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Conditions under which $K_2(\mathcal{O}_F)$ is not generated by Dennis–Stein symbols

by

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Introduction. Let F be a number field, and $\mathcal{O}_F$ its ring of integers. Much is now known about the structure of $K_2(\mathcal{O}_F)$ but explicit computations are still quite rare. In part, the difficulty lies with the need to find sufficiently many explicit elements of $K_2(\mathcal{O}_F)$. Although $K_2(\mathcal{O}_F)$ is naturally identified with the tame kernel—i.e., the kernel of the tame map $K_2(F) \to \bigoplus k(\mathfrak{p})^*$ (see Section 1)—it is clearly preferable, for the purposes of an explicit computation, to describe it in terms of generators which are identifiable elements of $\mathrm{St}(\mathcal{O}_F)$ and not just products of symbols in $K_2(F)$ which vanish under the tame map. In this way we obtain presentations of the special linear groups $\mathrm{Sl}(n, \mathcal{O}_F)$, $n \geq 3$, for instance.

While $K_2(F)$ is generated by symbols $\{u, v\}, u, v \in F^*$, this is not generally true for arbitrary commutative rings. In particular, $K_2(\mathcal{O}_F)$ is rarely generated by symbols (see [4], for example). However, Mulders has proven in [8] that if $\mathcal{O}_F$ contains nontorsion units then it is very often the case that $K_2(\mathcal{O}_F)$ is generated by *Dennis–Stein symbols*. Like the symbols $\{u, v\}$, these are also described explicitly in terms of generators of the Steinberg group (see Section 1 again). Furthermore, except in the case of imaginary quadratic fields (where there are too few units), almost all explicit computations of $K_2(\mathcal{O}_F)$ are given in terms of Dennis–Stein symbols (see, for instance, the computations in [3]–[5] and [8]). These results raise the question of whether it is always possible to generate $K_2(\mathcal{O}_F)$ by Dennis–Stein symbols if there are infinitely many units available.

The purpose of this note is to answer this question in the negative; namely, we show that under certain (very rare) conditions (other than the obvious case of imaginary quadratic fields) $K_2(\mathcal{O}_F)$ is not generated by Dennis–Stein symbols. In particular, for certain biquadratic fields we prove (in Section 4) that $K_2(\mathcal{O}_F)$ is not generated by Dennis–Stein symbols. Thus,

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to describe $K_2(\mathcal{O}_F)$ for such fields it will be necessary to find other types of explicit elements.

1. Preliminaries: Symbols in K_2. We begin by recalling some of the basic facts about K_2 (see [7] for more details). For any ring R, the Steinberg group of R, $\mathrm{St}(n,R)$ ($n \geq 3$), is the group with generators $x_{ij}(a)$, with $a \in R$ and i,j distinct integers between 1 and n, and subject to the relations

$$x_{ij}(a)x_{ij}(b) = x_{ij}(a+b)$$

and

$$[x_{ij}(a), x_{kl}(b)] = \begin{cases} x_{il}(ab) & \text{if } j = k,\ i \neq l, \\ 1 & \text{if } j \neq k,\ i \neq l. \end{cases}$$

There is a natural surjective map $\phi_n : \mathrm{St}(n,R) \to \mathrm{E}(n,R)$, where $\mathrm{E}(n,R)$ is the subgroup of $\mathrm{Gl}(n,R)$ generated by elementary matrices $E_{ij}(a)$, sending $x_{ij}(a)$ to $E_{ij}(a)$. $K_2(n,R)$ is defined to be the kernel of ϕ_n and $K_2(R) = \lim_{n\to\infty} K_2(n,R)$. It follows from the definition that a set of generators of $K_2(R)$ will yield a presentation of $\mathrm{E}(R)$ (the infinite elementary group). If $R = \mathcal{O}_F$, the ring of integers in a number field F which is not imaginary quadratic, then it is known that $K_2(n,R) = K_2(R)$ for $n \geq 3$ (see [13]) and that $\mathrm{E}(n,R) = \mathrm{Sl}(n,R)$ for all $n \geq 3$ (see [7]). Thus in this case a set of generators for $K_2(R)$ belonging to $K_2(3,R)$ will give a presentation of $\mathrm{Sl}(n,R)$ for all $n \geq 3$.

Now suppose that R is a commutative ring. Given a pair of units $u, v \in R^*$, one can construct the *symbol* $\{u,v\} \in K_2(R)$ as follows: Let

$$w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u), \quad h_{ij}(u) = w_{ij}(u)w_{ij}(-1).$$

Then $\{u,v\} = h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1}$.

These symbols satisfy the following relations:

(a) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$ for $u_1, u_2, v \in R^*$,
(b) $\{u,v\}\{v,u\} = 1$ for $u, v \in R^*$,
(c) $\{u, 1-u\} = 1$ if $u, 1-u \in R^*$.

The theorem of Matsumoto says that for a field F, $K_2(F)$ has the following presentation: The generators are the symbols $\{u,v\}$ with $u, v \in F^*$ and the relations are (a), (b), (c) above.

