Conditions under which $K_2(O_F)$ is not generated by Dennis–Stein symbols

by

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Introduction. Let $F$ be a number field, and $O_F$ its ring of integers. Much is now known about the structure of $K_2(O_F)$ but explicit computations are still quite rare. In part, the difficulty lies with the need to find sufficiently many explicit elements of $K_2(O_F)$. Although $K_2(O_F)$ is naturally identified with the tame kernel—i.e., the kernel of the tame map $K_2(F) \to \bigoplus k(p)^*$ (see Section 1)—it is clearly preferable, for the purposes of an explicit computation, to describe it in terms of generators which are identifiable elements of $\text{St}(O_F)$ and not just products of symbols in $K_2(F)$ which vanish under the tame map. In this way we obtain presentations of the special linear groups $\text{Sl}(n, O_F)$, $n \geq 3$, for instance.

While $K_2(F)$ is generated by symbols $\{u, v\}, u, v \in F^*$, this is not generally true for arbitrary commutative rings. In particular, $K_2(O_F)$ is rarely generated by symbols (see [4], for example). However, Mulders has proven in [8] that if $O_F$ contains non-torsion units then it is very often the case that $K_2(O_F)$ is generated by Dennis–Stein symbols. Like the symbols $\{u, v\}$, these are also described explicitly in terms of generators of the Steinberg group (see Section 1 again). Furthermore, except in the case of imaginary quadratic fields (where there are too few units), almost all explicit computations of $K_2(O_F)$ are given in terms of Dennis–Stein symbols (see, for instance, the computations in [3]–[5] and [8]). These results raise the question of whether it is always possible to generate $K_2(O_F)$ by Dennis–Stein symbols if there are infinitely many units available.

The purpose of this note is to answer this question in the negative; namely, we show that under certain (very rare) conditions (other than the obvious case of imaginary quadratic fields) $K_2(O_F)$ is not generated by Dennis–Stein symbols. In particular, for certain biquadratic fields we prove (in Section 4) that $K_2(O_F)$ is not generated by Dennis–Stein symbols. Thus,
to describe $K_2(\mathcal{O}_F)$ for such fields it will be necessary to find other types of explicit elements.

1. Preliminaries: Symbols in $K_2$. We begin by recalling some of the basic facts about $K_2$ (see [7] for more details). For any ring $R$, the Steinberg group of $R$, $\text{St}(n,R)$ ($n \geq 3$), is the group with generators $x_{ij}(a)$, with $a \in R$ and $i, j$ distinct integers between 1 and $n$, and subject to the relations

\[ x_{ij}(a)x_{ij}(b) = x_{ij}(a+b) \]

and

\[ [x_{ij}(a), x_{kl}(b)] = \begin{cases} x_{il}(ab) & \text{if } j = k, \ i \neq l, \\ 1 & \text{if } j \neq k, \ i \neq l. \end{cases} \]

There is a natural surjective map $\phi_n : \text{St}(n, R) \to \text{E}(n, R)$, where $\text{E}(n, R)$ is the subgroup of $\text{GL}(n, R)$ generated by elementary matrices $E_{ij}(a)$, sending $x_{ij}(a)$ to $E_{ij}(a)$. $K_2(n, R)$ is defined to be the kernel of $\phi_n$ and $K_2(R) = \lim_{n \to \infty} K_2(n, R)$. It follows from the definition that a set of generators of $K_2(R)$ will yield a presentation of $\text{E}(R)$ (the infinite elementary group). If $R = \mathcal{O}_F$, the ring of integers in a number field $F$ which is not imaginary quadratic, then it is known that $K_2(n, R) = K_2(R)$ for $n \geq 3$ (see [13]) and that $\text{E}(n, R) = \text{Sl}(n, R)$ for all $n \geq 3$ (see [7]). Thus in this case a set of generators for $K_2(R)$ belonging to $K_2(3, R)$ will give a presentation of $\text{Sl}(n, R)$ for all $n \geq 3$.

Now suppose that $R$ is a commutative ring. Given a pair of units $u, v \in R^*$, one can construct the symbol $\{u, v\} \in K_2(R)$ as follows: Let

\[ w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u), \quad h_{ij}(u) = w_{ij}(u)w_{ij}(-1). \]

Then $\{u, v\} = h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1}$.

These symbols satisfy the following relations:

(a) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$ for $u_1, u_2, v \in R^*$,
(b) $\{u, v\}\{v, u\} = 1$ for $u, v \in R^*$,
(c) $\{u, 1-u\} = 1$ if $u, 1-u \in R^*$.

The theorem of Matsumoto says that for a field $F$, $K_2(F)$ has the following presentation: The generators are the symbols $\{u, v\}$ with $u, v \in F^*$ and the relations are (a), (b), (c) above.

