geq 2$, let $\Delta_n^d = \{(x_1, \ldots, x_d) : x_i \text{ integers and } 1 \leq x_i \leq n \ \forall i\}$ be the set of integer lattice points in a cube in $\mathbb{R}^d$. If $\alpha = (a_1, \ldots, a_d)$ and $\beta = (b_1, \ldots, b_d)$ are two points in Δ_n^d, we say that α *is visible from* β if either $\alpha = \beta$ or there is no lattice point in Δ_n^d on the line segment joining α and β. It is not difficult to verify that if $\alpha \neq \beta$, then $\alpha = (a_1, \ldots, a_d)$ is visible from $\beta = (b_1, \ldots, b_d)$ if and only if $\gcd(a_1 - b_1, \ldots, a_d - b_d) = 1$.

If A and B are subsets of Δ_n^d, one says that A *is visible from* B if each point of A is visible from some point of B.

Let $f_d(n)$ be defined by

$$f_d(n) = \min\{|S| : S \subset \Delta_n^d,\ \Delta_n^d \text{ is visible from } S\}.$$

That is, $f_d(n)$ is the least number of points that can be selected from Δ_n^d such that every point of Δ_n^d is visible from at least one of the selected points.

Professor Imre Z. Ruzsa informed the authors that the problem of finding the exact order of $f_d(n)$ is one of the problems in the list compiled by L. & W. Moser.

For the two-dimensional case, it was proved in [1] that

THEOREM 1*. *For all* $n > n_0$,

$$\frac{\log n}{2 \log\log n} < f_2(n) < 4 \log n. \tag{1}$$

Here, the second inequality was established by the greedy algorithm, while the first one follows by using the Chinese Remainder Theorem.

The following result was also proved in [1].

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THEOREM 2*. *One can explicitly describe a set* $S_n \subset \Delta_n^2$ *such that* Δ_n^2 *is visible from* S_n *and* $|S_n| = O((\log n)^\alpha)$ *where* α *has the property that the Jacobsthal function* $g(n)$ *satisfies* $g(n) = O((\log n)^\alpha)$.

Here, the Jacobsthal function $g(n)$ is defined to be the least integer with the property that among any $g(n)$ consecutive integers $a+1, \ldots, a+g(n)$, there is at least one which is relatively prime to n. Erdős [3] was the first to establish that $g(n) = O((\log n)^\alpha)$ for some finite α. Since then, several mathematicians (see, for example, [5]–[9]) have taken up the problem of improving the estimate of Erdős. Currently, the best known result in this direction, due to Iwaniec [6], implies $g(n) = O((\log n)^2)$. Even though it is expected that $g(n) = O((\log n)^{1+\varepsilon})$ for any $\varepsilon > 0$, it seems that to prove $g(n) = O((\log n)^\alpha)$ with some $\alpha < 2$ would be very difficult. Erdős, Gruber and Hammer [4] asked for a replacement of S_n in Theorem 2* by a set S_n' which would satisfy $|S_n'| = O(\log n)$ as is expected from Theorem 1*. In connection with this problem, even if the expected order of $g(n)$ is established, Abbott's explicit construction falls short of our target.

In [2], Adhikari and Balasubramanian could give explicit construction of a set $S_n' \subset \Delta_n^2$ from which Δ_n^2 is visible, where S_n' satisfies

$$|S_n'| = O\left(\frac{\log n \cdot \log\log\log n}{\log\log n}\right).$$

One observes that the order of $|S_n'|$ not only satisfies (1), but also it improves on it by improving the upper bound of $f_2(n)$ thereof.

In a conference in RIMS, Kyoto, several mathematicians asked the first author about the answer to the similar question in higher dimensions.

In the present paper, we prove the following

THEOREM 3. *One can give an explicit description of a set* $X_n \subset \Delta_n^3$ *from which* Δ_n^3 *is visible, where* X_n *satisfies*

$$|X_n| = O\left(\frac{\log n}{\log\log n}\right).$$

It is easy to see that the proof for the lower bound $\frac{\log n}{2\log\log n}$ for $f_2(n)$ in [1] goes through in higher dimensions to yield the same lower bound for $f_d(n)$ for $d \geq 3$. Again, as will be clear from the proof of Theorem 3 (see Remark 1 after the proof of the theorem), for dimensions $d > 3$ by trivial modifications of our proof one obtains the same result as in Theorem 3. Thus, for $d \geq 3$, the order problem for $f_d(n)$ is solved up to a constant factor. For $d = 2$, it remains an open question whether the order of $f_2(n)$ obtained in [2] can be improved or not.