A *Steinberg symbol* on a field F is a map

$$c : F^* \times F^* \to A$$

where A is an abelian group, having the property that c is bimultiplicative, $c(x,y)c(y,x) = 1$ and $c(x, 1-x) = 0$ if $x \neq 0, 1$. Thus Matsumoto's theorem says that given a Steinberg symbol c on F, there is a unique homomorphism $K_2(F) \to A$ carrying the symbol $\{x,y\}$ to $c(x,y)$; or, equivalently, the map $F^* \times F^* \to K_2(F), (x,y) \mapsto \{x,y\}$ is the universal Steinberg symbol.

For any number field F the inclusion $\mathcal{O}_F \to F$ induces a monomorphism $\varrho_F : K_2(\mathcal{O}_F) \to K_2(F)$. For any nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$, let $\tau_{\mathfrak{p}} : K_2(F) \to k(\mathfrak{p})^*$ be the *tame symbol* (it is a Steinberg symbol), determined by the formula

$$\tau_{\mathfrak{p}}(a,b) = (-1)^{v_{\mathfrak{p}}(a)v_{\mathfrak{p}}(b)} a^{v_{\mathfrak{p}}(b)} b^{-v_{\mathfrak{p}}(a)} \pmod{\mathfrak{p}}$$

where $k(\mathfrak{p})$ is the residue field at $\mathfrak{p}$ and $v_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic valuation on F. Now let

$$T_F : K_2(F) \to \bigoplus_{\mathfrak{p}\text{ prime}} k(\mathfrak{p})^*$$

be the map $z \mapsto \{\tau_{\mathfrak{p}}(z)\}_{\mathfrak{p}}$. Then T_F is surjective and ϱ_F induces a natural isomorphism from $K_2(\mathcal{O}_F)$ onto $\operatorname{Ker}(T_F)$, the *tame kernel.*

For a commutative ring R, if $a, b \in R$ with $1+ab \in R^*$, the *Dennis–Stein symbol* $\langle a, b\rangle \in K_2(R)$ is defined by the formula

$$\langle a,b\rangle = x_{21}\left(\frac{-b}{1+ab}\right) x_{12}(a) x_{21}(b) x_{12}\left(\frac{-a}{1+ab}\right) h_{12}(1+ab)^{-1}$$

(see [1], [2]). These are related to the symbols $\{u, v\}$ by the formulae

$$\langle a,b\rangle = \begin{cases} \{-a, 1+ab\} & \text{if } a \in R^*,\\ \{1+ab, b\} & \text{if } b \in R^*. \end{cases}$$

Thus the symbol

$$\{u,v\} = \left\langle -u, \frac{1-v}{u}\right\rangle = \left\langle \frac{u-1}{v}, v\right\rangle$$

is also a Dennis–Stein symbol if $u, v \in R^* - \{1\}$.

2. The homomorphism I. Let $n \in \mathbb{N}$ and suppose that F is a number field containing the nth roots of unity and let S be a finite set of primes containing the infinite primes of F and the primes of F which divide n. Let $\mu_n \subset F$ be the group of nth roots of unity. For an abelian group A, we write A/n for $A \otimes \mathbb{Z}/n\mathbb{Z}$.

Based on the ideas of Tate in [12], Keune in [6] introduced a homomorphism

$$I : \mu_n \otimes Cl(\mathcal{O}_S) \to K_2(\mathcal{O}_S)/n$$

(where $\mathcal{O}_S$ is the ring of S-integers of F) defined as follows:

$$I(\zeta \otimes [\mathfrak{a}]) = z^n \pmod{K_2(\mathcal{O}_S)^n}$$

where $z \in K_2(F)$ is any element satisfying $\tau_{\mathfrak{p}}(z) \equiv \zeta^{v_{\mathfrak{p}}(\mathfrak{a})} \pmod{\mathfrak{p}}$ for all $\mathfrak{p} \notin S$. He proved (see Section 3 of [6]) that this map is injective and, furthermore, it fits into an exact sequence

$$0 \to \mu_n \otimes Cl(\mathcal{O}_S) \xrightarrow{I} K_2(\mathcal{O}_S)/n \xrightarrow{\lambda} \bigoplus_{\mathfrak{p}\in S_0} \mu_n \to \mu_n \to 0.$$

Here S_0 denotes the set of finite and infinite real primes of S and λ is induced by the Hilbert symbols of order n for each of the primes of S_0. Thus, I is an isomorphism precisely when S_0 is a singleton. Furthermore, from the construction of I (see [6], Section 3) it follows that the image of I is precisely the group

$$\frac{K_2(\mathcal{O}_S) \cap K_2(F)^n}{K_2(\mathcal{O}_S)^n}.$$

The case of immediate interest in this paper is when $n = p^r$ with p prime and $r \geq 1$ and S consists of the infinite primes of F together with the finite primes which divide p. In this case $Cl(\mathcal{O}_S) = Cl(\mathcal{O}_F[1/p])$ and $K_2(\mathcal{O}_S) = K_2(\mathcal{O}_F[1/p])$. However, from the localisation sequence for K-theory, we deduce that the natural map $K_2(\mathcal{O}_F) \to K_2(\mathcal{O}_F[1/p])$ is injective and induces an isomorphism on p-Sylow subgroups (since the order of $k(\mathfrak{p})^*$ is prime to p if $\mathfrak{p} \mid p$). Thus, in this case, the exact sequence above takes the form

$$0 \to \mu_{p^r} \otimes Cl\left(\mathcal{O}_F\left[\frac{1}{p}\right]\right) \xrightarrow{I} K_2(\mathcal{O}_F)/p^r \xrightarrow{\lambda} \bigoplus_{\mathfrak{p} \in S_0} \mu_{p^r} \to \mu_{p^r} \to 0.$$