A Steinberg symbol on a field $F$ is a map

\[ c : F^* \times F^* \to A \]

where $A$ is an abelian group, having the property that $c$ is bimultiplicative, $c(x, y)c(y, x) = 1$ and $c(x, 1-x) = 0$ if $x \neq 0, 1$. Thus Matsumoto’s theorem says that given a Steinberg symbol $c$ on $F$, there is a unique homomorphism $K_2(F) \to A$ carrying the symbol $\{x, y\}$ to $c(x, y)$; or, equivalently, the map $F^* \times F^* \to K_2(F), (x, y) \mapsto \{x, y\}$ is the universal Steinberg symbol.
For any number field $F$ the inclusion $OF \to F$ induces a monomorphism $\varrho_F : K_2(OF) \to K_2(F)$. For any nonzero prime ideal $p$ of $OF$, let $\tau_p : K_2(F) \to k(p)^*$ be the tame symbol (it is a Steinberg symbol), determined by the formula

$$\tau_p(a, b) = (-1)^{v_p(a)v_p(b)} a^{v_p(b)} b^{-v_p(a)} \pmod{p}$$

where $k(p)$ is the residue field at $p$ and $v_p$ is the $p$-adic valuation on $F$. Now let

$$T_F : K_2(F) \to \bigoplus_{p \text{ prime}} k(p)^*$$

be the map $z \mapsto \{\tau_p(z)\}_p$. Then $T_F$ is surjective and $\varrho_F$ induces a natural isomorphism from $K_2(OF)$ onto $\ker(T_F)$, the tame kernel.

For a commutative ring $R$, if $a, b \in R$ with $1 + ab \in R^*$, the Dennis–Stein symbol $\langle a, b \rangle \in K_2(R)$ is defined by the formula

$$\langle a, b \rangle = x_{21} \left( \frac{-b}{1 + ab} \right) x_{12}(a)x_{21}(b) x_{12} \left( \frac{-a}{1 + ab} \right) h_{12}(1 + ab)^{-1}$$

(see [1], [2]). These are related to the symbols $\{u, v\}$ by the formulae

$$\langle a, b \rangle = \begin{cases} \{-a, 1 + ab\} & \text{if } a \in R^*, \\ \{1 + ab, b\} & \text{if } b \in R^*. \end{cases}$$

Thus the symbol

$$\{u, v\} = \left( -u, \frac{1 - v}{u} \right) = \left( \frac{u - 1}{v}, v \right)$$

is also a Dennis–Stein symbol if $u, v \in R^* - \{1\}$.

2. The homomorphism $I$. Let $n \in \mathbb{N}$ and suppose that $F$ is a number field containing the $n$th roots of unity and let $S$ be a finite set of primes containing the infinite primes of $F$ and the primes of $F$ which divide $n$. Let $\mu_n \subset F$ be the group of $n$th roots of unity. For an abelian group $A$, we write $A/n$ for $A \otimes \mathbb{Z}/n\mathbb{Z}$.

Based on the ideas of Tate in [12], Keune in [6] introduced a homomorphism

$$I : \mu_n \otimes Cl(\mathcal{O}_S) \to K_2(\mathcal{O}_S)/n$$

(where $\mathcal{O}_S$ is the ring of $S$-integers of $F$) defined as follows:

$$I(\zeta \otimes [a]) = z^n \pmod{K_2(\mathcal{O}_S)^n}$$

where $z \in K_2(F)$ is any element satisfying $\tau_p(z) \equiv \zeta^{v_p(a)} \pmod{p}$ for all $p \not\in S$. He proved (see Section 3 of [6]) that this map is injective and, furthermore, it fits into an exact sequence

$$0 \to \mu_n \otimes Cl(\mathcal{O}_S) \xrightarrow{I} K_2(\mathcal{O}_S)/n \xrightarrow{\lambda} \bigoplus_{p \in S} \mu_n \to \mu_n \to 0.$$
Here $S_0$ denotes the set of finite and infinite real primes of $S$ and $\lambda$ is induced by the Hilbert symbols of order $n$ for each of the primes of $S_0$. Thus, $I$ is an isomorphism precisely when $S_0$ is a singleton. Furthermore, from the construction of $I$ (see [6], Section 3) it follows that the image of $I$ is precisely the group
\[ K_2(O_S) \cap K_2(F)^n / K_2(O_S)^n. \]

