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Density of M-sets in arithmetic progression

by

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Following Motzkin, for a given set M of positive integers, a set S of nonnegative integers is called an *M-set* if $a, b \in S$ implies $a - b \notin M$. In an unpublished problem collection, Motzkin posed the problem of determining the quantity

$$\mu(M) = \sup_S \overline{\delta}(S),$$

where the supremum is taken over the class of all M-sets S. As usual, the upper density of S, $\overline{\delta}(S)$, is defined by $\limsup_{n\to\infty} S(n)/n$, where $S(x)$ denotes the number of elements in S less than or equal to x.

Cantor and Gordon [1] solved the problem for the cases where M has at most two elements besides obtaining partial results for the general case. Haralambis [2], besides giving some general estimates, determined $\mu(M)$ for most members of the families $\{1, j, k\}$ and $\{1, 2, j, k\}$. In this note, we determine $\mu(M)$ in the case where the elements of M are in *arithmetic progression*.

We recall the following results proved by Cantor and Gordon [1]:

Lemma A. *If $M_1 = \{m_1, m_2, \ldots\}$ and $M_2 = \{dm_1, dm_2, \ldots\}$, where d is a positive integer, then $\mu(M_1) = \mu(M_2)$.*

Lemma B. *Let $M = \{m_1, m_2, \ldots\}$ and let c and m be positive, relatively prime integers with*

$$d = \min_k |cm_k|_m,$$

where $|x|_m$ denotes the absolute value of the absolutely least remainder of x (mod m). Then $\mu(M) \ge d/m$.

This implies that for any set $M = \{m_1, m_2, \ldots\}$,

$$\mu(M) \ge \sup_{\gcd(c,m)=1} \frac{1}{m} \min_k |cm_k|_m.$$

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Lemma C. *Let M be a given set of positive integers*, $\alpha \in [0,1]$, *and suppose that for any M-set S with $0 \in S$ there exists a positive integer k* (*possibly dependent on S*) *such that $S(k) \le (k+1)\alpha$. Then $\mu(M) \le \alpha$.*

Theorem. *If $M = \{a, a+d, a+2d, \ldots, a+(n-1)d\}$ with* $\gcd(a,d) = 1$ *and $n \ge 1$, then*

$$\mu(M) = \begin{cases} \dfrac{2a+(n-1)(d-1)}{2\{2a+(n-1)d\}} & \text{if } d \text{ is odd},\\ \dfrac{1}{2} & \text{if } d \text{ is even}.\end{cases}$$

Proof. By Lemma A, it suffices to consider the case where $\gcd(a,d) = 1$. If d is even, then $\{1,3,5,\ldots\}$ is an M-set, and $\mu(M) = 1/2$.

Suppose now that d is odd. If we write m for $2a + (n-1)d$, then $\gcd(d,m) = 1$, and so $dx_0 \equiv 1 \pmod{m}$ for some integer x_0. Let $dx_0 = 1 + mq$. Then x_0 and mq, and hence x_0 and $(n-1)q$, are of opposite parity, and so

$$\begin{aligned} ax_0 &= \frac{m-(n-1)d}{2}x_0 \\ &= \frac{m\{x_0-(n-1)q\}-(n-1)}{2} \equiv \frac{m-(n-1)}{2} \pmod{m}. \end{aligned}$$

Therefore, for $0 \le k \le n-1$,

$$(a+kd)x_0 \equiv \frac{m}{2} + \left(k - \frac{n-1}{2}\right) \pmod{m},$$

and by Lemma B, we have

$$\mu(M) \ge \frac{2a+(n-1)(d-1)}{2\{2a+(n-1)d\}}.$$

Conversely, let S be any M-set with $0 \in S$, and for $m = 2a + (n-1)d$ let

$$\bigcup_{i=1}^{(m-n-1)/2} A_i \cup B$$

be a partition of $\{0,1,\ldots,m-1\}$, where $B = \{0, a, a+d, \ldots, a+(n-1)d\}$ and $A_i = \{id, a+(n+i-1)d\}, 1 \le i \le (m-n-1)/2$, the elements of A_i taken modulo m. Hence, $|S \cap B| = 1$ and $|S \cap A_i| \le 1$ for each i.

Therefore,

$$S(m-1) \le 1 + \frac{m-(n+1)}{2} = \frac{2a+(n-1)(d-1)}{2}$$

for any M-set S.

Thus, by Lemma C, $\mu(M) \le \{2a + (n-1)(d-1)\}/(2m)$. Therefore,

$$\mu(M) = \frac{2a + (n-1)(d-1)}{2\{2a + (n-1)d\}}. \ \blacksquare$$

We observe that the results of Cantor and Gordon [1] for $\mu(M)$ when $|M| \le 2$ follow easily from this theorem. Also, as $n \to \infty$, $\mu(M) \to (d-1)/(2d)$ if d is odd and to $1/2$ if d is even.

References

[1] D. G. Cantor and B. Gordon, *Sequences of integers with missing differences*, J. Combin. Theory Ser. A 14 (1973), 281–287.

[2] N. M. Haralambis, *Sets of integers with missing differences*, ibid. 23 (1977), 22–33.

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