3. Main results. Throughout this section, p will denote a fixed prime number and F will denote a number field containing the p^rth roots of unity ($r \geq 1$). $\zeta = \zeta_{p^r}$ denotes a fixed primitive p^rth root of unity in F and $\mu = \mu_{p^r}$ the cyclic group generated by ζ. For any number field K, $\mathcal{H}_K$ will denote the maximal abelian unramified extension of K in which all primes dividing p split completely. For an abelian extension K/L of number fields and any prime $\mathfrak{p}$ of L which does not ramify in K, $\left(\frac{K/L}{\mathfrak{p}}\right) \in \mathrm{Gal}(K/L)$ will denote the Frobenius of $\mathfrak{p}$.

In [8] and in his Ph.D. thesis (University of Nijmegen, 1992), Mulders proved the following: if $r = 1$ and $\sqrt[p]{\zeta} \notin F$, and if $F(\sqrt[p]{\mathcal{O}_F^*}) \not\subset \mathcal{H}_F$ and F is not imaginary quadratic then the image of I is generated by classes of Dennis–Stein symbols. In this section we prove our main result; namely, that if $\sqrt[p]{\zeta} \notin F$, but $F(\sqrt[p^r]{\mathcal{O}_F^*}) \subset \mathcal{H}_F$ and if, furthermore, for each $u \in \mathcal{O}_F^*$ if $L = F(\sqrt[p^r]{u})$ then $L(\sqrt[p^{2r}]{u}) \subset \mathcal{H}_L$, then the image of I is not generated by Dennis–Stein symbols (see Section 4 for examples of fields in which these conditions hold).

For any number field L containing F and any prime ideal $\mathfrak{p}$ of L, define $\varepsilon_{\mathfrak{p}}$ by

$$\varepsilon_{\mathfrak{p}} = \begin{cases} |k(\mathfrak{p})^*|/p^r & \text{if } \mathfrak{p} \text{ does not divide } p, \\ 0 & \text{if } \mathfrak{p} \text{ divides } p. \end{cases}$$

(If $\mathfrak{p}$ does not divide p, then the map $\mu \mapsto k(\mathfrak{p})$ is injective and so p^r divides $|k(\mathfrak{p})^*|$.)

LEMMA 3.1. *Suppose that* L/F *is a finite extension and* $x \in L^*$. *Let* $K = L(\sqrt[p^r]{x})$ *and for any prime ideal* $\mathfrak{p}$ *of* L *which does not ramify in* K *define* $l_\mathfrak{p} \in \mathbb{Z}/p^r\mathbb{Z}$ *by the formula*

$$\left(\frac{K/L}{\mathfrak{p}}\right)(\sqrt[p^r]{x}) = \zeta^{l_\mathfrak{p}}\ \sqrt[p^r]{x}.$$

Then

(i) $x^{\varepsilon_\mathfrak{p}} \equiv \zeta^{l_\mathfrak{p}} \pmod{\mathfrak{p}}$.

(ii) *If* K/L *is unramified then* $\sum l_\mathfrak{p} v_\mathfrak{p}(a) \equiv 0 \pmod{p^r}$ *for all* $a \in L^*$.

Proof. (i) Let $\mathfrak{P}$ be a prime ideal of K lying over $\mathfrak{p}$. Then

$$\zeta^{l_\mathfrak{p}}\ \sqrt[p^r]{x} = \left(\frac{K/L}{\mathfrak{p}}\right)(\sqrt[p^r]{x}) \equiv (\sqrt[p^r]{x})^{|k(\mathfrak{p})|} \pmod{\mathfrak{P}}$$

and thus

$$\zeta^{l_\mathfrak{p}} \equiv (\sqrt[p^r]{x})^{|k(\mathfrak{p})^*|} \pmod{\mathfrak{P}}.$$

If $\mathfrak{p}$ does not divide p, this gives

$$\zeta^{l_\mathfrak{p}} \equiv x^{\varepsilon_\mathfrak{p}} \pmod{\mathfrak{p}}$$

(since p^r then divides $|k(\mathfrak{p})^*|$) proving (i). If $\mathfrak{p}$ divides p then $\zeta \equiv 1 \pmod{\mathfrak{p}}$ and (i) holds by definition.

(ii) Since K/L is unramified, there are well defined homomorphisms

$$Cl(\mathcal{O}_L) \to \mathrm{Gal}(K/L) \to \mathbb{Z}/p^r\mathbb{Z}, \qquad [\mathfrak{p}] \mapsto \left(\frac{K/L}{\mathfrak{p}}\right) \mapsto l_\mathfrak{p}.$$

Part (ii) of the lemma simply says that this map is trivial on principal ideals.