The case of immediate interest in this paper is when $n = p^r$ with $p$ prime and $r \geq 1$ and $S$ consists of the infinite primes of $F$ together with the finite primes which divide $p$. In this case $Cl(O_S) = Cl(O_F[1/p])$ and $K_2(O_S) = K_2(O_F[1/p])$. However, from the localisation sequence for $K$-theory, we deduce that the natural map $K_2(O_F) \to K_2(O_F[1/p])$ is injective and induces an isomorphism on $p$-Sylow subgroups (since the order of $k(p)^*$ is prime to $p$ if $p \mid p$). Thus, in this case, the exact sequence above takes the form
\[ 0 \to \mu_{p^r} \to Cl\left(O_F \left[ \frac{1}{p} \right]\right) \xrightarrow{I} K_2(O_F)/p^r \xrightarrow{\lambda} \bigoplus_{p \in S_0} \mu_{p^r} \to 0. \]

3. Main results. Throughout this section, $p$ will denote a fixed prime number and $F$ will denote a number field containing the $p^r$th roots of unity ($r \geq 1$). $\zeta = \zeta_{p^r}$ denotes a fixed primitive $p^r$th root of unity in $F$ and $\mu = \mu_{p^r}$ the cyclic group generated by $\zeta$. For any number field $K$, $H_K$ will denote the maximal abelian unramified extension of $K$ in which all primes dividing $p$ split completely. For an abelian extension $K/L$ of number fields and any prime $p$ of $L$ which does not ramify in $K$, $(K/L_p) \in Gal(K/L)$ will denote the Frobenius of $p$.

In [8] and in his Ph.D. thesis (University of Nijmegen, 1992), Mulders proved the following: if $r = 1$ and $\sqrt[p^r]{\zeta} \notin F$, and if $F(\sqrt[p^r]{O_F^*}) \notin H_F$ and $F$ is not imaginary quadratic then the image of $I$ is generated by classes of Dennis–Stein symbols. In this section we prove our main result; namely, that if $\sqrt[p^r]{\zeta} \notin F$, but $F(\sqrt[p^r]{O_F}) \subset H_F$ and if, furthermore, for each $u \in O_F^*$ if $L = F(\sqrt[p^r]{u})$ then $L(\sqrt[p^r]{u}) \subset H_L$, then the image of $I$ is not generated by Dennis–Stein symbols (see Section 4 for examples of fields in which these conditions hold).

For any number field $L$ containing $F$ and any prime ideal $p$ of $L$, define $\varepsilon_p$ by
\[ \varepsilon_p = \begin{cases} |k(p)^*|/p^r & \text{if } p \text{ does not divide } p, \\ 0 & \text{if } p \text{ divides } p. \end{cases} \]

(If $p$ does not divide $p$, then the map $\mu \mapsto k(p)$ is injective and so $p^r$ divides $|k(p)^*|$.)
Lemma 3.1. Suppose that $L/F$ is a finite extension and $x \in L^*$. Let $K = L(\sqrt[p]{x})$ and for any prime ideal $\mathfrak{p}$ of $L$ which does not ramify in $K$ define $l_\mathfrak{p} \in \mathbb{Z}/p^r\mathbb{Z}$ by the formula
\[
\left( \frac{K/L}{\mathfrak{p}} \right) (\sqrt[p]{x}) = \zeta^{l_\mathfrak{p}} \sqrt[p]{x}.
\]
Then

(i) $x^{p^r} \equiv \zeta^{l_\mathfrak{p}} (\mod \mathfrak{p})$.

(ii) If $K/L$ is unramified then $\sum l_\mathfrak{p} \varepsilon_\mathfrak{p}(a) \equiv 0 (\mod p^r)$ for all $a \in L^*$.

Proof. (i) Let $\mathfrak{P}$ be a prime ideal of $K$ lying over $\mathfrak{p}$. Then
\[
\zeta^{l_\mathfrak{p}} \sqrt[p]{x} = \left( \frac{K/L}{\mathfrak{p}} \right) (\sqrt[p]{x}) \equiv (\sqrt[p]{x})^{k(\mathfrak{p})|} (\mod \mathfrak{P})
\]
and thus
\[
\zeta^{l_\mathfrak{p}} \equiv (\sqrt[p]{x})^{k(\mathfrak{p})|} (\mod \mathfrak{P}).
\]
If $\mathfrak{p}$ does not divide $p$, this gives
\[
\zeta^{l_\mathfrak{p}} \equiv x^{p^r} \quad (\mod \mathfrak{p})
\]
(since $p^r$ then divides $|k(\mathfrak{p})|$) proving (i). If $\mathfrak{p}$ divides $p$ then $\zeta \equiv 1 \quad (\mod \mathfrak{p})$ and (i) holds by definition.