Notations. For real x we write $[x]$ for the integral part of x. We also use the notations $l_i(x), i \geq 1$, defined as follows:

$$l_1(x) = \log x \quad \text{and} \quad l_i(x) = \log(l_{i-1}(x)) \quad \text{for } i \geq 2.$$

2. Proof of Theorem 3. Let n be large and

$$s = \left[D \sqrt{\frac{l_1(n)}{l_2(n)}} \right],$$

where D is a positive number such that $\sum_p 1/p^2 + 2/D^2 < 1$.

We take X_n to be the set $\{(a,b,1) : 1 \leq a, b \leq s\} \cup \{(2,2,2)\}$. Given any $(x,y,z) \in \Delta_n^3$, we show that (x,y,z) is visible from some point in X_n.

First we observe that given any $(x,y,z) \in \Delta_n^3$, if $z = 1$, then (x,y,z) is visible from $(2,2,2)$.

Now, we assume that $z \neq 1$. Then

$$\begin{aligned}
\sum_{a,b=1}^{s} \sum_{((x-a),(y-b),(z-1))>1} 1 &\leq \sum_{a,b=1}^{s} \sum_{\substack{p \text{ prime} \\ p \mid ((x-a),(y-b),(z-1))}} 1 \\
&= \sum_{p \mid (z-1)} \sum_{\substack{1 \leq a,b \leq s \\ p \mid (x-a),\, p \mid (y-b)}} 1 \leq \sum_{p \mid (z-1)} \left(\frac{s}{p} + 1 \right)^2 \\
&= s^2 \sum_{p \mid (z-1)} \frac{1}{p^2} + 2s \sum_{p \mid (z-1)} \frac{1}{p} + \sum_{p \mid (z-1)} 1 \\
&< s^2 \sum_{p} \frac{1}{p^2} + 2s \sum_{p<n} \frac{1}{p} + 2\frac{l_1(n)}{l_2(n)} \\
&< s^2 \sum_{p} \frac{1}{p^2} + 4s l_2(n) + 2\frac{l_1(n)}{l_2(n)} \\
&= \left(\sum_{p} \frac{1}{p^2} + \frac{2}{D^2} \right) s^2 + 4s l_2(n) < s^2,
\end{aligned}$$

for all sufficiently large n.

Therefore, there exist a, b with $1 \leq a, b \leq s$ such that $((x-a), (y-b), (z-1)) = 1$, which further implies that (x,y,z) is visible from $(a,b,1)$.

Hence, Δ_n^3 is visible from X_n as claimed and we get our theorem.

REMARK 1. It is clear from our proof that for $d > 3$, X_n could be replaced by $X_n^d = \{(a_1, \ldots, a_{d-1}, 1) : 1 \leq a_i \leq s\} \cup \{(2,2,\ldots,2)\}$ where $s = [D l_1(n)/l_2(n)]^{1/(d-1)}$ with a suitable D.

REMARK 2. The O-constant in Theorem 3, which is about D^2, can be clearly brought down by handling the constants a little more carefully. Similarly, a glance at Abbott's proof for the lower bound in Theorem 1* makes it clear that there, too, we can have a better constant than 1/2. However here we are not much interested in those constants.

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References

[1] H. L. Abbott, *Some results in Combinatorial Geometry*, Discrete Math. 9 (1974), 199–204.
[2] S. D. Adhikari and R. Balasubramanian, *On a question regarding visibility of lattice points*, Mathematika 43 (1996), 155–158.
[3] P. Erdős, *On the integers relatively prime to n and on a number-theoretic function considered by Jacobsthal*, Math. Scand. 10 (1962), 163–170.
[4] P. Erdős, P. M. Gruber and J. Hammer, *Lattice Points*, Pitman Monographs Surveys Pure Appl. Math. 39, Wiley, New York, 1989.
[5] H. Iwaniec, *On the error term in the linear sieve*, Acta Arith. 19 (1971), 1–30.
[6] —, *On the problem of Jacobsthal*, Demonstratio Math. 11 (1978), 225–231.
[7] H. J. Kanold, *Über eine zahlentheoretische Funktion von Jacobsthal*, Math. Ann. 170 (1967), 314–326.
[8] H. Stevens, *On Jacobsthal's $g(n)$-function*, ibid. 226 (1977), 95–97.
[9] R. C. Vaughan, *On the order of magnitude of Jacobsthal's function*, Proc. Edinburgh Math. Soc. 20 (1976-77), 329–331.

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