Note that if $\mathfrak{p}$ does not divide p then there is a natural isomorphism $\mu \to \mu_{p^r}(k(\mathfrak{p}))$. Since $\varepsilon_\mathfrak{p} = 0$ if $\mathfrak{p}$ divides p, it follows that for all $\mathfrak{p}$ there is a well-defined homomorphism $\varrho_\mathfrak{p} : k(\mathfrak{p})^* \to \mu$ satisfying

$$x^{\varepsilon_\mathfrak{p}} \equiv \varrho_\mathfrak{p}(x) \pmod{\mathfrak{p}}$$

for all $x \in k(\mathfrak{p})^*$. Let $\Phi = \Phi_L : K_2(L) \to \mu$ be the map $z \mapsto \prod \varrho_\mathfrak{p}(\tau_\mathfrak{p}(z))$. (Thus, if $\mathfrak{p}$ does not divide p, then $\varrho_\mathfrak{p} \circ \tau_\mathfrak{p}$ is just the map induced on K_2 by the Hilbert symbol of order p^r—see [10], Section III.5—and Φ is the product of these over all primes of L not dividing p.)

For $A, B \subseteq L^*$, let $\{A, B\}$ denote the subgroup of $K_2(L)$ generated by symbols $\{a, b\}$, $a \in A$, $b \in B$.

LEMMA 3.2. *Suppose that* $W \subset \mathcal{O}_L^*$ *is a subgroup with the property that* $L(\sqrt[p^r]{W}) \subset \mathcal{H}_L$. *Then*

$$\{W, L^*\} \subset \mathrm{Ker}(\Phi_L).$$

Proof. Suppose that $z = \{u, a\}$ with $u \in W$, $a \in L^*$. Let $K = L(\sqrt[p^r]{u})$. By definition of $\mathcal{H}_L$, K/L is unramified and primes above p split in this extension. For a prime ideal $\mathfrak{p}$ of L, $\tau_{\mathfrak{p}}(z) \equiv u^{v_{\mathfrak{p}}(a)} \pmod{\mathfrak{p}}$. Thus $\tau_{\mathfrak{p}}(z)^{\varepsilon_{\mathfrak{p}}} \equiv u^{\varepsilon_{\mathfrak{p}} v_{\mathfrak{p}}(a)} \equiv \zeta^{l_{\mathfrak{p}} v_{\mathfrak{p}}(a)} \pmod{\mathfrak{p}}$ by Lemma 3.1(i). It follows that for $\mathfrak{p}$ not dividing p, $\varrho_{\mathfrak{p}}(\tau_{\mathfrak{p}}(z)) = \zeta^{l_{\mathfrak{p}} v_{\mathfrak{p}}(a)} \in \mu$. On the other hand, if $\mathfrak{p}$ divides p then $l_{\mathfrak{p}} = 0$ since $\mathfrak{p}$ splits in K and this formula also holds in this case. Thus

$$\Phi(z) = \prod_{\mathfrak{p}} \varrho_{\mathfrak{p}}(\tau_{\mathfrak{p}}(z)) = \prod_{\mathfrak{p}} \zeta^{l_{\mathfrak{p}} v_{\mathfrak{p}}(a)} = \zeta^{\sum_{\mathfrak{p}} l_{\mathfrak{p}} v_{\mathfrak{p}}(a)} = 1$$

by Lemma 3.1(ii). This proves the lemma.

We will also need the following property of the map Φ:

Lemma 3.3. *For any extension L/F, the diagram*

$$\begin{array}{ccc} K_2(L) & \xrightarrow{\Phi_L} & \mu \\ \big\downarrow{\scriptstyle \mathrm{tr}_{L/F}} & & \big\downarrow{\scriptstyle =} \\ K_2(F) & \xrightarrow{\Phi_F} & \mu \end{array}$$

commutes, where $\mathrm{tr}_{L/F} : K_2(L) \to K_2(F)$ *is the K-theory transfer.*

Proof. For a number field E, let $T = T_E : K_2(E) \to \bigoplus_{\mathfrak{p}} k(\mathfrak{p})^*$ be the tame homomorphism. For any extension L/F it is known that the diagram

$$\begin{array}{ccc} K_2(L) & \xrightarrow{T_L} & \bigoplus_{\mathfrak{q}} k(\mathfrak{q})^* \\ \big\downarrow{\scriptstyle \mathrm{tr}_{L/F}} & & \big\downarrow{\scriptstyle \mathcal{N}_{L/F}} \\ K_2(F) & \xrightarrow{T_F} & \bigoplus_{\mathfrak{p}} k(\mathfrak{p})^* \end{array}$$

commutes, where $\mathcal{N}_{L/F}$ is the map $\{\alpha_{\mathfrak{q}}\}_{\mathfrak{q}} \mapsto \{\prod_{\mathfrak{q}|\mathfrak{p}} N^{\mathfrak{q}}_{\mathfrak{p}}(\alpha_{\mathfrak{q}})\}_{\mathfrak{p}}$ and $N^{\mathfrak{q}}_{\mathfrak{p}} = N_{k(\mathfrak{q})/k(\mathfrak{p})}$ (see [6], Section 4, for properties of the transfer).