(ii) Since $K/L$ is unramified, there are well defined homomorphisms
\[
Cl(O_L) \rightarrow \text{Gal}(K/L) \rightarrow \mathbb{Z}/p^r\mathbb{Z}, \quad [p] \mapsto \left( \frac{K/L}{\mathfrak{p}} \right) \mapsto l_\mathfrak{p}.
\]
Part (ii) of the lemma simply says that this map is trivial on principal ideals.

Note that if $\mathfrak{p}$ does not divide $p$ then there is a natural isomorphism $\mu \rightarrow \mu_{p^r}(k(\mathfrak{p}))$. Since $\varepsilon_\mathfrak{p} = 0$ if $\mathfrak{p}$ divides $p$, it follows that for all $\mathfrak{p}$ there is a well-defined homomorphism $\varrho_\mathfrak{p} : k(p)^* \rightarrow \mu$ satisfying
\[
x^{p^r} \equiv \varrho_\mathfrak{p}(x) \quad (\mod \mathfrak{p})
\]
for all $x \in k(p)^*$. Let $\Phi = \Phi_L : K_2(L) \rightarrow \mu$ be the map $z \mapsto \prod \varrho_\mathfrak{p}(\tau_\mathfrak{p}(z))$. (Thus, if $\mathfrak{p}$ does not divide $p$, then $\varrho_\mathfrak{p} \circ \tau_\mathfrak{p}$ is just the map induced on $K_2$ by the Hilbert symbol of order $p^r$—see [10], Section III.5—and $\Phi$ is the product of these over all primes of $L$ not dividing $p$.)

For $A, B \subseteq L^*$, let $\{A, B\}$ denote the subgroup of $K_2(L)$ generated by symbols $\{a, b\}$, $a \in A$, $b \in B$.

Lemma 3.2. Suppose that $W \subset O_L^*$ is a subgroup with the property that $L(\sqrt[p]{W}) \subset \mathcal{H}_L$. Then
\[
\{W, L^*\} \subset \text{Ker}(\Phi_L).
\]
Proof. Suppose that $z = \{u, a\}$ with $u \in W$, $a \in L^*$. Let $K = L(\sqrt[p]{u})$. By definition of $\mathcal{H}_L$, $K/L$ is unramified and primes above $p$ split in this
extension. For a prime ideal \( p \) of \( L \), \( \tau_p(z) \equiv u^v_p(a) \pmod{p} \). Thus \( \tau_p(z)^{\epsilon_p} \equiv u^v_p \zeta^{l_p v_p(a)} \pmod{p} \) by Lemma 3.1(i). It follows that for \( p \) not dividing \( p \), \( \varrho_p(\tau_p(z)) = \zeta^{l_p v_p(a)} \in \mu \). On the other hand, if \( p \) divides \( p \) then \( l_p = 0 \) since \( p \) splits in \( K \) and this formula also holds in this case. Thus

\[
\Phi(z) = \prod_p \varrho_p(\tau_p(z)) = \prod_p \zeta^{l_p v_p(a)} = \zeta^{\sum_p l_p v_p(a)} = 1
\]

by Lemma 3.1(ii). This proves the lemma.

We will also need the following property of the map \( \Phi \):

**Lemma 3.3.** For any extension \( L/F \), the diagram

\[
\begin{array}{ccc}
K_2(L) & \xrightarrow{\phi_L} & \mu \\
\downarrow{\text{tr}_{L/F}} & & \downarrow{=} \\
K_2(F) & \xrightarrow{\phi_F} & \mu
\end{array}
\]

commutes, where \( \text{tr}_{L/F} : K_2(L) \to K_2(F) \) is the K-theory transfer.

**Proof.** For a number field \( E \), let \( T_E : K_2(E) \to \bigoplus_p k(p)^* \) be the tame homomorphism. For any extension \( L/F \) it is known that the diagram

\[
\begin{array}{ccc}
K_2(L) & \xrightarrow{T_L} & \bigoplus_q k(q)^* \\
\downarrow{\text{tr}_{L/F}} & & \downarrow{N_L/F} \\
K_2(F) & \xrightarrow{T_F} & \bigoplus_p k(p)^*
\end{array}
\]

commutes, where \( N_L/F \) is the map \( \{ \alpha_q \}_{q} \to \{ \prod_{q \mid p} N_p^q(\alpha_q) \}_{p} \) and \( N_p^q = N_{k(q)/k(p)} \) (see [6], Section 4, for properties of the transfer).

Thus, in view of the definition of \( \Phi \), we reduce to showing that for any prime \( p \) of \( F \) not dividing \( p \) we have \( \prod_{q \mid p} (\alpha_q)^{x_q} = \prod_{q \mid p} N_p^q(\alpha_q)^{x_q} \). This follows from the fact that for \( q \mid p \) and \( x \in k(q)^* \), \( N_p^q(x)^{x_q} = x^{x_q} \), which is easily verified.

With these preliminaries, we can prove our main theorem:

**Theorem 3.4.** Suppose that \( F(\sqrt[p]{\mathcal{O}_F}) \subset \mathcal{H}_F \), \( \sqrt[p]{\zeta} \notin F \) and \( L(\sqrt[p]{u}) \subset \mathcal{H}_L \) for any \( u \in \mathcal{O}_F \) where \( L = F(\sqrt[p]{u}) \). Then the image of \( I \) is not generated by Dennis–Stein symbols.

**Proof.** Let \( E = F(\sqrt[p]{\zeta}) \) and note that the Artin map induces a surjective homomorphism

\[
\text{Cl}(\mathcal{O}_F[1/p]) \to \text{Gal}(E/F), \quad [p] \mapsto \left( \frac{E/F}{p} \right)
\]
(well-defined since primes above \( p \) split in \( E \)). Let \( \mathcal{C} \) denote the kernel of this map. Then \( \text{Cl}(\mathcal{O}_F[1/p])/\mathcal{C} \) is isomorphic to \( \text{Gal}(E/F) \) which is a nontrivial cyclic group of order \( p^s \) for some \( s \) with \( 1 \leq s \leq r \) and thus the image of \( \mu \otimes \mathcal{C} \rightarrow \mu \otimes \text{Cl}(\mathcal{O}_F[1/p]) \) has index \( p^s \).

We will show the following: If \( x \in K_2(\mathcal{O}_F) \) is of the form \( \{u,a\} \) for some \( u \in \mathcal{O}_F^\times \) and \( x \equiv I(y) \mod \{K_2(\mathcal{O}_F)^{p^r}\} \) for some \( y \in \mu \otimes \text{Cl}(\mathcal{O}_F[1/p]) \), then \( y \in \mu \otimes \mathcal{C} \). (In particular, if \( I(y) \) is represented by a Dennis–Stein symbol then \( y \in \mu \otimes \mathcal{C} \).)

Suppose, to the contrary, that \( \mathfrak{p} \) is a prime ideal of \( F \) such that \( \left( \frac{E/F}{\mathfrak{p}} \right) \neq 1 \) (and hence \( \mathfrak{p} \) does not divide \( p \) and \( \zeta \otimes [\mathfrak{p}] \) is a nontrivial element of \( \mu \otimes \text{Cl}(\mathcal{O}_F[1/p]) \)) and \( I(\zeta \otimes [\mathfrak{p}]) \cong \{u,a\} \mod K_2(\mathcal{O}_F)^{p^r} \) with \( u \in \mathcal{O}_F^\times \).

We will show this leads to a contradiction.

By construction of \( I \), there exists \( z \in K_2(F) \) satisfying

\[
\tau_q(z) = \begin{cases} 
\zeta \pmod{\mathfrak{q}} & \text{if } \mathfrak{q} = \mathfrak{p}, \\
1 \pmod{\mathfrak{q}} & \text{if } \mathfrak{q} \neq \mathfrak{p},
\end{cases}
\]

and \( z^{p^r} \equiv \{u,a\} \mod K_2(\mathcal{O}_F)^{p^r} \).

Thus \( z^{p^r} = \{u,a\}w^{p^r} \) for some \( w \in K_2(\mathcal{O}_F) \), and replacing \( z \) by \( zw^{-1} \) if necessary, we can assume that \( z^{p^r} = \{u,a\} \) (while still satisfying (1)). Now let \( L = F(\sqrt[p^r]{u}) \). Then \( a = N_{L/F}(b) \) for some \( b \in L \) (by [7], Corollary 15.11) and \( \{u,a\} = \text{tr}_{L/F}(\{u,b\}) = (\text{tr}_{L/F}(\sqrt[p^r]{u},b))^{p^r} \). Hence \( \text{tr}_{L/F}(\sqrt[p^r]{u},b) = z\{\zeta,c\} \) for some \( c \in F^\times \), since \( z^{-1} \text{tr}_{L/F}(\sqrt[p^r]{u},b) \) lies in the \( p^r \)-torsion subgroup of \( K_2(F) \) which equals \( \{\zeta,F^\times\} \) by [11], Theorem 1.8.