Thus, in view of the definition of Φ, we reduce to showing that for any prime $\mathfrak{p}$ of F not dividing p we have $\prod_{\mathfrak{q}|\mathfrak{p}} (\alpha_{\mathfrak{q}})^{\varepsilon_{\mathfrak{q}}} = \prod_{\mathfrak{q}|\mathfrak{p}} N^{\mathfrak{q}}_{\mathfrak{p}}(\alpha_{\mathfrak{q}})^{\varepsilon_{\mathfrak{p}}}$. This follows from the fact that for $\mathfrak{q} \mid \mathfrak{p}$ and $x \in k(\mathfrak{q})^*$, $N^{\mathfrak{q}}_{\mathfrak{p}}(x)^{\varepsilon_{\mathfrak{p}}} = x^{\varepsilon_{\mathfrak{q}}}$, which is easily verified.

With these preliminaries, we can prove our main theorem:

Theorem 3.4. *Suppose that $F(\sqrt[p^r]{\mathcal{O}_F^*}) \subset \mathcal{H}_F$, $\sqrt[p^r]{\zeta} \notin F$ and $L(\sqrt[p^{2r}]{u}) \subset \mathcal{H}_L$ for any $u \in \mathcal{O}_F^*$ where $L = F(\sqrt[p^r]{u})$. Then the image of I is not generated by Dennis–Stein symbols.*

Proof. Let $E = F(\sqrt[p^r]{\zeta})$ and note that the Artin map induces a surjective homomorphism

$$Cl(\mathcal{O}_F[1/p]) \to \mathrm{Gal}(E/F), \qquad [\mathfrak{p}] \mapsto \left(\frac{E/F}{\mathfrak{p}}\right)$$

(well-defined since primes above p split in E). Let $\mathcal{C}$ denote the kernel of this map. Then $Cl(\mathcal{O}_F[1/p])/\mathcal{C}$ is isomorphic to $\mathrm{Gal}(E/F)$ which is a nontrivial cyclic group of order p^s for some s with $1 \le s \le r$ and thus the image of $\mu \otimes \mathcal{C} \to \mu \otimes Cl(\mathcal{O}_F[1/p])$ has index p^s.

We will show the following: If $x \in K_2(\mathcal{O}_F)$ is of the form $\{u, a\}$ for some $u \in \mathcal{O}_F^*$ and $x \equiv I(y) \pmod{K_2(\mathcal{O}_F)^{p^r}}$ for some $y \in \mu \otimes Cl(\mathcal{O}_F[1/p])$, then $y \in \mu \otimes \mathcal{C}$. (In particular, if $I(y)$ is represented by a Dennis–Stein symbol then $y \in \mu \otimes \mathcal{C}$.)

Suppose, to the contrary, that $\mathfrak{p}$ is a prime ideal of F such that $\left(\frac{E/F}{\mathfrak{p}}\right) \ne 1$ (and hence $\mathfrak{p}$ does not divide p and $\zeta \otimes [\mathfrak{p}]$ is a nontrivial element of $\mu \otimes Cl(\mathcal{O}_F[1/p])$) and $I(\zeta \otimes [\mathfrak{p}]) \cong \{u, a\} \pmod{K_2(\mathcal{O}_F)^{p^r}}$ with $u \in \mathcal{O}_F^*$. We will show this leads to a contradiction.

By construction of I, there exists $z \in K_2(F)$ satisfying

$$\tau_{\mathfrak{q}}(z) = \begin{cases} \zeta \pmod{\mathfrak{q}} & \text{if } \mathfrak{q} = \mathfrak{p}, \\ 1 \pmod{\mathfrak{q}} & \text{if } \mathfrak{q} \ne \mathfrak{p}, \end{cases} \tag{1}$$

and $z^{p^r} \equiv \{u, a\} \pmod{K_2(\mathcal{O}_F)^{p^r}}$.

Thus $z^{p^r} = \{u, a\} w^{p^r}$ for some $w \in K_2(\mathcal{O}_F)$, and replacing z by zw^{-1} if necessary, we can assume that $z^{p^r} = \{u, a\}$ (while still satisfying (1)). Now let $L = F(\sqrt[p^r]{u})$. Then $a = N_{L/F}(b)$ for some $b \in L$ (by [7], Corollary 15.11) and $\{u, a\} = \mathrm{tr}_{L/F}(\{u, b\}) = (\mathrm{tr}_{L/F}\{\sqrt[p^r]{u}, b\})^{p^r}$. Hence $\mathrm{tr}_{L/F}(\{\sqrt[p^r]{u}, b\}) = z\{\zeta, c\}$ for some $c \in F^*$, since $z^{-1}\mathrm{tr}_{L/F}(\{\sqrt[p^r]{u}, b\})$ lies in the p^r-torsion subgroup of $K_2(F)$ which equals $\{\zeta, F^*\}$ by [11], Theorem 1.8.

Thus $\Phi_L\{\sqrt[p^r]{u}, b\} = \Phi_F(z)\Phi_F\{\zeta, c\}$ by Lemma 3.3. Now $\Phi_F\{\zeta, c\} = 1$ by Lemma 3.2 and $\Phi_F(z) = \zeta^{\varepsilon_{\mathfrak{p}}}$ by (1). However, by Lemma 3.1(i) we get

$$\zeta^{\varepsilon_{\mathfrak{p}}} \equiv \zeta^{e_{\mathfrak{p}}} \pmod{\mathfrak{p}}$$

where $e_{\mathfrak{p}} \in \mathbb{Z}/p^r\mathbb{Z}$ is defined by

$$\left(\frac{E/F}{\mathfrak{p}}\right) \sqrt[p^r]{\zeta} = \zeta^{e_{\mathfrak{p}}} \sqrt[p^r]{\zeta}$$

and thus $e_{\mathfrak{p}} \ne 0$ by choice of $\mathfrak{p}$. Furthermore since $\mathfrak{p}$ does not divide p, ζ has order p^r in $k(\mathfrak{p})^*$ and thus

$$\Phi_L\{\sqrt[p^r]{u}, b\} = \zeta^{\varepsilon_{\mathfrak{p}}} = \zeta^{e_{\mathfrak{p}}} \ne 1,$$

contradicting Lemma 3.2 since $L(\sqrt[p^r]{\sqrt[p^r]{u}}) = L(\sqrt[p^{2r}]{u}) \subset \mathcal{H}_L$ by hypothesis. This proves the theorem.

Note that the proof establishes the slightly stronger fact that, under the given hypotheses, the image of I is not generated by elements of the form $\{u, a\}$ with $u \in \mathcal{O}_F^*$. We also obtain immediately:

Corollary 3.5. *If F satisfies the conditions of the last theorem and if furthermore F is totally imaginary and there is only one prime of F above p, then $K_2(\mathcal{O}_F)$ is not generated by Dennis–Stein symbols.*

Proof. The hypotheses imply that S_0 is a singleton (see Section 2) and thus I is an isomorphism. So $K_2(\mathcal{O}_F)/p^r$, and hence $K_2(\mathcal{O}_F)$ itself, is not generated by Dennis–Stein symbols.

In the positive direction, however, we can prove the following result, which guarantees that a large part of the image of I will be generated by Dennis–Stein symbols, even under the conditions of the last theorem:

Theorem 3.6. *Suppose that F is not imaginary quadratic, $\sqrt[p]{\zeta} \notin F$ and $\sqrt[p^r]{\mathcal{O}_F^*} \subset \mathcal{H}_F$. Let $E = F(\sqrt[p^r]{\zeta})$ and let $\mathcal{C}$ be the kernel of the Artin map $Cl(\mathcal{O}_F[1/p]) \to \mathrm{Gal}(E/F)$. Then $I(\mu \otimes \mathcal{C})$ is generated by Dennis–Stein symbols.*

Proof. Let ϕ denote the isomorphism from $Cl(\mathcal{O}_F[1/p])$ to $\mathrm{Gal}(\mathcal{H}_F/F)$ and for any intermediate field K let ϕ_K be the map $x \mapsto \phi(x)|_K \in \mathrm{Gal}(K/F)$. So $\mathcal{C} = \mathrm{Ker}(\phi_E)$.

Fix a unit $u \in \mathcal{O}_F^*$ such that $u \notin \langle\zeta\rangle(\mathcal{O}_F^*)^p$. Let $L = F(\sqrt[p^r]{u})$. By choice of u, E/F and L/F are linearly disjoint subextensions of $\mathcal{H}_F/F$, and $\mathrm{Gal}(L/F)$ and $\mathrm{Gal}(E/F)$ are cyclic extensions of order p^r. Let G be the unique subgroup of $\mathrm{Gal}(L/F)$ of index p. Then $\mathcal{C} \not\subset \phi_L^{-1}(G)$ since we can choose $\tau \in \mathrm{Gal}(\mathcal{H}_F/F)$ with $\tau|_L \notin G$ but $\tau|_E = 1$ so that $\phi^{-1}(\tau) \in \mathcal{C} - \phi_L^{-1}(G)$. Thus $\mathcal{C}_1 = \phi_L^{-1}(G) \cap \mathcal{C}$ has index p in $\mathcal{C}$ so $\mathcal{C}$ is generated as a group by $\mathcal{C} - \mathcal{C}_1$.

Now let $x \in \mathcal{C} - \mathcal{C}_1$. Let $\sigma = \phi(x)$. So $\sigma|_E = 1$ and $\sigma|_L$ has order p^r. Choose $\sigma_1 \in \mathrm{Gal}(\mathcal{H}/F)$ such that $\sigma_1|_E$ has order p^r.

Choose distinct primes $\mathfrak{p}$, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ not dividing p such that $x = [\mathfrak{p}]$ and $\phi([\mathfrak{p}_1]) = \sigma_1$ and $[\mathfrak{p}_2] = ([\mathfrak{p}][\mathfrak{p}_1])^{-1}$ in $Cl(\mathcal{O}_F)$. Let $\sigma_2 = \phi([\mathfrak{p}_2])$.