Thus \( \Phi_L(\sqrt[p^r]{u},b) = \Phi_F(\zeta)\Phi_F(\zeta,c) \) by Lemma 3.3. Now \( \Phi_F(\zeta,c) = 1 \) by Lemma 3.2 and \( \Phi_F(\zeta) = \zeta^{e_p} \) by (1). However, by Lemma 3.1(i) we get

\[
\zeta^{e_p} \equiv \zeta^{e_p} \pmod{\mathfrak{p}}
\]

where \( e_p \in \mathbb{Z}/p^r\mathbb{Z} \) is defined by

\[
\left( \frac{E/F}{\mathfrak{p}} \right)^{p^r}\zeta = \zeta^{e_p} \sqrt[p^r]{\zeta}
\]

and thus \( e_p \neq 0 \) by choice of \( \mathfrak{p} \). Furthermore since \( \mathfrak{p} \) does not divide \( p \), \( \zeta \) has order \( p^r \) in \( k(\mathfrak{p})^{p^r} \) and thus

\[
\Phi_L(\sqrt[p^r]{u},b) = \zeta^{e_p} = \zeta^{e_p} \neq 1,
\]

contradicting Lemma 3.2 since \( L(\sqrt[p^r]{u}) = L(\sqrt[p^r]{u}) \subset \mathcal{H}_L \) by hypothesis. This proves the theorem.

Note that the proof establishes the slightly stronger fact that, under the given hypotheses, the image of \( I \) is not generated by elements of the form \( \{u,a\} \) with \( u \in \mathcal{O}_F^\times \). We also obtain immediately:
Corollary 3.5. If $F$ satisfies the conditions of the last theorem and if furthermore $F$ is totally imaginary and there is only one prime of $F$ above $p$, then $K_2(O_F)$ is not generated by Dennis–Stein symbols.

Proof. The hypotheses imply that $S_0$ is a singleton (see Section 2) and thus $I$ is an isomorphism. So $K_2(O_F)/p^r$, and hence $K_2(O_F)$ itself, is not generated by Dennis–Stein symbols.

In the positive direction, however, we can prove the following result, which guarantees that a large part of the image of $I$ will be generated by Dennis–Stein symbols, even under the conditions of the last theorem:

Theorem 3.6. Suppose that $F$ is not imaginary quadratic, $q\sqrt{H} \not\in F$ and $\sqrt{O_F^*}/F$. Let $E = F(q\sqrt{H})$ and let $C$ be the kernel of the Artin map $\mathrm{Cl}(O_F[1/p]) \to \text{Gal}(E/F)$. Then $I(\mu \otimes C)$ is generated by Dennis–Stein symbols.

Proof. Let $\phi$ denote the isomorphism from $\text{Cl}(O_F[1/p])$ to $\text{Gal}(H_F/F)$ and for any intermediate field $K$ let $\phi_K$ be the map $x \mapsto \phi(x)|_K \in \text{Gal}(K/F)$. So $C = \ker(\phi_E)$.

Fix a unit $u \in O_F^*$ such that $u \not\in (\zeta)(O_F^*)^p$. Let $L = F(q\sqrt{H})$. By choice of $u$, $E/F$ and $L/F$ are linearly disjoint subextensions of $H_F/F$, and $\text{Gal}(L/F)$ and $\text{Gal}(E/F)$ are cyclic extensions of order $p^r$. Let $G$ be the unique subgroup of $\text{Gal}(L/F)$ of index $p$. Then $C \not\subseteq \phi_L^{-1}(G)$ since we can choose $\tau \in \text{Gal}(H_F/F)$ with $\tau|_L \not\in G$ but $\tau|_E = 1$ so that $\phi^{-1}(\tau) \in C - \phi_L^{-1}(G)$. Thus $C_1 = \phi_L^{-1}(G) \cap C$ has index $p$ in $C$ so $C$ is generated as a group by $C - C_1$.

Now let $x \in C - C_1$. Let $\sigma = \phi(x)$. So $\sigma|_E = 1$ and $\sigma|_L$ has order $p^r$. Choose $\sigma_1 \in \text{Gal}(H/F)$ such that $\sigma_1|_E$ has order $p^r$.

Choose distinct primes $p$, $p_1$ and $p_2$ not dividing $p$ such that $x = [p]$ and $\phi([p_1]) = \sigma_1$ and $[p_2] = ([p][p_1])^{-1}$ in $\text{Cl}(O_F)$. Let $\sigma_2 = \phi([p_2])$.

Since $\phi_E([p]) = 1$, $\zeta^{p^r} \equiv 1 \pmod{p}$ by Lemma 3.1(i) and hence $p^r$ divides $\varepsilon_p$ and thus $p^{2r}$ divides $|k_p|^*$. On the other hand, $\phi_L([p])$ has order $p^r$ and so $u^{p^r} \equiv \zeta^{p^r} \pmod{p}$ where $l_p$ is a generator of $\mathbb{Z}/p^r\mathbb{Z}$ (by Lemma 3.1(i) again) and thus $p^{2r}$ divides the order of $u \in k_p^*$.

For $i = 1, 2$, $\phi_E([p_i]) = \sigma_i|_E$ has order $p^r$ and thus $\zeta^{p^r} \equiv \zeta^{p^r} \pmod{p_1}$ where $\varepsilon_{p_1}$ is a generator of $\mathbb{Z}/p^r\mathbb{Z}$. Thus $\varepsilon_{p_i}$ is not divisible by $p$. It follows that $p^r$ is the exact power of $p$ dividing $|k_p|^*$ for $i = 1, 2$. Thus the order of $u$ in $k_p^*$ is of the form $p^{k_i s_i}$ where $0 \leq k_i \leq r$ and $p$ does not divide $s_i$.

It follows that there exists $t \in \mathbb{Z}$ such that $u^t \equiv \zeta \pmod{p}$ and $u^t \equiv 1 \pmod{p_i}$ for $i = 1, 2$. Let $w = u^t$ and let $a \in O_F$ be a generator of the principal ideal $pp_1p_2$. Let $z = \{a, w\} \in K_2(F)$. Then $\tau_p(z) = w \pmod{p}$ while $\tau_{p_1}(z) = w \pmod{p_1}$ and $\tau_{p_2}(z) = 1 \pmod{p_2}$ if $q \neq p, p_1, p_2$ then $q$ does not divide $a$ and $\tau_q(z) = 1$. Thus $I(\zeta \otimes [p]) = I(\zeta \otimes w) \equiv z^{p^r}$.
Since \( wp^r \equiv 1 \pmod{a} \). So \( I(\zeta \otimes x) \) is represented by a Dennis–Stein symbol as required.

**Corollary 3.7.** If the hypotheses of Theorem 3.4 are satisfied and \( \sqrt[p]{\zeta} \not\in F \) then the subgroup of the image of \( I \) generated by Dennis–Stein symbols is precisely \( I(\mu \otimes \mathcal{C}) \) and has index \( p^r = [E : F] \).

**Proof.** This follows from the proof of Theorem 3.4 where it is shown that the subgroup of the image of \( I \) which is generated by Dennis–Stein symbols is contained in \( I(\mu \otimes \mathcal{C}) \), together with Theorem 3.6

**Remarks.** Consider again the case where \( r = 1 \). Suppose that the hypotheses of Theorem 3.6 do not hold, in the sense that \( \sqrt[p]{\zeta} \not\in F \) and \( \sqrt[p]{\mathcal{O}_F} \not\subset H_F \). Then Mulders shows (in [9], Section 2.4) that the image of \( I \) is generated by Dennis–Stein symbols. However, if \( \sqrt[p]{\zeta} \not\in F \) but \( \sqrt[p]{\mathcal{O}_F} \subset H_F \) then the methods used in the proof of Theorem 3.4 can be used to show that the image of \( I \) is not generated by Dennis–Stein symbols of the type constructed by Mulders (namely Dennis–Stein symbols of the form \( \{a, w^p\} \) with \( u \in \mathcal{O}_F \)). However, Theorem 3.4 only proves that the image of \( I \) is not generated by Dennis–Stein symbols of any kind under stronger hypotheses; namely \( \sqrt[p]{\zeta} \not\in F \) but \( \sqrt[p]{\mathcal{O}_F} \subset H_F \) and for all \( u \in \mathcal{O}_F \), \( F(\sqrt[p]{u}) \subset H_L \). Thus it remains open whether the image of \( I \) can be generated by Dennis–Stein symbols under the hypotheses of Theorem 3.6.

**4. Examples.** In order to construct examples of fields satisfying the hypotheses of Theorem 3.4, it suffices to find fields in which the primes above \( p \) split in appropriate extensions.

**Lemma 4.1.** Suppose that \( F \) is totally imaginary, \( \zeta \in F \) and \( \sqrt[p]{\zeta} \not\in F \). Let \( u_1, \ldots, u_s \) be a system of fundamental units of \( F \). Then \( F \) satisfies the hypotheses of Theorem 3.4 if and only if every prime above \( p \) in \( F \) splits completely in each of the extensions \( F(\sqrt[p]{\zeta}) \) and in \( F(\sqrt[p]{\zeta}) \).