Since $\phi_E([\mathfrak{p}]) = 1$, $\zeta^{\varepsilon_{\mathfrak{p}}} \equiv 1 \pmod{\mathfrak{p}}$ by Lemma 3.1(i) and hence p^r divides $\varepsilon_{\mathfrak{p}}$ and thus p^{2r} divides $|k(\mathfrak{p})^*|$. On the other hand, $\phi_L([\mathfrak{p}])$ has order p^r and so $u^{\varepsilon_{\mathfrak{p}}} \equiv \zeta^{l_{\mathfrak{p}}} \pmod{\mathfrak{p}}$ where $l_{\mathfrak{p}}$ is a generator of $\mathbb{Z}/p^r\mathbb{Z}$ (by Lemma 3.1(i) again) and thus p^{2r} divides the order of $u \in k(\mathfrak{p})^*$.

For $i = 1, 2$, $\phi_E([\mathfrak{p}_i]) = \sigma_i|_E$ has order p^r and thus $\zeta^{\varepsilon_{\mathfrak{p}_i}} \equiv \zeta^{e_{\mathfrak{p}_i}} \pmod{\mathfrak{p}_i}$ where $e_{\mathfrak{p}_i}$ is a generator of $\mathbb{Z}/p^r\mathbb{Z}$. Thus $\varepsilon_{\mathfrak{p}_i}$ is not divisible by p. It follows that p^r is the exact power of p dividing $|k(\mathfrak{p}_i)^*|$ for $i = 1, 2$. Thus the order of u in $k(\mathfrak{p}_i)^*$ is of the form $p^{k_i}s_i$ where $0 \le k_i \le r$ and p does not divide s_i.

It follows that there exists $t \in \mathbb{Z}$ such that $u^t \equiv \zeta \pmod{\mathfrak{p}}$ and $u^t \equiv 1 \pmod{\mathfrak{p}_i}$ for $i = 1, 2$. Let $w = u^t$ and let $a \in \mathcal{O}_F$ be a generator of the principal ideal $\mathfrak{p}\mathfrak{p}_1\mathfrak{p}_2$. Let $z = \{a, w\} \in K_2(F)$. Then $\tau_{\mathfrak{p}}(z) = w \pmod{\mathfrak{p}} = \zeta \pmod{\mathfrak{p}}$ while $\tau_{\mathfrak{p}_i}(z) = w \pmod{\mathfrak{p}_i} = 1 \pmod{\mathfrak{p}_i}$ and if $\mathfrak{q} \neq \mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2$ then $\mathfrak{q}$ does not divide a and $\tau_{\mathfrak{q}}(z) = 1$. Thus $I(\zeta \otimes [\mathfrak{p}]) = I(\zeta \otimes x) \equiv z^{p^r}$

(mod $(K_2(\mathcal{O}_F))^{p^r}$). But

$$z^{p^r} = \{a, w^{p^r}\} = \left\langle -a, \frac{1-w^{p^r}}{a} \right\rangle$$

since $w^{p^r} \equiv 1 \pmod{a}$. So $I(\zeta \otimes x)$ is represented by a Dennis–Stein symbol as required.

COROLLARY 3.7. *If the hypotheses of Theorem* 3.4 *are satisfied and* $\sqrt[p]{\zeta} \notin F$ *then the subgroup of the image of* I *generated by Dennis–Stein symbols is precisely* $I(\mu \otimes \mathcal{C})$ *and has index* $p^r = [E : F]$.

Proof. This follows from the proof of Theorem 3.4 where it is shown that the subgroup of the image of I which is generated by Dennis–Stein symbols is contained in $I(\mu \otimes \mathcal{C})$, together with Theorem 3.6

REMARKS. Consider again the case where $r = 1$. Suppose that the hypotheses of Theorem 3.6 do *not* hold, in the sense that $\sqrt[p]{\zeta} \notin F$ and $\sqrt[p]{\mathcal{O}_F^*} \not\subset \mathcal{H}_F$. Then Mulders shows (in [9], Section 2.4) that the image of I is generated by Dennis–Stein symbols. However, if $\sqrt[p]{\zeta} \notin F$ but $\sqrt[p]{\mathcal{O}_F^*} \subset \mathcal{H}_F$ then the methods used in the proof of Theorem 3.4 can be used to show that the image of I is not generated by Dennis–Stein symbols *of the type constructed by Mulders* (namely Dennis–Stein symbols of the form $\{a, u^p\}$ with $u \in \mathcal{O}_F^*$). However, Theorem 3.4 only proves that the image of I is not generated by Dennis–Stein symbols of any kind under stronger hypotheses; namely $\sqrt[p]{\zeta} \notin F$ but $\sqrt[p]{\mathcal{O}_F^*} \subset \mathcal{H}_F$ *and* for all $u \in \mathcal{O}_F^*$, $F(\sqrt[p^2]{u}) \subset \mathcal{H}_L$. Thus it remains open whether the image of I can be generated by Dennis–Stein symbols under the hypotheses of Theorem 3.6.

4. Examples. In order to construct examples of fields satisfying the hypotheses of Theorem 3.4, it suffices to find fields in which the primes above p split in appropriate extensions.