**Proof.** It follows from the hypotheses that every prime above \( p \) splits completely in the Galois extension \( F(\sqrt[p]{\zeta}, \sqrt[p]{u_1}, \ldots, \sqrt[p]{u_s}) = F(\sqrt[p]{\mathcal{O}_F})/F \). Now, if \( L \) is any field containing \( \zeta \) and if \( u \) is a unit of \( L \), then \( L(\sqrt[p]{u})/L \) is an abelian extension and the only primes that may ramify in \( L \) are primes above \( p \) or primes at infinity. Thus if it is known that these primes split completely in \( L(\sqrt[p]{u}) \) then \( L(\sqrt[p]{u})/L \) is unramified abelian and thus contained in \( H_L \). Thus \( F(\sqrt[p]{\mathcal{O}_F})/F \) is unramified and for each \( u \in \mathcal{O}_F \), the abelian extension \( F(\sqrt[p]{u})/F(\sqrt[p]{u}) \) is unramified.
In the case \( p = 2, r = 1 \) we can construct biquadratic fields with the necessary properties:

**Lemma 4.2.** Suppose that \( d ≡ 1 \pmod{8}, d > 1 \) (and squarefree) and suppose that \( f ≡ 7 \pmod{8}, f > 0 \) (squarefree) with the property that if \( u \) is a fundamental unit of \( K = \mathbb{Q}(\sqrt{T}) \) then the prime above 2 in \( K \) splits completely in \( K(\sqrt{u}) \). Then the biquadratic field

\[
F = \mathbb{Q}(\sqrt{-2d}, \sqrt{f})
\]

satisfies the conditions of Theorem 3.4 for \( p = 2, r = 1 \).

**Proof.** Clearly the prime 2 totally ramifies in \( F \) since 2 ramifies in each of the quadratic subextensions \( \mathbb{Q}(\sqrt{-2d}), \mathbb{Q}(\sqrt{T}) \) and \( \mathbb{Q}(\sqrt{-2df}) \). So \( 2\mathcal{O}_F = p^3 \) for some prime ideal \( p \) of \( \mathcal{O}_F \).

\( F \) is totally imaginary and so the rank of the group of units is 1. Furthermore, \( u \) is clearly a fundamental unit of \( F \) because \( \sqrt{u}, \sqrt{-u} \notin \mathbb{F} \) (since the prime above 2 in \( K \) ramifies in \( F \) but splits in \( K(\sqrt{u}) \) and \( K(\sqrt{-1}) \)) and hence also in \( K(\sqrt{-u}) \). The conditions on \( f \) thus guarantee that \( p \) splits completely in \( F(\sqrt{u}) \).

\( p \) splits in \( F(\sqrt{-2}) \) since \( F(\sqrt{-2}) \supset \mathbb{Q}(\sqrt{-2}, \sqrt{-2d}) \supset \mathbb{Q}(\sqrt{d}) \) and 2 splits in this last field since \( d ≡ 1 \pmod{8} \).

\( p \) splits in \( F(\sqrt{2}) \) since \( F(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{-2d}, \sqrt{T}) \supset \mathbb{Q}(\sqrt{-2df}) \) and 2 splits in this last field since \( -df ≡ 1 \pmod{8} \).

Thus \( p \) splits completely in \( F(\sqrt{2}, \sqrt{-2}) = F(\sqrt{-1}) \).

Since \( \mathcal{O}_F^* = \{-1\} \times \langle u \rangle \), it follows from Lemma 4.1 that \( F \) satisfies the hypotheses of Theorem 3.4.

**Remarks on Lemma 4.2.** (i) Since there is exactly one prime above 2 in \( F \) and no real infinite primes, the map

\[
I : \mu_2 \otimes \text{Cl}(\mathcal{O}_F[1/2]) \to K_2(\mathcal{O}_F)/2
\]

is an isomorphism in this case and thus \( K_2(\mathcal{O}_F)/2 \), and hence \( K_2(\mathcal{O}_F) \) itself is not generated by Dennis–Stein symbols.

(ii) According to the proof of Theorem 3.4 if \( p \) is a prime ideal of \( \mathcal{O}_F \) not splitting in \( F(\sqrt{-1}) \) then \( I(-1 \otimes [p]) \) is a nontrivial element of \( K_2(\mathcal{O}_F)/2 \) not represented by an element of the form \( \{w, a\} \) where \( w \in \mathcal{O}_F^* \). In particular, if \( I(-1 \otimes [p]) \) is represented by \( z \in K_2(\mathcal{O}_F) \) then \( z \notin \{-1, F^*\} \), which is the 2-torsion part of \( K_2(F) \), and thus \( z \) has order divisible by 4.

(iii) The only number less than 2000 satisfying the conditions for \( f \) in the lemma is \( f = 1751 = 17 \cdot 103 \) (verified using the computer programme PARI/GP) and thus the smallest example of such a field is

\[
F = \mathbb{Q}(\sqrt{-34}, \sqrt{17}) = \mathbb{Q}(\sqrt{-34}, \sqrt{-206}).
\]
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Received on 11.8.1998
and in revised form on 19.10.1998
(3440)