LEMMA 4.1. *Suppose that* F *is totally imaginary*, $\zeta \in F$ *and* $\sqrt[p]{\zeta} \notin F$. *Let* $u_1, \ldots, u_s$ *be a system of fundamental units of* F. *Then* F *satisfies the hypotheses of Theorem* 3.4 *if and only if every prime above* p *in* F *splits completely in each of the extensions* $F(\sqrt[p^{2r}]{u_i})$ *and in* $F(\sqrt[p^{2r}]{\zeta})$.

Proof. It follows from the hypotheses that every prime above p splits completely in the Galois extension $F(\sqrt[p^{2r}]{\zeta}, \sqrt[p^{2r}]{u_1}, \ldots, \sqrt[p^{2r}]{u_s}) = F(\sqrt[p^{2r}]{\mathcal{O}_F^*})/F$. Now, if L is any field containing ζ and if u is a unit of L, then $L(\sqrt[p^r]{u})/L$ is an abelian extension and the only primes that may ramify in L are primes above p or primes at infinity. Thus if it is known that these primes split completely in $L(\sqrt[p^r]{u})$ then $L(\sqrt[p^r]{u})/L$ is unramified abelian and thus contained in $\mathcal{H}_L$. Thus $F(\sqrt[p^r]{\mathcal{O}_F^*})/F$ is unramified and for each $u \in \mathcal{O}_F^*$, the abelian extension $F(\sqrt[p^{2r}]{u})/F(\sqrt[p^r]{u})$ is unramified.

In the case $p = 2$, $r = 1$ we can construct biquadratic fields with the necessary properties:

LEMMA 4.2. *Suppose that* $d \equiv 1 \pmod 8$, $d > 1$ (*and squarefree*) *and suppose that* $f \equiv 7 \pmod 8$, $f > 0$ (*squarefree*) *with the property that if* u *is a fundamental unit of* $K = \mathbb{Q}(\sqrt{f})$ *then the prime above* 2 *in* K *splits completely in* $K(\sqrt[4]{u})$. *Then the biquadratic field*

$$F = \mathbb{Q}(\sqrt{-2d}, \sqrt{f})$$

satisfies the conditions of Theorem 3.4 *for* $p = 2$, $r = 1$.

Proof. Clearly the prime 2 totally ramifies in F since 2 ramifies in each of the quadratic subextensions $\mathbb{Q}(\sqrt{-2d})$, $\mathbb{Q}(\sqrt{f})$ and $\mathbb{Q}(\sqrt{-2df})$. So $2\mathcal{O}_F = \mathfrak{p}^4$ for some prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$.

F is totally imaginary and so the rank of the group of units is 1. Furthermore, u is clearly a fundamental unit of F because $\sqrt{u}, \sqrt{-u} \notin F$ (since the prime above 2 in K ramifies in F but splits in $K(\sqrt{u})$ and $K(\sqrt{-1})$ and hence also in $K(\sqrt{-u})$). The conditions on f thus guarantee that $\mathfrak{p}$ splits completely in $F(\sqrt[4]{u})$.

$\mathfrak{p}$ splits in $F(\sqrt{-2})$ since $F(\sqrt{-2}) \supset \mathbb{Q}(\sqrt{-2}, \sqrt{-2d}) \supset \mathbb{Q}(\sqrt{d})$ and 2 splits in this last field since $d \equiv 1 \pmod 8$.

$\mathfrak{p}$ splits in $F(\sqrt{2})$ since $F(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{-2d}, \sqrt{f}) \supset \mathbb{Q}(\sqrt{-df})$ and 2 splits in this last field since $-df \equiv 1 \pmod 8$.

Thus $\mathfrak{p}$ splits completely in $F(\sqrt{2}, \sqrt{-2}) = F(\sqrt[4]{-1})$.

Since $\mathcal{O}_F^* = \langle -1 \rangle \times \langle u \rangle$, it follows from Lemma 4.1 that F satisfies the hypotheses of Theorem 3.4.

REMARKS ON LEMMA 4.2. (i) Since there is exactly one prime above 2 in F and no real infinite primes, the map

$$I : \mu_2 \otimes Cl(\mathcal{O}_F[1/2]) \to K_2(\mathcal{O}_F)/2$$

is an isomorphism in this case and thus $K_2(\mathcal{O}_F)/2$, and hence $K_2(\mathcal{O}_F)$ itself is not generated by Dennis–Stein symbols.

(ii) According to the proof of Theorem 3.4 if $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_F$ not splitting in $F(\sqrt{-1})$ then $I(-1 \otimes [\mathfrak{p}])$ is a nontrivial element of $K_2(\mathcal{O}_F)/2$ not represented by an element of the form $\{w, a\}$ where $w \in \mathcal{O}_F^*$. In particular, if $I(-1 \otimes [\mathfrak{p}])$ is represented by $z \in K_2(\mathcal{O}_F)$ then $z \notin \{-1, F^*\}$, which is the 2-torsion part of $K_2(F)$, and thus z has order divisible by 4.

(iii) The only number less than 2000 satisfying the conditions for f in the lemma is $f = 1751 = 17 \cdot 103$ (verified using the computer programme PARI/GP) and thus the smallest example of such a field is

$$F = \mathbb{Q}(\sqrt{-34}, \sqrt{1751}) = \mathbb{Q}(\sqrt{-34}, \sqrt{-206}).$$

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